Calculus BC

Test #3 Review

1. Consider the function $f(x)=\sqrt{2+x}$.

(a) Compute the 4rd degree Taylor Polynomial of f centered at x=0.

(b) Compute the 4rd degree Taylor Polynomial of f centered at $x=2$.

(c) Use both polynomials to approximate $f(1.5)$.

2. Consider the function $f(x)=2x-\cos 2x$.

(a) Find $P_3(x)$, the third degree Taylor polynomial of f centered at $a=0$.

(b) Find $Q_3(x)$, the third degree Taylor polynomial of f centered at $a=\frac{\pi}{2}$.

(c) Use both polynomials to approximate $f(1.8)$.

3. Let f be the function given by $f(x)=\cos\left(2x+\frac{\pi}{6}\right)$ and let $P(x)$ be the third-degree Taylor polynomial for f about $x=0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|<\frac{1}{12,000}$.

4. Let $f(x)=\sqrt{x+1}$. Compute the second degree Maclaurin polynomial to estimate $\sqrt{1.1}$. What is the error associated with this estimation.

5. Let $f(x)=\ln(x+5)-\ln 5$. Compute the third degree Maclaurin polynomial. What is the error in this estimate provided $|x|<0.1$?

Find the radius and interval of convergence of the power series.

6. $\sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{(n+1)^2}$

8. $\sum_{n=0}^{\infty} n!\,(x+3)^n$

9. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

Represent the following function by power series. State the radius and interval of convergence.

10. $f(x) = \frac{6}{3-2x}$

11. $f(x) = \frac{1}{x^2 - x - 2}$

12. $f(x) = \frac{1}{1-x^2}$

13. $f(x) = \frac{1}{(1+x)^2}$.

14. $f(x) = \ln(x^2 + 1)$

15. $f(x) = \int \frac{1}{1+x^5}\,dx$

Find the first four nonzero terms and the general term for the Maclaurin series for each of the following, and find the interval of convergence for each series.

16. $f(x) = \sin(2x)$

17. $f(x) = e^{-2x}$

18. $f(x) = x\cos\left(x^3\right)$

19. $g(x) = \frac{x^2}{1+x}$

20. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

Test #3 Review: Answers

1. (a) $f(x) = \sqrt{2+x}$ $\qquad f(0) = \sqrt{2}$

$f'(x) = \frac{1}{2}(2+x)^{-\frac{1}{2}}$ $\qquad f'(0) = \frac{1}{2\sqrt{2}}$

$f''(x) = -\frac{1}{4}(2+x)^{-\frac{3}{2}}$ $\qquad f''(0) = -\frac{1}{4\sqrt{8}} = -\frac{1}{8\sqrt{2}}$

$f'''(x) = \frac{3}{8}(2+x)^{-\frac{5}{2}}$ $\qquad f'''(0) = \frac{3}{8\sqrt{32}} = \frac{3}{32\sqrt{2}}$

$f^{(4)}(x) = \frac{-15}{16}(2+x)^{-\frac{7}{2}}$ $\qquad f^{(4)}(0) = \frac{-15}{16\sqrt{128}} = \frac{-15}{128\sqrt{2}}$

$$P_4(x) = \sqrt{2} + \left(\frac{1}{2\sqrt{2}}\right)\cdot x + \left(-\frac{1}{8\sqrt{2}}\right)\cdot\frac{x^2}{2!} + \left(\frac{3}{32\sqrt{2}}\right)\cdot\frac{x^3}{3!} + \left(\frac{-15}{128\sqrt{2}}\right)\cdot\frac{x^4}{4!}$$

$$= \sqrt{2} + \frac{x}{2\sqrt{2}} - \frac{x^2}{(8\sqrt{2})2!} + \frac{3x^3}{(32\sqrt{2})3!} - \frac{15x^4}{(128\sqrt{2})4!}$$

(b) $f(2) = 2$

$f'(2) = \frac{1}{4}$

$f''(2) = -\frac{1}{32}$

$f'''(2) = \frac{3}{256}$

$f^{(4)}(2) = \frac{-15}{2048}$

$$P_4(x) = 2 + \left(\frac{1}{4}\right)(x-2) + \left(-\frac{1}{32}\right)\cdot\frac{(x-2)^2}{2!} + \left(\frac{3}{256}\right)\frac{(x-2)^3}{3!} + \left(\frac{-15}{2048}\right)\frac{(x-2)^4}{4!}$$

$$= 2 + \frac{(x-2)}{4} - \frac{(x-2)^2}{(32)2!} + \frac{3(x-2)^3}{(256)3!} - \frac{15(x-2)^4}{(2048)(4!)}$$

