Test #2: Differentiation

1. $\lim_{h\to 0} \frac{\sqrt[3]{5+h}-\sqrt[3]{5}}{h}$

$f(x)=\sqrt[3]{x}$

$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}$

$\boxed{f'(5)=\frac{1}{3\sqrt[3]{25}}}$

2. $\lim_{h\to 0} \frac{\cos\left(\frac{3\pi}{4}+h\right)-\cos\frac{3\pi}{4}}{h}$

$f(x)=\cos x$

$f'(x)=-\sin x$

$\boxed{f'\left(\frac{3\pi}{4}\right)=-\sin\left(\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}}$

3. $\lim_{h\to 0} \frac{\ln(3+h)-\ln 3}{h}$

$f(x)=\ln x$

$f'(x)=\frac{1}{x}$

$\boxed{f'(3)=\frac{1}{3}}$

4. $f(x)=\begin{cases} 4-x^2, & x<1 \\ 2x+2, & x\geq 1 \end{cases}$

$\boxed{\text{Continuity}}$

$\lim_{x\to 1^-} 4-x^2 \qquad \lim_{x\to 1^+} 2x+2$

$4-(1)^2 \qquad 2(1)+2$

$3 \neq 4$

Since $f(x)$ is not continuous at $x=1$, $f(x)$ is not differentiable at $x=1$ either.

5. $f(x) = \begin{cases} 3+(x+2)^{1/3}, & x \geq -2 \\ 3-(x+2)^{2/3}, & x < -2 \end{cases}$

$f'(x) = \begin{cases} \frac{1}{3}(x+2)^{-2/3}, & x > -2 \\ -\frac{2}{3}(x+2)^{-1/3}, & x < -2 \end{cases}$

$\boxed{\text{Continuous}}$

$\lim_{x \to -2^-} 3-(x+2)^{2/3}$ $\quad$ $\lim_{x \to -2^+} 3+(x+2)^{1/3}$

$3 = 3$

$\boxed{\text{Differentiable}}$

$\lim_{x \to -2^-} -\frac{2}{3}(x+2)^{-1/3}$ $\quad$ $\lim_{x \to -2^+} \frac{1}{3}(x+2)^{-2/3}$

undefined $\quad$ undefined

Since $f(-2) = \lim_{x \to -2} f(x) = 3$, $f(x)$ is continuous at $x = -2$.

Since $\lim_{x \to -2^-} f'(x)$ & $\lim_{x \to -2^+} f'(x)$ are undefined, $f(x)$ is not differentiable at $x = -2$.

6. $f(x) = \begin{cases} \sqrt{x} - 3, & x > 1 \\ \frac{1}{2}x - \frac{5}{2}, & x \leq 1 \end{cases}$

$f'(x) = \begin{cases} \frac{1}{2}x^{-1/2}, & x > 1 \\ \frac{1}{2}, & x \leq 1 \end{cases}$

$\boxed{\text{Continuous}}$

$\lim_{x \to 1^-} \frac{1}{2}x - \frac{5}{2}$ $\quad$ $\lim_{x \to 1^+} \sqrt{x} - 3$

$-2 = -2$

$\boxed{\text{Differentiable}}$

$\lim_{x \to 1^-} \frac{1}{2}$ $\quad$ $\lim_{x \to 1^+} \frac{1}{2}x^{-1/2}$

$\frac{1}{2} = \frac{1}{2}$

Since $f(1) = \lim_{x \to 1} f(x) = -2$ and $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$,

$f(x)$ is continuous and differentiable at $x = 1$.

5. $f'(x)$ is positive when $f(x)$ increases.

$f'(x) > 0$ on $(-6, -4)$ and $(2, 4)$

6. $f'(x)$ is negative when $f(x)$ decreases

$f'(x) < 0$ on $(-4, -2)$ and $(-2, 2)$.

7. $f'(x)$ equals zero when $f(x)$ has horizontal tangents.

$f'(x) = 0$ when $x = -4$.

8. $f'(x)$ is undefined at endpoints, sharp points, vertical tangents, and discontinuities.

$f'(x) =$ undefined when $x = -6, -2, 2, 4$.

