Limits and Continuity

Rate of Change

1. The traffic flow at a particular intersection is modeled by the function \( f \) defined by \( f(t) = 25 + 6 \cos\left(\frac{\pi}{3}t\right) \) for \( 0 \leq t \leq 120 \). What is the average rate of change of the traffic flow over the time interval \( 30 \leq t \leq 40 \).

   (A) 0.743  (B) 0.851  (C) 0.935  (D) 1.176

2. The rate of change of the altitude of a hot air balloon rising from the ground is given by \( y(t) = t^3 - 3t^2 + 3t \) for \( 0 \leq t \leq 10 \). What is the average rate of change in altitude of the balloon over the time interval \( 0 \leq t \leq 10 \).

   (A) 56  (B) 73  (C) 85  (D) 94

Free Response Questions

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) ) (ft/sec)</td>
<td>0</td>
<td>28</td>
<td>43</td>
<td>67</td>
<td>82</td>
<td>85</td>
<td>74</td>
<td>.58</td>
<td>42</td>
<td>35</td>
</tr>
</tbody>
</table>

3. The table above shows the velocity of a car moving on a straight road. The car’s velocity \( v \) is measured in feet per second.

   (a) Find the average velocity of the car from \( t = 60 \) to \( t = 90 \).

   (b) The instantaneous rate of change of \( f \) (See Ch. 2.1 for an explanation of instantaneous rate of change) with respect to \( x \) at \( x = a \) can be approximated by finding the average rate of change of \( f \) near \( x = a \). Approximate the instantaneous rate of change of \( f \) at \( x = 40 \) using two points, \( x = 30 \) and \( x = 50 \).
Limit of a Function and One Sided Limits

1. \( \lim_{x \to \pi/6} \sec^2 x = \)

   (A) \( \frac{3}{4} \)  
   (B) \( \frac{\sqrt{3}}{2} \)  
   (C) \( \frac{4}{3} \)  
   (D) \( \frac{2\sqrt{3}}{3} \)

2. If \( f(x) = \begin{cases} x^2 + 3, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases} \), then \( \lim_{x \to 1} f(x) = \)

   (A) 1  
   (B) 2  
   (C) 3  
   (D) 4

3. \( \lim_{x \to 1} \frac{|x-1|}{1-x} = \)

   (A) -2  
   (B) -1  
   (C) 1  
   (D) nonexistent

4. Let \( f \) be a function given by \( f(x) = \begin{cases} 3-x^2, & \text{if } x < 0 \\ 2-x, & \text{if } 0 \leq x < 2 \\ \sqrt{x-2}, & \text{if } x > 2 \end{cases} \).

Which of the following statements are true about \( f \)?

I. \( \lim_{x \to 0} f(x) = 2 \)

II. \( \lim_{x \to 2} f(x) = 0 \)

III. \( \lim_{x \to 1} f(x) = \lim_{x \to 0} f(x) \)

(A) I only  
(B) II only  
(C) II and III only  
(D) I, II, and III
Free Response Questions

Questions 5-11 refer to the following graph.

The figure above shows the graph of \( y = f(x) \) on the closed interval \([-4, 9]\).

5. Find \( \lim_{x \to -4} \cos(f(x)) \).

6. Find \( \lim_{x \to 2^-} f(x) \).

7. Find \( \lim_{x \to 2^+} f(x) \).

8. Find \( \lim_{x \to 2} f(x) \).

9. Find \( f(2) \).

10. Find \( \lim_{x \to 5^-} \arctan(f(x)) \).

11. Find \( \lim_{x \to 5^-} [x f(x)] \).
Calculating Limits Using the Limit Laws

1. \( \lim_{x \to \pi/3} \frac{\sin(\pi - x)}{\pi - x} = \)

   (A) \(-1\)  (B) \(0\)  (C) \(\frac{\sqrt{3}}{2}\)  (D) \(1\)

2. \( \lim_{x \to 0} \frac{\sin 3x}{\sin 2x} = \)

   (A) \(\frac{2}{3}\)  (B) \(1\)  (C) \(\frac{3}{2}\)  (D) nonexistent

3. \( \lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} = \)

   (A) \(\frac{1}{8}\)  (B) \(\frac{1}{4}\)  (C) \(\frac{1}{2}\)  (D) nonexistent

4. \( \lim_{x \to 1} \frac{\sqrt{3 + x} - 2}{x^3 - 1} = \)

   (A) \(\frac{1}{12}\)  (B) \(\frac{1}{6}\)  (C) \(\sqrt{3}\)  (D) nonexistent

5. \( \lim_{\theta \to 0} \frac{\theta + \theta \cos \theta}{\sin \theta \cos \theta} = \)

   (A) \(\frac{1}{4}\)  (B) \(\frac{1}{2}\)  (C) \(1\)  (D) \(2\)
6. \( \lim_{x \to 0} \frac{\tan 3x}{x} = \)

(A) 0  \hspace{1cm} (B) \frac{1}{5}  \hspace{1cm} (C) 1  \hspace{1cm} (D) 3

7. \( \lim_{x \to 3} \frac{x - 3}{x - 3} = \)

(A) \frac{1}{9}  \hspace{1cm} (B) \frac{1}{9}  \hspace{1cm} (C) -9  \hspace{1cm} (D) 9

**Free Response Questions**

8. If \( \lim_{x \to 0} \frac{\sqrt{2 + ax} - \sqrt{2}}{x} = \sqrt{2} \) what is the value of \( a \)?

9. Find \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \), if \( f(x) = \sqrt{2x + 1} \).

10. Find \( \lim_{x \to 0} \frac{f(x) - g(x)}{\sqrt{g(x) + 7}} \), if \( \lim_{x \to 0} f(x) = 2 \) and \( \lim_{x \to 0} g(x) = -3 \).

