Test #3 - Applications of Derivative

1. (a) $f'(x) = 0$ at $x = 0, 3, 5$

   (b) $f'(x)$ positive : $(0,3)\ (4,5)$

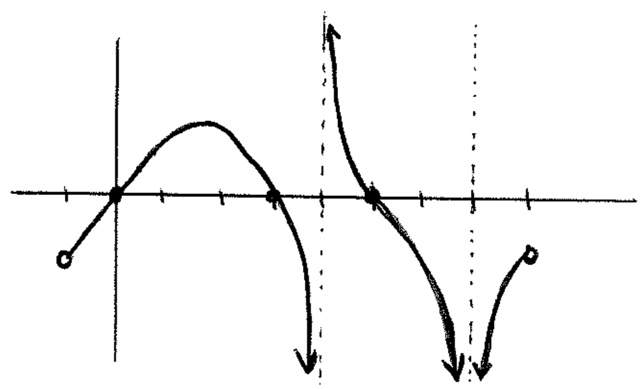
   $f'(x)$ negative : $(-1,0)\ (3,4)(5,7)\ (7,8)$

   (c) $f''(x)$ positive : $(-1,2)\ (7,8)$

   $f''(x)$ negative : $(2,4)\ (4,7)$

   (d) $f'(x)$ DNE at $x = -1, 4, 7, 8$

   (e)



2. (a) $f(x)$ increasing : $(-3,7)\ (9,10)$

   $f(x)$ decreasing : $(-4,-3)\ (7,9)$

   (b) $x = -3, 9 \Rightarrow$ Relative minimums

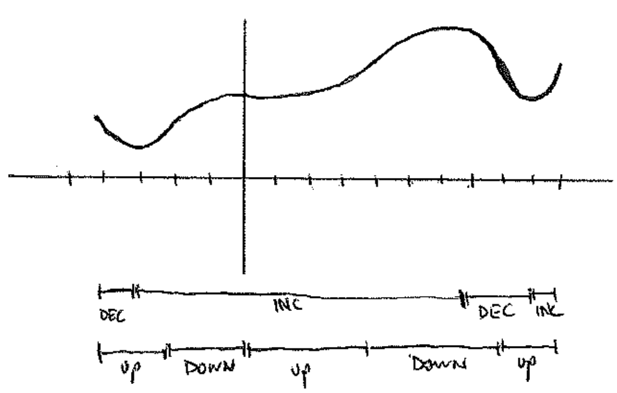
   $x = 7 \Rightarrow$ Relative maximum

   (c) $f(x)$ concave up : $(-4,-2)\ (0,4)(8,10)$

   $f(x)$ concave down : $(-2,0)(4,8)$

   (d) POI at $x = -2, 0, 4, 8$

   (e)



3. (a) $f(x) = x\sqrt{6-x}$

$f'(x) = (6-x)^{1/2} + x \cdot \frac{1}{2}(6-x)^{-1/2} \cdot -1$

$f'(x) = \dfrac{3(4-x)}{2\sqrt{6-x}}$

$f(x)$ is continuous on $[0,6]$ and differentiable on $(0,6)$.

$\left.\begin{array}{l} f(0) = 0 \\ f(6) = 0 \end{array}\right\}$ Rolle's guarantees some value on $[0,6]$ such that $f'(x) = 0$.

$f'(x) = 0 \Rightarrow 3(4-x) = 0$

$x = 4 \longrightarrow (4, 4\sqrt{2})$

(b) $y = -\sin 2x$

$y' = -2\cos 2x$

y is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$.

$\left.\begin{array}{l} f(-\pi) = 0 \\ f(\pi) = 0 \end{array}\right\}$ Rolle's guarantees some value on $[-\pi, \pi]$ such that $y' = 0$

$y' = 0 \Rightarrow -2\cos(2x) = 0$

$2x = \pi/2 \quad 2x = -\pi/2 \quad 2x = 3\pi/2 \quad 2x = -3\pi/2$

$x = \pi/4 \quad x = -\pi/4 \quad x = 3\pi/4 \quad x = -3\pi/4$

$\left(\frac{\pi}{4}, -1\right) \quad \left(\frac{-\pi}{4}, 1\right) \quad \left(\frac{3\pi}{4}, 1\right) \quad \left(\frac{-3\pi}{4}, -1\right)$

(c) $y = 2\sec x = \dfrac{2}{\cos x} \Rightarrow$ y is undefined when $\cos x = 0$. y is undefined at $x = -\pi/2, \pi/2$ on $[-\pi, \pi]$. Since y is not continuous on $[-\pi, \pi]$, Rolle's theorem does not apply.

