1. Let $R$ be the shaded region in the first quadrant enclosed by the $y$-axis and the graphs of $y = 1 - x^3$ and $y = \sin(x^2)$, as shown in the figure above.

(a) Find the area of $R$.

\[
\text{Area} = \int_0^A [(1-x^3) - \sin(x^2)] \, dx = 0.533 \quad (0.534)
\]

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin(x^2)$ intersect. Find $k$ and determine whether this line divides $R$ into two regions of equal area. Show the work that leads to your conclusion.

\[
k = B \quad y = 1 - x^3 \Rightarrow x = \sqrt[3]{1-y} \quad y = \sin(x^2) \Rightarrow x = \sqrt{\arcsin(y)}
\]

\[
\int_0^A (1-x^3-k) \, dx = 0.257
\]

\[
\int_0^A (k-\sin(x^2)) \, dx = 0.277
\]

The regions have unequal areas.

\[
\int_0^B \sqrt{\arcsin(y)} \, dy = 0.277
\]

\[
\int_0^1 \sqrt{1-y} \, dy = 0.257
\]

OR

The regions have unequal areas.

(c) Find the volume of the solid generated when $R$ is revolved about the line $y = -3$.

\[
\text{Volume} = \pi \int_0^A \left[ (1-x^3-3)^2 - (\sin(x^2)-3)^2 \right] \, dx = 11.841 \quad (11.840)
\]
2. A planetary rover travels on a flat surface. The path of the rover for the time interval \(0 \leq t \leq 2\) hours is shown in the rectangular coordinate system above. The rover starts at the point with coordinates \((6, 5)\) at time \(t = 0\). The coordinates \((x(t), y(t))\) of the position of the rover change at rates given by

\[
x'(t) = -12 \sin\left(2t^2\right)
\]
\[
y'(t) = 10 \cos\left(1 + \sqrt{t}\right),
\]

where \(x(t)\) and \(y(t)\) are measured in meters and \(t\) is measured in hours.

(a) Find the acceleration vector of the rover at time \(t = 1\). Find the speed of the rover at time \(t = 1\).

\[
\mathbf{a}(1) = \left\langle x''(1), y''(1) \right\rangle = \left\langle 19.975, -4.546 \right\rangle
\]

\[
\text{Speed} = \sqrt{\left[x'(1)\right]^2 + \left[y'(1)\right]^2} = 11.678
\]

(b) Find the total distance that the rover travels over the time interval \(0 \leq t \leq 1\).

\[
\text{Distance} = \int_0^1 \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2} \, dt = 6.703 \quad \text{(6.703)}
\]

(c) Find the \(y\)-coordinate of the position of the rover at time \(t = 1\).

\[
y(1) = 5 + \int_0^1 y'(t) \, dt = 4.057 \quad \text{(4.056)}
\]

(d) The rover receives a signal at each point where the line tangent to its path has slope \(
\frac{1}{2}
\). At what times \(t\), for \(0 \leq t \leq 2\), does the rover receive a signal?

\[
\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad \Rightarrow \quad \frac{10 \cos\left(1 + \sqrt{t}\right)}{-12 \sin\left(2t^2\right)} = \frac{1}{2} \quad \Rightarrow \quad t = 1.072
\]
3. The twice-differentiable function $W$ models the volume of water in a reservoir at time $t$, where $W(t)$ is measured in gigaliters (GL) and $t$ is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \leq t \leq 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

(a) Use the tangent line approximation to $W$ at time $t = 30$ to predict the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

$$W(32) \approx 0.6(32-30) + 125 = 126$$

(b) Use a left Riemann sum, with the three subintervals indicated by the data in the table, to approximate

$$\int_0^{30} W'(t) \, dt. \text{ Use this approximation to estimate the volume of water } W(t), \text{ in gigaliters, in the reservoir at time } t = 0. \text{ Show the computations that lead to your answer.}$$

$$\int_0^{30} W'(t) \, dt \approx 10 \cdot W'(0) + 12 \cdot W'(10) + 8 \cdot W'(22) = 22.4$$

$$\int_0^{30} W'(t) \, dt = W(30) - W(0) \Rightarrow W(0) = W(30) - \int_0^{30} W'(t) \, dt$$

$$w(0) = 125 - 22.4 = 102.6$$

(c) Explain why there must be at least one time $t$, other than $t = 10$, such that $W'(t) = 0.7$ GL/day.

$W'$ differentiable $\Rightarrow W'$ continuous

$W'(30) = 0.6 < 0.7$ \hspace{1cm} \text{ By IVT, there must be at least one } t, \text{ on}$

$W'(22) = 1.0 > 0.7$ \hspace{1cm} 22 $\leq t \leq 30$, such that $W'(t) = 0.7$.

(d) The equation $A = 0.3W^{2/3}$ gives the relationship between the area $A$, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of $A$, in square kilometers per day, with respect to $t$ when $t = 30$ days.

$$\frac{dA}{dt} = (0.3) \cdot \frac{2}{3} W^{-\frac{2}{3}} \cdot \frac{dW}{dt} = \frac{1}{5\sqrt[3]{W}} \cdot \frac{dw}{dt}$$

$$\frac{dA}{dt} \bigg|_{t=30} = \frac{1}{5\sqrt[3]{125}} \cdot \frac{1}{2} = \frac{1}{50}$$
5. Let \( f \) be the function satisfying \( f'(x) = 4x - 2xf(x) \) for all real numbers \( x \), with \( f(0) = 5 \) and \( \lim_{x \to \infty} f(x) = 2 \).