(c) CENTERED @ $x=0$: $P_4(1.5) = 1.86491645$

CENTERED @ $x=2$: $P_4(1.5) = 1.870830536$

2. (a) $f(x) = 2x - \cos 2x$ $\quad f(0) = -1$

$f'(x) = 2 + 2\sin 2x$ $\quad f'(0) = 2$

$f''(x) = 4\cos 2x$ $\quad f''(0) = 4$

$f'''(x) = -8\sin 2x$ $\quad f'''(0) = 0$

$$P_3(x) = -1 + (2)x + (4)\frac{x^2}{2!} + (0)\frac{x^3}{3!}$$

$$= -1 + 2x + 2x^2$$

(b) $f\left(\frac{\pi}{2}\right) = \pi + 1$

$f'\left(\frac{\pi}{2}\right) = 2$

$f''\left(\frac{\pi}{2}\right) = -4$

$f'''\left(\frac{\pi}{2}\right) = 0$

$$Q_3(x) = (\pi+1) + 2\left(x - \frac{\pi}{2}\right) + (-4)\cdot\frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + 0\cdot\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}$$

$$= (\pi+1) + 2\left(x - \frac{\pi}{2}\right) - 2\left(x - \frac{\pi}{2}\right)^2$$

(c) $P_3(1.8) = 9.08$

$Q_3(1.8) = 4.494931352$

3. (a) $f(x) = \cos\left(2x + \frac{\pi}{6}\right)$ $\quad f(0) = \frac{\sqrt{3}}{2}$

$f'(x) = -2\sin\left(2x + \frac{\pi}{6}\right)$ $\quad f'(0) = -1$

$f''(x) = -4\cos\left(2x + \frac{\pi}{6}\right)$ $\quad f''(0) = -2\sqrt{3}$

$f'''(x) = 8\sin\left(2x + \frac{\pi}{6}\right)$ $\quad f'''(0) = 4$

$$P(x) = \frac{\sqrt{3}}{2} + (-1)x + (-2\sqrt{3})\cdot\frac{x^2}{2!} + (4)\cdot\frac{x^3}{3!}$$

$$= \frac{\sqrt{3}}{2} - x - \sqrt{3}\cdot x^2 + \frac{2}{3}x^3$$

(b) $|R_3(x)| \leq \left|\frac{x^4}{4!}\cdot \max f^{(4)}(z)\right|$ $\quad 0 < z < \frac{1}{10}$ $\quad \max f^{(4)}(z) = 16$ $\quad f^{(4)}(x) = 16\cos\left(2x + \frac{\pi}{6}\right)$

$|R_3(x)| \leq \left|\frac{x^4}{4!}\cdot 16\right|$ $\quad \left|R_3\left(\frac{1}{10}\right)\right| \leq \left|\frac{(1/10)^4}{4!}\cdot 16\right| = \frac{1}{15000} < \frac{1}{12000}$

4. $f(x) = \sqrt{x+1}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}}$ $\quad f''(0) = -\frac{1}{4}$

$$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$P_2(0.1) = 1.04875$$

$0 < z < 0.1$ for $f^{(3)}(z) = \frac{3}{8}(x+1)^{-\frac{5}{2}}$

$\left|\max f^{(3)}(z)\right| = \frac{3}{8}(0+1)^{-\frac{5}{2}} = \frac{3}{8}$

$$|R_2(x)| \le \left| \frac{x^3}{3!} \cdot \max f^{(3)}(z) \right|$$

$$|R_2(0.1)| \le \left| \frac{(0.1)^3}{3!} \cdot \frac{3}{8} \right| = \frac{1}{16000} \approx .0000625$$

$$1.04875 < \sqrt{1.1} < 1.0488125$$

5. $f(x) = \ln(x+5) - \ln 5$ $\quad f(0) = 0$

$f'(x) = \frac{1}{x+5}$ $\quad f'(0) = \frac{1}{5}$

$f''(x) = \frac{-1}{(x+5)^2}$ $\quad f''(0) = -\frac{1}{25}$

$f'''(x) = \frac{2}{(x+5)^3}$ $\quad f'''(0) = \frac{2}{125}$

$$P_3(x) = 0 + \frac{1}{5}x + \left(-\frac{1}{25}\right)\frac{x^2}{2!} + \left(\frac{2}{125} \cdot \frac{x^3}{3!}\right)$$

$$= \frac{1}{5}x - \frac{1}{50}x^2 + \frac{1}{375}x^3$$

$$|R_3(x)| \le \left| \frac{x^4}{4!} \cdot \max f^{(4)}(z) \right|$$

$-0.1 < z < 0.1$ $\quad f^{(4)}(z) = \frac{-6}{(x+5)^4}$

$\left|\max f^{(4)}(z)\right| = \left|\frac{-6}{(-0.1+5)^4}\right| = \frac{6}{(4.9)^4}$

$$|R_3(x)| \le \left| \frac{(0.1)^4}{4!} \cdot \frac{6}{(4.9)^4} \right| \approx 4.34 \times 10^{-8}$$