(d) $f(x) = \dfrac{x^2 - 2x - 3}{x+2} = \dfrac{(x-3)(x+1)}{x+2}$

$f'(x) = \dfrac{x^2 + 4x - 1}{(x+2)^2}$

$f(x)$ is continuous on $[-1, 3]$ and differentiable on $(-1, 3)$.

$\left.\begin{array}{l} f(-1) = 0 \\ f(3) = 0 \end{array}\right\}$ Rolle's guarantees some value on $[-1, 3]$ such that $f'(x) = 0$

$f'(x) = 0 \Rightarrow x^2 + 4x - 1 = 0$

$x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$

$x = -2 \pm \sqrt{5}$

$x = -2 + \sqrt{5} \longrightarrow \left(-2+\sqrt{5}, -2\sqrt{5} - 10\right)$

3. (e) $f(x) = (1-x^2)^{1/3}$

$f'(x) = \frac{1}{3}(1-x^2)^{-2/3} \cdot -2x$

$$f'(x) = \frac{-2x}{3(1-x^2)^{2/3}}$$

$f(x)$ is continuous on $[-3,3]$ but $f'(x)$ does not exist at $x = \pm 1$. Therefore, Rolle's theorem does not apply.

4. (a) $f(x) = \ln(x-1)$

$$f'(x) = \frac{1}{x-1}$$

$f(x)$ is continuous on $[2,4]$ and differentiable on $(2,4)$.

$$\frac{f(4)-f(2)}{4-2} = \frac{\ln(3)-\ln(1)}{2} = \frac{\ln(3)}{2}$$

MVT: $$\frac{1}{x-1} = \frac{\ln(3)}{2}$$

$$x-1 = \frac{2}{\ln(3)}$$

$$x = \frac{2}{\ln(3)} + 1 \approx 2.820$$

(b) $y = (3x+15)^{2/3}$

$y' = \frac{2}{3}(3x+15)^{-1/3} \cdot 3$

$$y' = \frac{2}{(3x+15)^{1/3}}$$

y is continuous on $[-7,4]$ but y' does not exist at $x = -5$. Therefore MVT does not apply.

(c) $$y = \frac{x+1}{x}$$

$$y' = -\frac{1}{x^2}$$

y is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - 1/2} = -1$$

MVT: $$-\frac{1}{x^2} = -1$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x = 1$$

(d) $f(x) = \sin x - \cos x$

$f'(x) = \cos x + \sin x$

$f(x)$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$.

$$\frac{f(2\pi) - f(0)}{2\pi - 0} = \frac{-1 - -1}{2\pi} = 0$$

MVT: $\cos x + \sin x = 0$

$\cos x = -\sin x$

$x = 3\pi/4, \ 7\pi/4$

(e) $f(x) = \begin{cases} \sin x, & -\infty < x < 1 \\ \frac{1}{2}x, & 1 \le x < \infty \end{cases}$

$\lim_{x \to 1^-} \sin x \neq \lim_{x \to 1^+} \frac{1}{2}x$

$.841 \neq \frac{1}{2}$

$f(x)$ is not continuous since there exists a jump discontinuity at $x = 1$. Therefore MVT does not apply.

5. (a) $f(x) = (x-2)^2(x+3)$

Domain: $\mathbb{R}$

$f'(x) = 2(x-2)(x+3) + (x-2)^2$

$f'(x) = (x-2)[2(x+3) + (x-2)]$

$f'(x) = (x-2)(3x+4)$

CV: $f'(x) = 0 \Rightarrow x = 2, -4/3$

$f'(x)$

$-\infty$		$-4/3$		2		∞
	$+$		$-$		$+$	

Relative Maximum $(-4/3, 500/27)$ since $f'(x)$ changes from positive to negative.

Relative Minimum $(2, 0)$ since $f'(x)$ changes from negative to positive.