(a) Find the value of \( \int_{0}^{\infty} (4x - 2xf(x)) \, dx \). Show the work that leads to your answer.

\[
\int_{0}^{\infty} (4x - 2xf(x)) \, dx = \lim_{b \to \infty} \int_{0}^{b} f'(x) \, dx = \lim_{b \to \infty} \left[ f(x) \right]_{0}^{b} = \lim_{b \to \infty} f(b) - f(0) = 2 - 5 = -3
\]

(b) Use Euler's method to approximate \( f(-1) \), starting at \( x = 0 \), with two steps of equal size.

\[
f\left(\frac{1}{2}\right) = 5 + \frac{1}{2} \left[ 4(0) - 2(0) \cdot f(0) \right] = 5 + \frac{1}{2} \cdot 0 = 5
\]

\[
f(-1) = 5 + \frac{1}{2} \left[ 4(-\frac{1}{2}) - 2(-\frac{1}{2}) \cdot f\left(-\frac{1}{2}\right) \right] = 5 + \frac{1}{2} \cdot 3 = \frac{11}{2}
\]

(c) Find the particular solution \( y = f(x) \) to the differential equation \( \frac{dy}{dx} = 4x - 2xy \) with the initial condition \( f(0) = 5 \).

\[
\frac{1}{4-2y} \, dy = x \, dx
\]

\[
-\frac{1}{2} \ln |4-2y| = \frac{1}{2} x^2 + C
\]

\[
-\frac{1}{2} \ln |6| = C
\]

\[
-\frac{1}{2} \ln (6) = C
\]

If \( f(0) = 5 \) \( \Rightarrow \) \( y = 2 + 3e^{-x^2} \)
5. The graph of the continuous function $f$, consisting of three line segments and a semicircle, is shown above. Let $g$ be the function given by $g(x) = \int_{-2}^{x} f(t) \, dt$.

(a) Find $g(-6)$ and $g(3)$.

$$g(-6) = \int_{-2}^{-6} f(t) \, dt = - \int_{0}^{-6} f(t) \, dt = - \left[ \frac{1}{2} \cdot 4 \cdot 5 \right] = -10$$

$$g(3) = \int_{-2}^{3} f(t) \, dt = \frac{1}{2} \pi (2)^2 + \left[ \frac{1}{2} \cdot 1 \cdot 2 \right] = 2\pi - 1$$

(b) Find $g'(0)$.

$$g'(x) = f(x)$$

$$g'(0) = f(0) = 2$$

(c) Find all values of $x$ on the open interval $-6 < x < 3$ for which the graph of $g$ has a horizontal tangent. Determine whether $g$ has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

Horizontal Tangents $\implies g'(x) = f(x) = 0 \implies x = -2, 2$

At $x = -2$, $g$ has no local extreme since $g'(x)$ does not change sign.

At $x = 2$, $g$ has a local maximum since $g'(x)$ changes from positive to negative.

(d) Find all values of $x$ on the open interval $-6 < x < 3$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

$$g''(x) = f'(x)$$

$$g''(x) = 0 \implies x = 0$$

$$g''(x) = \text{undefined} \implies x = -4, -2, 2$$

POI at $x = -4, -2, 0$ since $g''(x)$ changes sign at these values.

At $x = -4$ and $x = 0$, $g''(x)$ changes from positive to negative.

At $x = -2$, $g''(x)$ changes from negative to positive.
6. The function $f$ satisfies the equation

$$f'(x) = f(x) + x + 1$$

and $f(0) = 2$. The Taylor series for $f$ about $x = 0$ converges to $f(x)$ for all $x$.

(a) Write an equation for the line tangent to the curve $y = f(x)$ at the point where $x = 0$.

$$f'(0) = f(0) + 0 + 1 = 3$$

Tangent: $y - 2 = 3x$

(b) Find $f''(0)$ and find the second-degree Taylor polynomial for $f$ about $x = 0$.

$$f''(x) = f'(x) + 1$$
$$f''(0) = f'(0) + 1 = 4$$

$$P_2(x) = 2 + 3x + 4 \cdot \frac{x^2}{2!} = 2 + 3x + 2x^2$$

(c) Find the fourth-degree Taylor polynomial for $f$ about $x = 0$.

$$f^{(4)}(x) = f'''(x) \Rightarrow f^{(4)}(0) = 4$$
$$P_4(x) = 2 + 3x + 4 \cdot \frac{x^2}{2!} + 4 \cdot \frac{x^3}{3!} + 4 \cdot \frac{x^4}{4!}$$
$$= 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4$$

(d) Find $f^{(n)}(0)$, the $n$th derivative of $f$ at $x = 0$, for $n \geq 2$. Use the Taylor series for $f$ about $x = 0$ and the Taylor series for $e^x$ about $x = 0$ to find a polynomial expression for $f(x) - 4e^x$.

$$f^{(n)}(0) = 4 \text{ for } n \geq 2$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$f(x) = 2 + 3x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + ...$$
$$4e^x = 4 + 4x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + ...$$

$$f(x) - 4e^x = -2 - x$$