The Definition of Continuity

A function $f(x)$ is continuous at $c$ if

I. $\lim_{x \to c} f(x)$ exists

II. $f(c)$ exists

III. $\lim_{x \to c} f(x) = f(c)$

The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.

Let $g$ be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5. \end{cases}$

Is $g$ continuous at $x = -3$? Use the definition of continuity to explain your answer.

1. $\lim_{x \to -3^-} g(x) = \lim_{x \to -3} f(x) = \sqrt{25 - (-3)^2} = 4$
   
   $\lim_{x \to -3^+} (-3) + 7 = 4$

11. $g(-3) = f(-3) = 4$

$g(x)$ is continuous at $x = -3$ since $g(-3) = \lim_{x \to -3} g(x) = 4$. 
Discontinuities:

1. Removable
   \[ \lim_{{x \to c}} f(x) = L \quad \text{but} \quad \lim_{{x \to c}} f(x) \neq f(c) \]
   
   Occurs when:
   \[ \lim_{{x \to c}} f(x) = L \quad \text{is constant} \]

   Example:
   \[ \lim_{{x \to -1}} \frac{x^2 - 1}{x + 1} = \frac{0}{0} = \text{INDETERMINATE} \]
   
   \[ \lim_{{x \to -1}} \frac{2x}{1} = -2 \]
   
   Hole at \((-1, -2)\)

2. Non Removable
   - Jump (piecewise or \( \frac{|x-c|}{x-c} \))
     \[ \lim_{{x \to c^-}} f(x) = a \quad \text{and} \quad \lim_{{x \to c^+}} f(x) = b \]
     
   - (Vertical)
     Asymptote (Infinite Discontinuity)
     \[ \lim_{{x \to c}} f(x) = \pm \infty \]
Intermediate Value Theorem

If \( f \) is a continuous function on the closed interval \([a, b]\) and \( k \) is any number between \( f(a) \) and \( f(b) \), then there exists at least one value of \( c \) on \([a, b]\) such that \( f(c) = k \). In other words, on a continuous function, if \( f(a) < f(b) \), any \( y \) value greater than \( f(a) \) and less than \( f(b) \) is guaranteed to exist on the function \( f \).

Let \( f \) be a continuous function on the closed interval \([-2, 7]\). If \( f(-2) = 5 \) and \( f(7) = -3 \), then the Intermediate Value Theorem guarantees that

\( f' \) guarantees any/all \( y \)-values such that \(-3 \leq y \leq 5\).

(A) \( f'(c) = 0 \) for at least one \( c \) between \(-2 \) and \( 7 \)

(B) \( f'(c) = 0 \) for at least one \( c \) between \(-3 \) and \( 5 \)

(C) \( f(c) = 0 \) for at least one \( c \) between \(-3 \) and \( 5 \)

(D) \( f(c) = 0 \) for at least one \( c \) between \(-2 \) and \( 7 \)
Mean Value Theorem for Derivatives

If the function $f$ is continuous on the close interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there exists at least one number $c$ between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The slope of the tangent line is equal to the slope of the secant line.

Let $f$ be the function given by $f(x) = \frac{x}{x + 2}$. What are the values of $c$ that satisfy the Mean Value Theorem on the closed interval $[-1, 2]$?

(A) $-4$ only
(B) $0$ only
(C) $0$ and $\frac{3}{2}$
(D) $-4$ and $0$

$$f'(x) = \frac{(x+2) - x}{(x+2)^2}$$

$$= \frac{2}{(x+2)^2}$$

**MVT:**

$$\frac{2}{(x+2)^2} = \frac{\frac{1}{2} + \frac{1}{2}}{3}$$

$$\frac{2}{(x+2)^2} = \frac{1}{2}$$

$$(x+2)^2 = 4$$

$$x+2 = \pm 2$$

$$x = 0, -4$$
Rolle’s Theorem (Special Case of Mean Value Theorem)

If the function \( f \) is continuous on the close interval \([a, b]\) and differentiable on the open interval \((a, b)\), and \( f(a) = f(b) \), then there exists at least one number \( c \) between \( a \) and \( b \) such that

\[
f''(c) = \frac{f(b) - f(a)}{b - a} = 0
\]

Let \( f \) be the function given by \( f(x) = \sin(\pi x) \). What are the values of \( c \) that satisfy Rolle’s Theorem on the closed interval \([0, 2]\)?

\( f(0) = \sin(0) = 0 \)

\( f(2) = \sin(2\pi) = 0 \)

(A) \( \frac{1}{4} \) only

(B) \( \frac{1}{2} \) only

(C) \( \frac{1}{4} \) and \( \frac{1}{2} \)

(D) \( \frac{1}{2} \) and \( \frac{3}{2} \)

\[
f'(x) = \pi \cos(\pi x)
\]

\[
\text{Rolle's: } \pi \cos(\pi x) = 0
\]

\[
\pi x = \frac{\pi}{2} \quad \pi x = \frac{3\pi}{2}
\]

\[
x = \frac{1}{2} \quad x = \frac{3}{2}
\]
Extreme Value Theorem

If the function $f$ is continuous on the closed interval $[a, b]$, then the absolute extrema of the function $f$ on the closed interval will occur at the endpoints or critical values of $f$.

"After identifying critical values, create a table with endpoints and critical values. Calculate the $y$-value at each of these $x$ values to identify the extrema.

The figure above shows the graph of $f'$, the derivative of the function $f$, for $-4 \leq x \leq 7$. The graph of $f'$ has horizontal tangent lines at $x = -1$, $x = 3$, and $x = 5$.