9. $f(x)$ is increasing when $f'(x)$ is positive (above x-axis)

$f(x)$ is increasing on $(-\infty, -6.6)$ and $(0, 3.6)$.

10. $f(x)$ is decreasing when $f'(x)$ is negative (below x-axis)

$f(x)$ is decreasing on $(-6.6, 0)$ and $(3.6, \infty)$

11. Horizontal tangents when $f'(x) = 0$ (x-intercepts)

Horizontal tangents when $x = -6.6, 0, 3.6$

12. $g(x) = \ln\left[\dfrac{\sqrt{x^2+4}}{(6x-5)^2}\right] = \frac{1}{2}\ln(x^2+4) - 2\ln(6x-5)$

$$g'(x) = \frac{1}{2}\cdot\frac{1}{x^2+4}\cdot 2x - 2\cdot\frac{1}{6x-5}\cdot 6$$

$$= \frac{x}{x^2+4} - \frac{12}{6x-5}$$

$$= \frac{x(6x-5) - 12(x^2+4)}{(x^2+4)(6x-5)}$$

$$= \frac{-6x^2-5x-48}{(x^2+4)(6x-5)}$$

13. $f(x) = \sqrt{\dfrac{2x+5}{7x-9}} = \boxed{\left(\dfrac{2x+5}{7x-9}\right)^{1/2}} = \dfrac{(2x+5)^{1/2}}{(7x-9)^{1/2}} = (2x+5)^{1/2}(7x-9)^{-1/2}$

$$f'(x) = \frac{1}{2}\left(\frac{2x+5}{7x-9}\right)^{-1/2}\left[\frac{(7x-9)(2)-(2x+5)(7)}{(7x-9)^2}\right]$$

$$= \frac{1}{2}\,\frac{(2x+5)^{-1/2}}{(7x-9)^{-1/2}}\left[\frac{14x-18-14x-35}{(7x-9)^2}\right]$$

$$= \frac{1}{2}\,\frac{(7x-9)^{1/2}}{(2x+5)^{1/2}}\left[\frac{-53}{(7x-9)^2}\right]$$

$$= \frac{-53}{2(2x+5)^{1/2}(7x-9)^{3/2}}$$

$$= \frac{-53}{2\sqrt{(2x+5)(7x-9)^3}}$$

14. $f(x) = \boxed{\ln\left(xe^{7x}\right)} = \ln x + 7x \ln e = \boxed{\ln x + 7x}$

$$f'(x) = \frac{1}{xe^{7x}}\left[e^{7x} + xe^{7x} \cdot 7\right]$$

$$= \frac{e^{7x} + 7xe^{7x}}{xe^{7x}}$$

$$= \frac{e^{7x}(1+7x)}{xe^{7x}}$$

$$= \frac{1+7x}{x}$$

$$f'(x) = \frac{1}{x} + 7$$

$$= \frac{1+7x}{x}$$

15. $f(x) = \sec^2 x \cdot \tan x = [\sec x]^2 \cdot \tan x$

$$f'(x) = 2[\sec x] \cdot \sec x \tan x \cdot \tan x + [\sec x]^2 \cdot \sec^2 x$$

$$= 2\sec^2 x \tan^2 x + \sec^4 x$$

$$= \sec^2 x\left[2\tan^2 x + \sec^2 x\right].$$

16. $f(x) = \ln(5x^2+9)^3 = 3\ln(5x^2+9)$

$$f'(x) = 3 \cdot \frac{1}{5x^2+9} \cdot 10x$$

$$= \frac{30x}{5x^2+9}$$

17. $$y = \frac{x}{\sqrt{x^2-1}} = \boxed{x\,(x^2-1)^{-1/2}}$$

$$\frac{dy}{dx} = 1\cdot(x^2-1)^{-1/2} + x\cdot -\frac{1}{2}(x^2-1)^{-3/2}\cdot 2x$$