11. Find \( \lim_{x \to \sqrt{3}} g(x) \), if \( \lim_{x \to \sqrt{3}} \frac{1}{x^2 + g(x)} = \frac{1}{5} \).
Continuity and Intermediate Value Theorem

1. Let \( f \) be a function defined by \( f(x) = \begin{cases} \frac{x^2 - a^2}{x-a}, & \text{if } x \neq a \\ 4, & \text{if } x = a \end{cases} \). If \( f \) is continuous for all real numbers \( x \), what is the value of \( a \)?

   (A) \( \frac{1}{2} \) \hspace{1cm} (B) \( 0 \) \hspace{1cm} (C) \( 1 \) \hspace{1cm} (D) \( 2 \)

2. The graph of a function \( f \) is shown above. If \( \lim_{x \to a} f(x) \) exists and \( f \) is not continuous at \( x = a \), then \( a = \)

   (A) \( -1 \) \hspace{1cm} (B) \( 0 \) \hspace{1cm} (C) \( 2 \) \hspace{1cm} (D) \( 4 \)

3. If \( f(x) = \begin{cases} \frac{\sqrt{3x} - 1 - \sqrt{2x}}{x-1}, & \text{for } x \neq 1 \\ a, & \text{for } x = 1 \end{cases} \), and if \( f \) is continuous at \( x = 1 \), then \( a = \)

   (A) \( \frac{1}{4} \) \hspace{1cm} (B) \( \frac{\sqrt{2}}{4} \) \hspace{1cm} (C) \( \sqrt{2} \) \hspace{1cm} (D) \( 2 \)
4. Let \( f \) be a continuous function on the closed interval \([-2, 7]\). If \( f(-2) = 5 \) and \( f(7) = -3 \), then the Intermediate Value Theorem guarantees that

(A) \( f'(c) = 0 \) for at least one \( c \) between \(-2\) and \( 7 \)
(B) \( f''(c) = 0 \) for at least one \( c \) between \(-3\) and \( 5 \)
(C) \( f(c) = 0 \) for at least one \( c \) between \(-3\) and \( 5 \)
(D) \( f(c) = 0 \) for at least one \( c \) between \(-2\) and \( 7 \)

Free Response Questions

5. Let \( g \) be a function defined by \( g(x) = \begin{cases} \frac{\pi \sin x}{x}, & \text{if } x < 0 \\ a - bx, & \text{if } 0 \leq x < 1 \\ \arctan x, & \text{if } x \geq 1 \end{cases} \). If \( g \) is continuous for all real numbers \( x \), what are the values of \( a \) and \( b \)?

6. Evaluate \( \lim_{a \to 0} \frac{-1 + \sqrt{1 + a}}{a} \).

7. What is the value of \( a \), if \( \lim_{x \to 0} \frac{\sqrt{ax + 9} - 3}{x} = 1 \)?
Limits and Asymptotes

1. \[ \lim_{x \to \infty} \frac{3 + 2x^2 - x^4}{3x^4 - 5} = \]
   
   (A) \(-2\) \hspace{1cm} (B) \(-\frac{1}{3}\) \hspace{1cm} (C) \(\frac{1}{5}\) \hspace{1cm} (D) 1

2. What is \( \lim_{x \to -\infty} \frac{x^2 + x - 8}{2x^3 + 3x - 1} = \)
   
   (A) \(-\frac{1}{2}\) \hspace{1cm} (B) 0 \hspace{1cm} (C) \(\frac{1}{2}\) \hspace{1cm} (D) 2

3. Which of the following lines is an asymptote of the graph of \( f(x) = \frac{x^2 + 5x + 6}{x^2 - x - 12} \) ?
   
   I. \( x = -3 \)
   II. \( x = 4 \)
   III. \( y = 1 \)

   (A) II only \hspace{1cm} (B) III only \hspace{1cm} (C) II and III only \hspace{1cm} (D) I, II, and III

4. If the horizontal line \( y = 1 \) is an asymptote for the graph of the function \( f \), which of the following statements must be true?
   
   (A) \( \lim_{x \to \infty} f(x) = 1 \)
   (B) \( \lim_{x \to 1} f(x) = \infty \)
   (C) \( f(1) \) is undefined
   (D) \( f(x) = 1 \) for all \( x \)
5. If $x = 1$ is the vertical asymptote and $y = -3$ is the horizontal asymptote for the graph of the function $f$, which of the following could be the equation of the curve?

(A) $f(x) = \frac{-3x^2}{x-1}$

(B) $f(x) = \frac{-3(x-1)}{x+3}$

(C) $f(x) = \frac{-3(x^2-1)}{x-1}$

(D) $f(x) = \frac{-3(x^2-1)}{(x-1)^2}$

6. What are all horizontal asymptotes of the graph of $y = \frac{6+3e^x}{3-3e^x}$ in the $xy$-plane?

(A) $y = -1$ only

(B) $y = 2$ only

(C) $y = -1$ and $y = 2$

(D) $y = 0$ and $y = 2$

Free Response Questions

7. Let $f(x) = \frac{3x-1}{x^3-8}$.

(a) Find the vertical asymptote(s) of $f$. Show the work that leads to your answer.

(b) Find the horizontal asymptote(s) of $f$. Show the work that leads to your answer.

8. Let $f(x) = \frac{\sin x}{x^2 + 2x}$.

(a) Find the vertical asymptote(s) of $f$. Show the work that leads to your answer.

(b) Find the horizontal asymptote(s) of $f$. Show the work that leads to your answer.