(b) $f(x) = \sec(x - \pi/2)$

Domain: $\mathbb{R}$ on $[0, 2\pi]$, $x \neq 0, \pi, 2\pi$

$f'(x) = \sec(x - \pi/2)\tan(x - \pi/2)$

$$f'(x) = \frac{\sin(x - \pi/2)}{\cos^2(x - \pi/2)}$$

CV: $f'(x) = 0 \Rightarrow \sin(x - \pi/2) = 0$

$x - \pi/2 = 0 \qquad x - \pi/2 = \pi$

$x = \pi/2 \qquad x = 3\pi/2$

$f'(x) = \text{UND} \Rightarrow \cos^2(x - \pi/2) = 0$

$\cancel{x = 0, \pi, 2\pi}$

$\downarrow$

Not in Domain

$f'(x)$

0		$\pi/2$		π		$3\pi/2$		2π
	$-$		$+$		$+$		$-$	

Relative maximum $(3\pi/2, -1)$ since $f'(x)$ changes from positive to negative.

Relative minimum $(\pi/2, 1)$ since $f'(x)$ changes from negative to positive.

(c) $f(x) = (x-1)^2\sqrt{x+1}$

Domain $x \geq -1$

$f'(x) = 2(x-1)(x+1)^{1/2} + (x-1)^2 \cdot \frac{1}{2}(x+1)^{-1/2}$

$f'(x) = \frac{1}{2}(x-1)(x+1)^{-1/2}\left[4(x+1) + (x-1)\right]$

$f'(x) = \dfrac{(x-1)(5x+3)}{2\sqrt{x+1}}$

$\underline{CV}$: $f'(x) = 0 \Rightarrow (x-1)(5x+3) = 0$

$x = 1, -3/5$

$f'(x) = \text{UND} \Rightarrow 2\sqrt{x+1} = 0$

$x = -1$

$f'(x)$: $+$ on $(-1, -3/5)$, $-$ on $(-3/5, 1)$, $+$ on $(1, \infty)$

Relative maximum $\left(-3/5, \frac{64\sqrt{10}}{125}\right)$ since $f(x)$ changes from increasing to decreasing.

Relative minimum $(1, 0)$ since $f(x)$ changes from decreasing to increasing.

(d) $f(x) = 4xe^{-x^2}$

Domain $\mathbb{R}$

$f'(x) = 4e^{-x^2} + 4x \cdot e^{-x^2} \cdot -2x$

$f'(x) = 4e^{-x^2} - 8x^2e^{-x^2}$

$f'(x) = 4e^{-x^2}\left[1 - 2x^2\right]$

$\underline{CV}$: $f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$

$x = \pm\sqrt{1/2} = \pm\frac{\sqrt{2}}{2}$

$f'(x) = \text{und} \Rightarrow e^{x^2} \neq 0$

$f'(x)$: $-$ on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$, $+$ on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $-$ on $\left(\frac{\sqrt{2}}{2}, \infty\right)$

Relative maximum $\left(\sqrt{2}/2, 2\sqrt{\frac{2}{e}}\right)$ since $f(x)$ changes from increasing to decreasing.

Relative minimum $\left(-\sqrt{2}/2, -2\sqrt{\frac{2}{e}}\right)$ since $f(x)$ changes from decreasing to increasing.

6. (a) $f(x) = x^4 - 8x^3 + 2$

$f'(x) = 4x^3 - 24x^2$

$f'(x) = 4x^2(x-6)$

$f'(x) = 0 \Rightarrow 4x^2(x-6) = 0$

$x = 0 \quad x = 6$

$f''(x) = 12x^2 - 48x$

$f''(0) = 0$

$f''(6) = 144 > 0$

Relative minimum $(6, -430)$ since $f'(6) = 0$ and $f''(6) > 0$.

The second derivative test fails at $x = 0$ since $f''(0) = 0$

$\underline{1^{st}\text{ Derivative Test}}$: $f'(x)$: $-$ on $(-\infty, 0)$, $-$ on $(0, 6)$