6. $\sum_{n=0}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{3^n(x-2)^n}\right| = 3|x-2|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 3|x-2|<1$$

$$= |x-2| < \frac{1}{3}$$

$R = \frac{1}{3}$

$\boxed{x=\frac{5}{3}}$ $\sum_{n=0}^{\infty} \frac{3^n\left(-\frac{1}{3}\right)^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ CONVERGES

$\boxed{x=\frac{7}{3}}$ $\sum_{n=0}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n} = \sum_{n=0}^{\infty}\frac{1}{n}$ DIVERGES

$$\frac{5}{3} \le x < \frac{7}{3}$$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(x-2)^n}\right| = |x-2|\cdot\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+2)^2}\right| = |x-2|<1$$

$R=1$

$\boxed{x=1}$ $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$ CONVERGES BY LIMIT COMP

$\boxed{x=3}$ $\sum_{n=0}^{\infty}\frac{(-1)^n(1)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$ CONVERGES BY ALT. SERIES TEST

$$1 \le x \le 3$$

8. $\sum_{n=0}^{\infty} n!(x+3)^n$

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x+3)^{n+1}}{n!\,(x+3)^n}\right| = |x+3|\cdot\lim_{n\to\infty}\left|n+1\right| = \infty$$

The series only converges at the center, $x=-3$.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

$$\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\cdot\frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \left|\frac{2x-1}{5}\right|\cdot\lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}\right| = \left|\frac{2x-1}{5}\right| < 1$$

$$= |x-1/2| < \frac{5}{2}$$

$R = \frac{5}{2}$

$\boxed{x=-2}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(2(-2)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ CONVERGENT BY ALT SERIES TEST

$\boxed{x=3}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(2(3)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ DIVERGENT P-SERIES

$-2 \le x < 3$

10. $\displaystyle\frac{6}{3-2x} = \frac{2}{1-2/3x} = \sum_{n=0}^{\infty}2\left(\frac{2}{3}x\right)^n = \sum_{n=0}^{\infty}\frac{2^{n+1}x^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{2^{n+2}x^{n+1}}{3^{n+1}}\cdot\frac{3^n}{2^{n+1}x^n}\right| = \left|\frac{2x}{3}\right| < 1 \qquad |x| < \frac{3}{2}$$

$R = \frac{3}{2}$

$\boxed{x=-\frac{3}{2}}$ $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+1}\left(-\frac{3}{2}\right)^n}{3^n} = \sum_{n=0}^{\infty}(-1)^n\cdot 2$ DIVERGES

$-\frac{3}{2} < x < \frac{3}{2}$

$\boxed{x=\frac{3}{2}}$ $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+1}\left(\frac{3}{2}\right)^n}{3^n} = \sum_{n=0}^{\infty}2$ DIVERGES

11. $f(x) = \dfrac{1}{x^2-x-2} = \dfrac{1/3}{x-2} + \dfrac{-1/3}{x+1}$

$\displaystyle\frac{1/3}{x-2} = \frac{-1/3}{2-x} = \frac{-1/6}{1-x/2} = \sum_{n=0}^{\infty}\left(\frac{-1}{6}\right)\left(\frac{x}{2}\right)^n = -\frac{1}{6}\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ $\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right| = \left|\frac{x}{2}\right| < 1$ $-2 < x < 2$

$\displaystyle\frac{-1/3}{x+1} = \frac{-1/3}{1-(-x)} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)(-x)^n = -\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n(x)^n$ $\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x| < 1$ $-1 < x < 1$

$$\frac{1}{x^2-x-2} = -\frac{1}{6}\sum_{n=0}^{\infty}\frac{x^n}{2^n} + -\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n x^n = -\frac{1}{3}\left[\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}}+(-1)^n\right)x^n\right]$$

$-1 < x < 1$

12. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} \qquad -1 < x < 1$

13. $\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \sum_{n=0}^{\infty} (-1)^n n x^{n-1} \qquad \frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty} (-1)^n n x^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1} \qquad R = 1$$