(a) Find all values of $x$, for $-4 \leq x \leq 7$, at which $f$ attains a relative minimum. Justify your answer. $x = 1$

(b) Find all values of $x$, for $-4 \leq x \leq 7$, at which $f$ attains a relative maximum. Justify your answer. $x = -2$

(c) At what value of $x$, for $-4 \leq x \leq 7$, does $f$ attain its absolute maximum? Justify your answer. $x = 7$

\[
\begin{array}{c|c|c}
\text{x} & f(x) & f'(x) \\
\hline
-4 & f(-4) & \int_{-4}^{x} f'(t) \, dt \\
-2 & f(-2) & \int_{-4}^{x} f'(t) \, dt \\
1 & f(1) & \int_{-4}^{x} f'(t) \, dt \\
5 & f(5) & \int_{-4}^{x} f'(t) \, dt \\
7 & f(7) & \int_{-4}^{x} f'(t) \, dt \\
\end{array}
\]

Since $f(x) = f(-4) + \int_{-4}^{x} f'(t) \, dt$ and $\int_{-4}^{x} f'(t) \, dt$ achieves the most positive area at $x = 7$.
Definition of the Derivative

The derivative of the function $f$, or instantaneous rate of change, is given by converting the slope of the secant line to the slope of the tangent line by making the change $h$, $\Delta x$, or $h$, approach zero.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate Definition

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x-c}$$

- $f(x) = 3x^2$
  - $f'(x) = 6x$

- $f(x) = \sqrt{x}$
  - $f'(x) = \frac{1}{2\sqrt{x}}$
  - $f'(9) = \frac{1}{6}$

- $f(x) = \frac{1}{x}$
  - $f'(x) = -\frac{2}{x^2}$
  - $f'(3) = -\frac{2}{9}$
Riemann Sum (Limit Definition of Area)

Let $f$ be a continuous function on the interval $[a, b]$. The area of the region bounded by the graph of the function $f$ and the $x$–axis (i.e. the value of the definite integral) can be found using

$$
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x
$$

Where $c_i$ is either the left endpoint ($c_i = a + (i-1)\Delta x$) or right endpoint ($c_i = a + i\Delta x$) and $\Delta x = (b - a)/n$.

\[\lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{2i}{n} \right)^3 \cdot \frac{2}{n} \]

$\Delta x = \frac{2}{n} \to b - a = 2$

$c_k = 1 + \frac{k}{n} \to a = 1, b = 3$

$f(x) = x^3$

\[\int_{1}^{3} x^3 \, dx = \left. \frac{x^4}{4} \right|_{1}^{3} = 2.0\]

\[\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{i} \cdot \frac{1}{n} \]

$\Delta x = \frac{1}{n} \to b - a = 1$

$c_i = i \cdot \frac{1}{n} \to a = 0, b = 1$

$f(x) = \sqrt{x}$

\[\int_{0}^{1} \sqrt{x} \, dx = \left[ \frac{2}{3} x^{3/2} \right]_{0}^{1} = \frac{2}{3}\]

\[\lim_{n \to \infty} \frac{1}{3} \sum_{k=1}^{n} \left( 2 + 3k \cdot \frac{1}{n} \right)^2 \cdot \frac{1}{n} \]

$\Delta x = \frac{1}{n} \to b - a = 1$

$c_i = k \cdot \frac{1}{n} \to a = 0, b = 1$

$f(x) = (2 + 3x)^2$

\[\frac{1}{3} \int_{0}^{1} (2 + 3x)^2 \, dx \]

$du = 3dx$

\[\frac{1}{3} \int_{0}^{2} u^2 \, du \]

\[\left[ \frac{1}{2} u^3 \right]_{2}^{5} = \frac{13}{3}\]
Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function $B$ models Ben’s position on the track, measured in meters from the western end of the track, at time $t$, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben’s velocity, $v(t)$, measured in meters per second, at selected times $t$.

(a) Use the data in the table to approximate Ben’s acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)| \, dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)| \, dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \leq t \leq 60$, must there be a time $t$ when Ben’s velocity is 2 meters per second? Justify your answer.

(d) A light is directly above the western end of the track. Ben rides so that at time $t$, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?
2009 Question 3 Form B

A continuous function $f$ is defined on the closed interval $-4 \leq x \leq 6$. The graph of $f$ consists of a line segment and a curve that is tangent to the $x$-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function $f$ is twice differentiable, with $f''(x) > 0$.

(a) Is $f$ differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of $a$, $-4 \leq a < 6$, is the average rate of change of $f$ on the interval $[a, 6]$ equal to 0? Give a reason for your answer.

(c) Is there a value of $a$, $-4 \leq a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value $c$, $a < c < 6$, at which $f''(c) = \frac{1}{3}$? Justify your answer.

(d) The function $g$ is defined by $g(x) = \int_{0}^{x} f(t) \, dt$ for $-4 \leq x \leq 6$. On what intervals contained in $[-4, 6]$ is the graph of $g$ concave up? Explain your reasoning.
2005 Question 2

The tide removes sand from Sandy Point Beach at a rate modeled by the function \( R \), given by

\[
R(t) = 2 + 5 \sin \left( \frac{4 \pi t}{25} \right).
\]

A pumping station adds sand to the beach at a rate modeled by the function \( S \), given by

\[
S(t) = \frac{15t}{1 + 3t}.
\]

Both \( R(t) \) and \( S(t) \) have units of cubic yards per hour and \( t \) is measured in hours for \( 0 \leq t \leq 6 \). At time \( t = 0 \), the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

(b) Write an expression for \( Y(t) \), the total number of cubic yards of sand on the beach at time \( t \).

(c) Find the rate at which the total amount of sand on the beach is changing at time \( t = 4 \).

(d) For \( 0 \leq t \leq 6 \), at what time \( t \) is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.