Shelf point at $(0, 2)$

(b) $f(x) = 2\cos x + \sin 2x$

$f'(x) = -2\sin x + 2\cos(2x)$

$f'(x) = -2\sin x + 2(1 - 2\sin^2 x)$

$f'(x) = -(4\sin^2 x + 2\sin x - 2)$

$f'(x) = -(2\sin x - 1)(2\sin x + 2)$

$f'(x) = 0 \Rightarrow -(2\sin x - 1)(2\sin x + 2) = 0$

$\sin x = 1/2 \qquad \sin x = -1$

$x = \pi/6,\ 5\pi/6 \qquad x = -\pi/2$

$f''(x) = -2\cos x - 4\sin(2x)$

$f''(\pi/6) = -3\sqrt{3} < 0$

$f''(5\pi/6) = 3\sqrt{3} > 0$

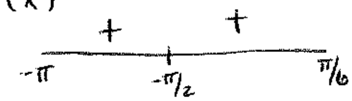
$f''(-\pi/2) = 0$

Relative maximum $\left(\pi/6, \frac{3\sqrt{3}}{2}\right)$ since $f'(\pi/6) = 0$ and $f''(\pi/6) < 0$.

Relative minimum $\left(5\pi/6, -\frac{3\sqrt{3}}{2}\right)$ since $f'(5\pi/6) = 0$ and $f''(5\pi/6) > 0$.

The second derivative test fails at $x = -\pi/2$ since $f''(-\pi/2) = 0$.

1st Derivative Test $\cdot$ $f'(x)$



Shelf point $(-\pi/2, 0)$.

(c) $f(x) = xe^x$

$f'(x) = e^x + xe^x$

$f'(x) = e^x(1 + x)$

$f'(x) = 0 \Rightarrow x = -1$

$f''(x) = e^x + e^x + xe^x$

$f''(x) = 2e^x + xe^x$

$f''(x) = e^x(2 + x)$

$f''(-1) = e^{-1} > 0$

Relative minimum $\left(-1, -\frac{1}{e}\right)$ since $f'(-1) = 0$ and $f''(-1) > 0$.

(d) $f(x) = -4x^2 \ln\left(\frac{1}{2}x\right)$

Domain $x > 0$

$f'(x) = -8x \ln\left(\frac{1}{2}x\right) + -4x^2 \cdot \frac{1}{\frac{1}{2}x} \cdot \frac{1}{2}$

$f'(x) = -8x \ln\left(\frac{1}{2}x\right) - 4x$

$f'(x) = -4x\left(2\ln(\frac{1}{2}x) + 1\right)$

Relative maximum $\left(\frac{2}{\sqrt{e}}, \frac{8}{e}\right)$ since $f'\left(\frac{2}{\sqrt{e}}\right) = 0$ and $f''\left(\frac{2}{\sqrt{e}}\right) < 0$.

$f'(x) = 0 \Rightarrow -4x\left(2\ln(\frac{1}{2}x) + 1\right)$

$\cancel{x = 0}$ Not in domain

$2\ln(\frac{1}{2}x) + 1 = 0$

$\ln(\frac{1}{2}x) = -\frac{1}{2}$

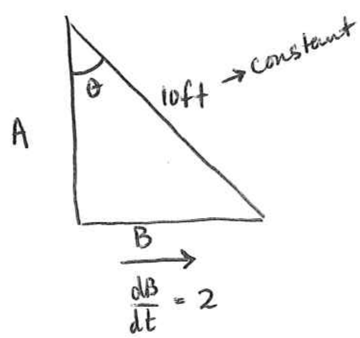
$\frac{1}{2}x = e^{-1/2} \Rightarrow x = 2e^{-1/2} = \frac{2}{\sqrt{e}}$

$f''(x) = -8\ln(\frac{1}{2}x) + -8x \cdot \frac{1}{\frac{1}{2}x} \cdot \frac{1}{2} - 4$

$f''(x) = -8\ln(\frac{1}{2}x) - 12 = -4\left(2\ln(\frac{1}{2}x) + 3\right)$

$f''\left(\frac{2}{\sqrt{e}}\right) < 0$

7. (a)



$\sin\theta = \frac{1}{10}B$

$\theta = \arcsin\left(\frac{1}{10}B\right)$

$\sin(\pi/4) = \frac{1}{10}B$

$\frac{\sqrt{2}}{2} = \frac{1}{10}B$

$5\sqrt{2} = B$

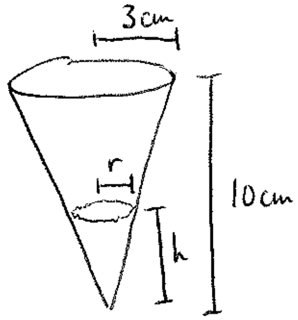
$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1-\left(\frac{1}{10}B\right)^2}} \cdot \frac{1}{10}\frac{dB}{dt}$$