$\boxed{x=-1}$ $\sum_{n=0}^{\infty} (-1)^{n+1} n (-1)^{n-1} = \sum_{n=0}^{\infty} n$ DIVERGES

$\boxed{x=1}$ $\sum_{n=0}^{\infty} (-1)^{n+1} n (1)^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} n$ DIVERGES

$-1 < x < 1$

14. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$

$-1 \leq x \leq 1$

15. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n (x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n}$

$$\int \frac{1}{1+x^5} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1} + C \qquad \text{when } x=0,\ C=0$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1} \qquad R = 1$$

$\boxed{x=-1}$ $\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{5n+1}}{5n+1} = \sum_{n=0}^{\infty} \frac{(-1)}{5n+1}$ DIVERGES

$\boxed{x=1}$ $\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{5n+1}}{5n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$ CONVERGES BY ALT SERIES TEST

$-1 < x \leq 1$

16. $$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$$

17. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^n}{n!} \qquad -\infty < x < \infty$$

18. $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

$$x\cos(x^3) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \qquad -\infty < x < \infty$$

19. $$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{x^2}{1+x} = x^2 \cdot \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2} \qquad -1 < x < 1$$

20.

(a) $f(x) = \sin(5x + \frac{\pi}{4})$ $\qquad f(0) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos(5x + \frac{\pi}{4}) \cdot 5 = 5\cos(5x + \frac{\pi}{4})$ $\qquad f'(0) = \frac{5\sqrt{2}}{2}$

$f''(x) = -25\sin(5x + \frac{\pi}{4})$ $\qquad f''(0) = \frac{-25\sqrt{2}}{2}$

$f'''(x) = -125\cos(5x + \frac{\pi}{4})$ $\qquad f'''(0) = \frac{-125\sqrt{2}}{2}$

$$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x + \left(\frac{-25\sqrt{2}}{2}\right)\left(\frac{x^2}{2!}\right) + \left(\frac{-125\sqrt{2}}{2}\right)\left(\frac{x^3}{3!}\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2 \cdot 2!}x^2 - \frac{125\sqrt{2}}{2 \cdot 3!}x^3$$

OR

$$= \frac{\sqrt{2}}{2}\left[1 + 5x - \frac{25x^2}{2!} - \frac{125x^3}{3!}\right]$$

(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$

(c) $|R_3(x)| \leq \left|\frac{x^4}{4!} \cdot \max f^4(z)\right|$ $\qquad 0 < z < \frac{1}{10}$ $\qquad f^4(z) = 625\sin(5x + \frac{\pi}{4})$

$\max f^{(4)}(z) = 625$

$$\left|R_3\left(\frac{1}{10}\right)\right| \leq \left|\frac{(1/10)^4}{4!} \cdot 625\right| = \frac{1}{384} < \frac{1}{100}$$

(d) $$G(x) = \int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right)dx$$

$$= \left[\frac{\sqrt{2}}{2}t + \frac{5\sqrt{2}}{4}t^2 - \frac{25\sqrt{2}}{12}t^3\right]_0^x$$

$$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$$

21. What is the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-4)^n}{2 \cdot 3^{n+1}}$?

(A) $\frac{1}{3}$ (B) $\frac{3}{2}$ (C) 3 (D) 4 (E) 6

22. Let f be a function that has derivatives of all orders for all real numbers, and let $P_3(x)$ be the third-degree Taylor polynomial for f about $x = 0$. The Taylor series for f about $x = 0$ converges at $x = 1$, and $\left|f^{(n)}(x)\right| \le \frac{n}{n+1}$ for $1 \le n \le 4$ and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $\left|f(1) - P_3(1)\right| \le k$?

(A) $\frac{4}{5}$

(B) $\frac{4}{5} \cdot \frac{1}{4!}$

(C) $\frac{4}{5} \cdot \frac{1}{3!}$

(D) $\frac{3}{4} \cdot \frac{1}{4!}$

(E) $\frac{3}{4} \cdot \frac{1}{3!}$

23. The function f has derivatives of all orders for all real numbers with $f(0) = 3$, $f'(0) = -4$, $f''(0) = 2$, and $f'''(0) = 1$. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. What is the third-degree Taylor polynomial for g about $x = 0$?

(A) $-4x + 2x^2 + \frac{1}{3}x^3$

(B) $-4x + x^2 + \frac{1}{6}x^3$

(C) $3x - 2x^2 + \frac{1}{3}x^3$

(D) $3x - 2x^2 + \frac{2}{3}x^3$

(E) $3 - 4x + x^2 + \frac{1}{6}x^3$

24. Let $P(x) = 3 - 3x^2 + 6x^4$ be the fourth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f^{(4)}(0)$?

(A) 0 (B) $\frac{1}{4}$ (C) 6 (D) 24 (E) 144

25. What is the radius of convergence for the series $\sum_{n=0}^{\infty} \frac{(x-5)^n}{2^n(2n+3)^2}$?

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 5

26. The Taylor series for a function f about $x = 0$ is given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n}$ and converges to f for all real numbers x. If the fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{1}{2}\right)$, what is the alternating series error bound?