$$= (x^2-1)^{-1/2} - x^2(x^2-1)^{-3/2}$$

$$= (x^2-1)^{-3/2}\left[(x^2-1)^1 - x^2\right]$$

$$= (x^2-1)^{-3/2}\left[-1\right]$$

$$= \frac{-1}{(x^2-1)^{3/2}}$$

18. $$h(x) = \boxed{\frac{1-\cos x}{\sin x}} = (1-\cos x)(\sin x)^{-1}$$

$$h'(x) = \frac{\sin x(\sin x) - (1-\cos x)\cos x}{\sin^2 x}$$

$$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$$

$$= \frac{1-\cos x}{\sin^2 x}$$

$$= \frac{1-\cos x}{1-\cos^2 x}$$

$$= \frac{1-\cos x}{(1-\cos x)(1+\cos x)}$$

$$= \frac{1}{1+\cos x}$$

19. $f(x) = \left[(x^2-1)^5 - x\right]^3$

$$f'(x) = 3\left[(x^2-1)^5 - x\right]^2 \cdot \left[5(x^2-1)^4 \cdot 2x - 1\right]$$

$$= 3\left[(x^2-1)^5 - x\right]^2 \cdot \left[10x(x^2-1)^4 - 1\right]$$

20. $f(x) = \arcsin(x^3+1)$

$$f'(x) = \frac{1}{\sqrt{1-(x^3+1)^2}} \cdot 3x^2$$

$$= \frac{3x^2}{\sqrt{1-(x^3+1)^2}}$$

$$= \frac{3x^2}{\sqrt{1-x^6-2x^3-1}}$$

$$= \frac{3x^2}{\sqrt{-x^6-2x^3}}$$

$$= \frac{3x^2}{\sqrt{-x^3(x^3+2)}}$$

$$= \frac{3x^2}{x\sqrt{-x(x^3+2)}}$$

$$= \frac{3x}{\sqrt{-x(x^3+2)}}$$

21. $g(x) = x^3 \sec^4(2x) = x^3[\sec(2x)]^4$

$$g'(x) = 3x^2[\sec(2x)]^4 + x^3 \cdot 4[\sec(2x)]^3 \cdot \sec(2x)\tan(2x) \cdot 2$$

$$= 3x^2 \sec^4(2x) + 8x^3 \sec^4(2x)\tan(2x)$$

$$= x^2 \sec^4(2x)\left[3 + 8x\tan(2x)\right]$$

22. $j(x) = \left[\dfrac{3x+2}{x-9}\right]^5 = \dfrac{(3x+2)^5}{(x-9)^5} = \boxed{(3x+2)^5(x-9)^{-5}}$

$$j'(x) = 5(3x+2)^4 \cdot 3\,(x-9)^{-5} + (3x+2)^5 \cdot -5(x-9)^{-6} \cdot 1$$

$$= 15(3x+2)^4(x-9)^{-5} - 5(3x+2)^5(x-9)^{-6}$$

$$= 5(3x+2)^4(x-9)^{-6}\left[3(x-9) - (3x+2)\right]$$

$$= \frac{5(3x+2)^4[-29]}{(x-9)^6}$$

$$= \frac{-145(3x+2)^4}{(x-9)^6}$$

23. $y = \dfrac{\sec^3(2x)}{x^2} = \dfrac{[\sec(2x)]^3}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \cdot 3[\sec(2x)]^2 \cdot \sec(2x)\tan(2x) \cdot 2 - [\sec(2x)]^3 \cdot 2x}{(x^2)^2}$$

$$= \frac{6x^2 \sec^3(2x)\tan(2x) - 2x\sec^3(2x)}{x^4}$$

$$= \frac{2x\sec^3(2x)\left[3x\tan(2x) - 1\right]}{x^4}$$

$$= \frac{2\sec^3(2x)\left[3x\tan(2x) - 1\right]}{x^3}$$

24. $f(x) = (2x+3)e^{x^2}$

$$f'(x) = 2 \cdot e^{x^2} + (2x+3)e^{x^2} \cdot 2x$$

$$= 2e^{x^2} + 4x^2e^{x^2} + 6xe^{x^2}$$

$$= 2e^{x^2}(2x^2+3x+1)$$

$$= 2e^{x^2}(2x+1)(x+1)$$

25. $y = \frac{1}{4}\arctan\left(\frac{x}{4}\right) = \frac{1}{4}\arctan\left(\frac{1}{4}x\right)$

$$\frac{dy}{dx} = \frac{1}{4} \cdot \frac{1}{1+\left(\frac{1}{4}x\right)^2} \cdot \frac{1}{4}$$