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/4} = \frac{1}{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}} \cdot \frac{1}{10}(2)$$

$$= \frac{1}{\sqrt{1/2}} \cdot \frac{1}{5}$$

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/4} = \frac{\sqrt{2}}{5} \text{ RAD/sec}$$

(b)



$$\frac{h}{r} = \frac{10}{3}$$

$$3h = 10r$$

$$r = \frac{3}{10}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{3}{10}h\right)^2 \cdot h$$

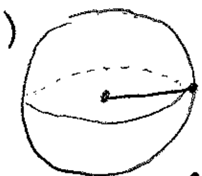
$$V = \frac{3}{100}\pi h^3$$

$$\frac{dV}{dt} = \frac{9}{100}\pi h^2 \cdot \frac{dh}{dt}$$

$$2 = \frac{9}{100}\pi (5)^2 \cdot \frac{dh}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=5} = \frac{8}{9\pi} \text{ cm/sec}$$

(c)



Surface Area $= S$

$$S = 4\pi r^2$$

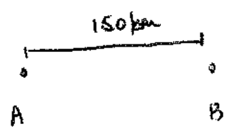
$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$-1 = 8\pi(5) \cdot \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=5} = \frac{-1}{40\pi} \text{ cm/min}$$

(d)

Noon



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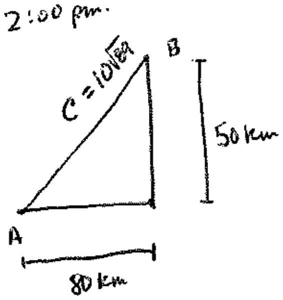


$$A^2 + B^2 = C^2$$

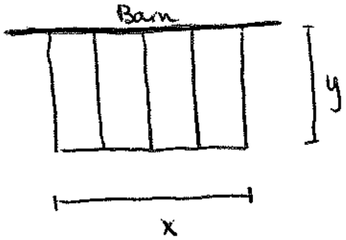
$$2A \cdot \frac{dA}{dt} + 2B \cdot \frac{dB}{dt} = 2c \cdot \frac{dc}{dt}$$

$$A \cdot \frac{dA}{dt} + B \cdot \frac{dB}{dt} = C \cdot \frac{dc}{dt}$$

$$80(-35) + 50(25) = 10\sqrt{89} \cdot \frac{dc}{dt}$$

$$\frac{dc}{dt} = \frac{-155}{\sqrt{89}} \text{ km/h}$$

8. (a)



$x + 5y = 150$

$x = 150 - 5y$

<u>Area</u>: $A = x \cdot y$

$A(y) = (150 - 5y) \cdot y$

$= 150y - 5y^2$

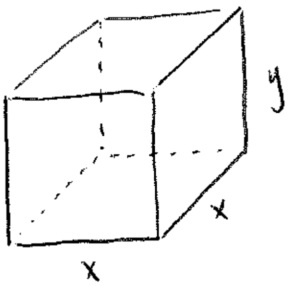
$A'(y) = 150 - 10y \longrightarrow$ $A''(y) = -10$ Always concave down

$150 - 10y = 0$

$y = 15$

$\boxed{\text{Dimensions}: x = 75, y = 15}$

(b)



$x^2 + 4xy = 108$

$y = \frac{108 - x^2}{4x}$

$y = 27x^{-1} - \frac{1}{4}x$

<u>Volume</u>: $V = x^2 \cdot y$

$V(x) = x^2\left(27x^{-1} - \frac{1}{4}x\right)$

$V(x) = 27x - \frac{1}{4}x^3$

$V'(x) = 27 - \frac{3}{4}x^2 \longrightarrow$ $V''(x) = -\frac{3}{2}x$, $V''(6) < 0 \Rightarrow \max$

$27 - \frac{3}{4}x^2 = 0$

$x^2 = 36$

$x = 6$

$\boxed{\text{Dimensions}: x = 6, y = 3}$

(c)