(A) $\frac{1}{2^4 \cdot 5!}$

(B) $\frac{1}{2^5 \cdot 6!}$

(C) $\frac{1}{2^6 \cdot 7!}$

(D) $\frac{1}{2^{10} \cdot 11!}$

27. The power series $\sum_{n=0}^{\infty} a_n(x-1)^n$ converges conditionally at $x = 5$. Which of the following statements about convergence of the series at $x = -4$ is true?

(A) The series converges absolutely at $x = -4$.

(B) The series converges conditionally at $x = -4$.

(C) The series diverges at $x = -4$.

(D) There is not enough information given to determine convergence of the series at $x = -4$.

28. Let f be a function with second derivative $f''(x) = \sqrt{1+3x}$. The coefficient of x^3 in the Taylor series for f about $x = 0$ is

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$ (E) $\frac{3}{2}$

(21) $\lim_{n\to\infty}\left|\dfrac{(x-4)^{n+1}}{2\cdot 3^{n+2}}\cdot\dfrac{2\cdot 3^{n+1}}{(x-4)^n}\right|$ $\boxed{C}$

$\left|\dfrac{x-4}{3}\right|\lim_{n\to\infty}|1| = \left|\dfrac{x-4}{3}\right|$ $\left|\dfrac{x-4}{3}\right| < 1$

$|x-4| < 3$

(22) $|f(1)-P_3(1)| \le \left|\dfrac{(1)^4}{4!}\cdot \max f^{(4)}(z)\right|$ $\boxed{B}$

$\le \left|\dfrac{1}{4!}\cdot\dfrac{4}{5}\right|$

(23) $g(0) = \int_0^0 f(t)\,dt = 0$

$g'(0) = f(0) = 3$

$g''(0) = f'(0) = -4$ $P_3(x) = 0 + 3\cdot x + -4\cdot\dfrac{x^2}{2!} + 2\cdot\dfrac{x^3}{3!}$ $\boxed{C}$

$g'''(0) = f''(0) = 2$ $= 3x - 2x^2 + \frac{1}{3}x^3$

$g^{(4)}(0) = f'''(0) = 1$

(24) Forth degree term: $f^{(4)}(0)\cdot\dfrac{x^4}{4!} = 6x^4$

$\boxed{E}$

$f^{(4)}(0)\cdot\dfrac{1}{24} = 6$

$f^{(4)}(0) = 144$

(25) $\lim_{n\to\infty}\left|\dfrac{(x-5)^{n+1}}{2^{n+1}(2n+5)^2}\cdot\dfrac{2^n(2n+3)^2}{(x-5)^n}\right|$ $\boxed{C}$

$\left|\dfrac{x-5}{2}\right|\cdot\lim_{n\to\infty}\left|\dfrac{(2n+3)^2}{(2n+5)^2}\right| = \left|\dfrac{x-5}{2}\right|$ $\left|\dfrac{x-5}{2}\right| < 1$

$|x-5| < 2$

(26) Alt series error $\left| R_4\left(\tfrac{1}{2}\right) \right| \le \left| \text{First Omitted Term} \right|$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \cdot x^{2n} = \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots$$

4th Degree First Omitted

$$\left| R_4\left(\tfrac{1}{2}\right) \right| \le \left| \frac{\left(\tfrac{1}{2}\right)^6}{7!} \right|$$

$\boxed{C}$

(27) If $\sum_{n=0}^{\infty} a_n (x-1)^n$ converges conditionally at $x=5$, this must be an endpoint of the interval of convergence

Therefore, since $c = 1$, the radius of convergence $R = 4$.

The interval of convergence is of the form: $-3 < x \le 5$, not knowing convergence at $x=-3$.

Since $x=-4$ is outside, the series diverges at $x=-4$.

$\boxed{C}$

(28)

$$f'''(x) = \frac{1}{2}(1+3x)^{-1/2} \cdot 3$$

$$f'''(0) = \frac{3}{2}$$

$$P(x) = \cdots + \frac{3}{2} \cdot \frac{x^3}{3!} + \cdots$$

$$= \cdots + \frac{1}{4} x^3 + \cdots$$

$\boxed{C}$