$$= \frac{1}{16} \cdot \frac{1}{1+\frac{x^2}{16}}$$

$$= \frac{1}{16} \cdot \frac{1}{\frac{16+x^2}{16}}$$

$$= \frac{1}{16} \cdot \frac{16}{16+x^2}$$

$$= \frac{1}{16+x^2}$$

26. $g(x) = \sin\left(\frac{2x+1}{x-3}\right)$

$$g'(x) = \cos\left(\frac{2x+1}{x-3}\right) \cdot \left[\frac{(x-3)(2) - (2x+1)(1)}{(x-3)^2}\right]$$

$$= \cos\left(\frac{2x+1}{x-3}\right)\left[\frac{2x-6-2x-1}{(x-3)^2}\right]$$

$$= \cos\left(\frac{2x+1}{x-3}\right)\left[\frac{-7}{(x-3)^2}\right]$$

19. $p(x) = \frac{x}{\sqrt{x^2+1}} = \boxed{\frac{x}{(x^2+1)^{1/2}}} = x(x^2+1)^{-1/2}$

$$p'(x) = \frac{(x^2+1)^{1/2} \cdot 1 - x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{\left((x^2+1)^{1/2}\right)^2}$$

$$= \frac{(x^2+1)^{1/2} - x^2(x^2+1)^{-1/2}}{x^2+1}$$

$$= \frac{(x^2+1)^{-1/2}\left[x^2+1-x^2\right]}{(x^2+1)^1}$$

$$p'(x) = \frac{1}{(x^2+1)^{3/2}}$$

$p(0) = 0$

$p'(0) = \frac{1}{(0^2+1)^{3/2}} = 1$

Tangent: $y = x$

Normal: $y = -x$

20. $g(x) = x^2 \cos x$

$$g'(x) = 2x\cos x + x^2 \cdot -\sin x$$

$$= 2x\cos x - x^2 \sin x$$

$$= x(2\cos x - x\sin x)$$

$g(\pi) = \pi^2\cos(\pi) = -\pi^2$

$g'(\pi) = \pi(2\cos(\pi) - \pi\sin(\pi)) = -2\pi$

Tangent: $y + \pi^2 = -2\pi(x - \pi)$

Normal: $y + \pi^2 = \frac{1}{2\pi}(x - \pi)$

21. $y = x^2 + \ln(4x-7)$

$\frac{dy}{dx} = 2x + \frac{1}{4x-7} \cdot 4$

$= 2x + \frac{4}{4x-7}$

$\left.\frac{dy}{dx}\right|_{(2,4)} = 2(2) + \frac{4}{4(2)-7} = 8$

Tangent: $y-4 = 8(x-2)$

Normal: $y-4 = -\frac{1}{8}(x-2)$

22. $f(x) = x^3 - 2x + 3$

$f'(x) = 3x^2 - 2$

$f'(2) = 3(2)^2 - 2$
$= 10$

$f(2) = 2^3 - 2(2) + 3$
$= 7$

$L(x) = 7 + 10(x-2)$

$L(1.9) = 7 + 10(1.9 - 2)$
$= 7 + 10(-0.1)$
$= 6$

23. $f(x) = x + \frac{1}{x} = x + x^{-1}$

$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

$f'(1) = 0$

$f(1) = 1 + \frac{1}{1}$
$= 2$

$L(x) = 2 + 0(x-1) = 2$

$L(1.11) = 2$

24. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f'(\pi) = \sec^2(\pi)$
$= 1$

$f(\pi) = \tan(\pi)$
$= 0$

$L(x) = 0 + 1(x - \pi) = x - \pi$

$L(3.1) = 3.1 - \pi$
$= -0.042$

25. $f(x) = \frac{x^4}{12} + \frac{x^3}{6} - 3x^2 - 2x + 4 = \frac{1}{12}x^4 + \frac{1}{6}x^3 - 3x^2 - 2x + 4$