$300 = \pi r^2 \cdot h$

$h = \frac{300}{\pi r^2}$

<u>Cost</u>: $C = 2(2\pi r^2) + 6(2\pi r \cdot h) = 4\pi r^2 + 12\pi r h$

$C(r) = 4\pi r^2 + 12\pi r\left(\frac{300}{\pi r^2}\right)$

$= 4\pi r^2 + 3600 r^{-1}$

$C'(r) = 8\pi r - 3600 r^{-2} \longrightarrow$ $C''(r) = 8\pi + 7200 r^{-3}$ Always positive

$8\pi r - \frac{3600}{r^2} = 0$

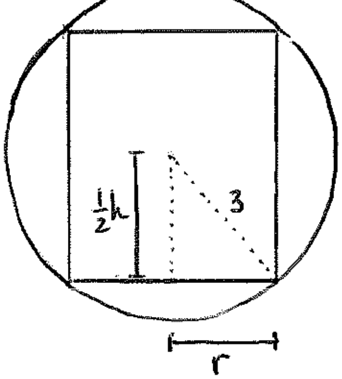
$8\pi r = \frac{3600}{r^2}$

$r^3 = \frac{3600}{8\pi}$

$r = \sqrt[3]{\frac{450}{\pi}}$

$\boxed{\text{Dimensions}: r = \sqrt[3]{\frac{450}{\pi}} \quad h = \frac{300}{\pi\left(\frac{450}{\pi}\right)^{2/3}}}$

(d)



$r^2 + (\frac{1}{2}h)^2 = 9$

$r^2 = 9 - \frac{1}{4}h^2$

Volume : $V = \pi r^2 h$

$V(h) = \pi(9 - \frac{1}{4}h^2)\cdot h$

$= \pi(9h - \frac{1}{4}h^3)$

$V'(h) = \pi(9 - \frac{3}{4}h^2)$ → $V''(h) = \pi(-\frac{3}{2}h)$ Always negative

$9 - \frac{3}{4}h^2 = 0$

$h^2 = 12$

$h = 2\sqrt{3}$

Dimensions : $h = 2\sqrt{3}$ in $r = \sqrt{6}$ in
MAX Volume : $12\pi\sqrt{3}$ in^3

9. (a) $s'(t) = v(t) = 6t^2 - 18t + 12$

$s''(t) = a(t) = 12t - 18$

$s(4) = 28$

$v(4) = 36$

$a(4) = 30$

(b) $v(t) = 6(t^2 - 3t + 2)$

$= 6(t-2)(t-1)$

$v(t) = 0 \Rightarrow t = 1, 2$

Particle at rest at $t = 1, 2$.

$v(t)$: $+$ on $(-\infty, 1)$, $-$ on $(1, 2)$, $+$ on $(2, \infty)$

Distance Increasing $(-\infty, 1)$ $(2, \infty)$

Decreasing $(1, 2)$

(c) $a(t) = 6(2t - 3)$

$a(t) = 0 \Rightarrow t = 3/2$

$a(t)$: $-$ on $(-\infty, 3/2)$, $+$ on $(3/2, \infty)$

Speeding Up : $(1, 3/2)$ $(2, \infty)$
Slowing Down : $(-\infty, 1)$ $(3/2, 2)$

(d) $v(t) = 12 \Rightarrow 6t^2 - 18t + 12 = 12$

$6t^2 - 18t = 0$

$6t(t-3) = 0$

Velocity equal to 12 at $t = 0, 3$.

(e) Displacement : $s(8) - s(0)$

$540 - -4$

544

Distance :

t	0	1	2	8
$s(t)$	-4	1	0	544

550

10. (a) Speed $= |v(t)|$

Speed is greatest at $t = 8$ since $|v(8)| = 6$ is the highest value on the graph of speed.

(b) At Rest at $t = 0, 4, 10$

Moving left on $4 < t < 10$

Moving right on $0 < t < 4$ and $10 < t < 11$

(c) Speed decreases when $v(t)$ and $a(t)$ have different signs. Given the graph of velocity, $v(t)$ and $a(t)$ have different signs where the graph of $v(t)$ is above the x-axis and decreasing or below the x-axis and increasing. Therefore speed decreases on $2 < t < 4$ and $8 < t < 10$.

Speed increases when $v(t)$ and $a(t)$ have the same sign. Given the graph of velocity, $v(t)$ and $a(t)$ have the same sign where the graph of $v(t)$ is above the x-axis and increasing or below the x-axis and decreasing. Therefore speed increases on $0 < t < 2$, $4 < t < 8$, and $10 < t < 11$.