$f'(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x - 2$

$f''(x) = x^2 + x - 6$

26. $y = x^2 \ln x$

$\frac{dy}{dx} = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$

$= 2x \ln x + x$

$\frac{d^2y}{dx^2} = 2 \ln x + 2x \cdot \frac{1}{x} + 1$

$= 2\ln x + 3$

27. $f(x) = e^x \sin x$

$f'(x) = e^x \sin x + e^x \cos x$

$= e^x(\sin x + \cos x)$

$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$= e^x(\sin x + \cos x + \cos x - \sin x)$

$= 2e^x \cos x$

28. $2x^3 = (3xy+1)^2$

$$6x^2 = 2(3xy+1)\left[3y + 3x\frac{dy}{dx}\right]$$

$$6x^2 = 6y(3xy+1) + 6x(3xy+1)\frac{dy}{dx}$$

$$6x^2 - 6y(3xy+1) = 6x(3xy+1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2 - 6y(3xy+1)}{6x(3xy+1)}$$

$$= \frac{6(x^2 - y(3xy+1))}{6x(3xy+1)}$$

$$= \frac{x^2 - y(3xy+1)}{x(3xy+1)} \quad \text{OR} \quad = \frac{x}{3xy+1} - \frac{y}{x}$$

29. $\sin(2x^2y^3) = 3x^3 + 1$

$$\cos(2x^2y^3)\left[4xy^3 + 2x^2 \cdot 3y^2 \cdot \frac{dy}{dx}\right] = 9x^2$$

$$4xy^3 + 6x^2y^2\frac{dy}{dx} = \frac{9x^2}{\cos(2x^2y^3)}$$

$$6x^2y^2\frac{dy}{dx} = \frac{9x^2}{\cos(2x^2y^3)} - 4xy^3$$

$$\frac{dy}{dx} = \frac{9x^2}{6x^2y^2\cos(2x^2y^3)} - \frac{4xy^3}{6x^2y^2}$$

$$= \frac{3}{2y^2\cos(2x^2y^3)} - \frac{2y}{3x}$$

$$= \frac{9x - 4y^3\cos(2x^2y^3)}{6xy^2\cos(2x^2y^3)}$$

30. $3x^2+3 = \ln(5xy^2)$

$$6x = \frac{1}{5xy^2}\left[5y^2 + 5x\cdot 2y\cdot\frac{dy}{dx}\right]$$

$$6x = \frac{5y^2}{5xy^2} + \frac{10xy}{5xy^2}\frac{dy}{dx}$$

$$6x = \frac{1}{x} + \frac{2}{y}\frac{dy}{dx}$$

$$6x - \frac{1}{x} = \frac{2}{y}\cdot\frac{dy}{dx}$$

$$3xy - \frac{y}{2x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2y - y}{2x}$$

$$= \frac{y(6x^2-1)}{2x}$$

31. $\lim\limits_{x\to 1} \dfrac{1-1/x}{1-1/x^2} = \lim\limits_{x\to 1} \dfrac{1-x^{-1}}{1-x^{-2}}$ Indeterminant Form : $\frac{0}{0}$

L'Hops : $\lim\limits_{x\to 1} \dfrac{x^{-2}}{2x^{-3}}$

$$\lim_{x\to 1} \frac{x^3}{2x^2}$$

$$\lim_{x\to 1} \frac{x}{2} = \frac{1}{2}$$

32. $\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right)$ Indeterminant Form: $\infty - \infty$

$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$

$\lim_{x\to 0} \frac{x - \sin x}{x \sin x}$

L'Hops: $\lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}$ Still indeterminant: $\frac{0}{0}$

L'Hops: $\lim_{x\to 0} \frac{\sin x}{\cos x + \cos x + x(-\sin x)}$

$\lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{0+2} = 0$

33. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$ Indeterminant Form: 1^{∞}

$y = \lim_{x\to\infty}\left(1+x^{-1}\right)^{3x}$

$\ln y = \lim_{x\to\infty} 3x \ln(1+x^{-1})$

$= \lim_{x\to\infty} \frac{3\ln(1+x^{-1})}{x^{-1}}$

$= \lim_{x\to\infty} \frac{3\cdot\frac{1}{1+x^{-1}}\cdot -x^{-2}}{-x^{-2}}$ ← L'Hops

$= \lim_{x\to\infty} \frac{3}{1+1/x}$

$\ln y = 3$

$y = e^3$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = e^3$