Sequences and Series :

1. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n} \rightarrow \sum_{n=1}^{\infty} 3\left(\frac{3}{5}\right)^n \rightarrow \sum_{n=0}^{\infty} \frac{9}{5}\left(\frac{3}{5}\right)^n$ $\boxed{C}$

$S = \frac{\frac{9}{5}}{1-\frac{3}{5}} = \frac{9}{\cancel{5}} \cdot \frac{\cancel{5}}{2}$

Geometric Series : $a = \frac{9}{5}$ $r = \frac{3}{5}$

2. $f(x) = \sum_{n=1}^{\infty} (\tan x)^n$ $\boxed{D}$

$f(1) = \sum_{n=1}^{\infty} (\tan 1)^n$

$\lim_{n\to\infty} (\tan 1)^n = \infty$ since $\tan 1 > 1$

3. $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \frac{2}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \frac{1}{n-1} + \frac{-1}{n+1}$ → Telescoping Series $\boxed{D}$

$\frac{2}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}$

$2 = (A+B)n + (B-A)$

$A+B=0$ $B-A=2$

$A=-1$ $B=1$

$S_2 = 1 - \frac{1}{3}$

$S_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

$S_4 = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}$

$S_5 = 1 + \frac{1}{2} - \cancel{\frac{1}{4}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$

$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1 + \frac{1}{2} = \frac{3}{2}$

4. $\frac{2}{21} + \frac{4}{63} + \frac{8}{189} + \cdots = \sum_{n=1}^{\infty} \frac{1}{7}\left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2}{21}\left(\frac{2}{3}\right)^n$ $\boxed{B}$

$r = \frac{\frac{4}{63}}{\frac{2}{21}} = \frac{\frac{8}{189}}{\frac{4}{63}} = \frac{2}{3}$ $a_1 = \frac{\frac{2}{21}}{\frac{2}{3}} = \frac{1}{7}$

$S = \frac{\frac{2}{21}}{1-\frac{2}{3}} = \frac{2}{\cancel{21}_{7}} \cdot \frac{\cancel{3}^{1}}{1} = \frac{2}{7}$

5. $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{3^{n-1}}{(4+n)^{20}}\right)\left(\frac{(7+n)^{20}}{3^n}\right)$ $\boxed{A}$

$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{(7+n)^{20}}{(4+n)^{20}}\right) = \lim_{n\to\infty} \frac{1}{3}\left(\frac{7+n}{4+n}\right)^{20} = \frac{1}{3}(1) = \frac{1}{3}$

6. I. $\left\{ \dfrac{(\cos n)^2}{(1.1)^n} \right\}$ $\quad \lim\limits_{n\to\infty} \dfrac{(\cos n)^2}{(1.1)^n} = \lim\limits_{n\to\infty} \dfrac{(\cos n)^2 \cdot 10^n}{11^n} = 0$ ✓

Since $|\cos n| \le 1$,
$\cos^2 n \le 1$.

$\boxed{D}$

II. $\left\{ \dfrac{e^n - 3}{3^n} \right\}$ $\quad \lim\limits_{n\to\infty} \dfrac{e^n - 3}{3^n} = \lim\limits_{n\to\infty} \left(\dfrac{e}{3}\right)^n - 3\left(\dfrac{1}{3}\right)^n = 0$ ✓

Convergent Geometric $\quad$ Convergent Geometric

III. $\left\{ \dfrac{n}{9+\sqrt{n}} \right\}$ $\quad \lim\limits_{n\to\infty} \dfrac{n}{9+\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1}{\frac{1}{2\sqrt{n}}} = \lim\limits_{n\to\infty} 2\sqrt{n} = \infty$ X

$\frac{\infty}{\infty}$ Apply L'Hopitals

7. I. $\sum\limits_{n=1}^{\infty} \dfrac{n}{10(n+1)}$ $\quad \lim\limits_{n\to\infty} \dfrac{n}{10n+10} = \dfrac{1}{10} \to$ Diverges by n^{th} term test

$\boxed{C}$

II. $\sum\limits_{n=1}^{\infty} \arctan n$ $\quad \lim\limits_{n\to\infty} \arctan n = \dfrac{\pi}{2} \to$ Diverges by n^{th} term test

III. $\sum\limits_{n=1}^{\infty} \dfrac{-6}{(-5)^n} = \sum\limits_{n=1}^{\infty} (-6)\left(-\dfrac{1}{5}\right)^n \longrightarrow$ Convergent Geometric

8. $\sum\limits_{n=1}^{\infty} \left(\dfrac{3}{n(n+3)} + \dfrac{1}{7^n} \right) = \sum\limits_{n=1}^{\infty} \dfrac{3}{n(n+3)} + \sum\limits_{n=1}^{\infty} \left(\dfrac{1}{7}\right)^n = \underbrace{\sum\limits_{n=1}^{\infty} \dfrac{1}{n} - \dfrac{1}{n+3}}_{\text{[Telescoping]}} + \underbrace{\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{7}\right)^n}_{\text{[GEOMETRIC]}}$

$\dfrac{3}{n(n+3)} = \dfrac{A}{n} + \dfrac{B}{n+3}$

$3 = (A+B)n + 3A$

$A + B = 0 \qquad 3A = 3$
$B = -1 \qquad A = 1$

$S_1 = 1 - \frac{1}{4}$
$S_2 = 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5}$
$S_3 = 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6}$
$S_4 = 1 - \cancel{\frac{1}{4}} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \cancel{\frac{1}{4}} - \frac{1}{7}$
$S_5 = 1 + \frac{1}{2} - \cancel{\frac{1}{5}} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} + \cancel{\frac{1}{5}} - \frac{1}{8}$
$S_6 = 1 + \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{6}} - \frac{1}{7} - \frac{1}{8} + \cancel{\frac{1}{6}} - \frac{1}{9}$
$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} \Rightarrow \lim\limits_{n\to\infty} S_n = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$S = \dfrac{\frac{1}{7}}{1 - \frac{1}{7}} = \dfrac{1}{7} \cdot \dfrac{7}{6} = \dfrac{1}{6}$

$\text{Sum} = \dfrac{11}{6} + \dfrac{1}{6} = 2$

Integral Test & P-Series :

1. If $\int_1^\infty \frac{dx}{x^2+1}$ converges to $\pi/4$, the series $\sum \frac{1}{n^2+1}$ also converges but is not necessarily equal to the value the integral converges to.

$\boxed{C}$

2. $\int_1^\infty \frac{1}{\sqrt[3]{x^p}}\,dx = \int_1^\infty \frac{1}{x^{p/3}}\,dx$

$\boxed{D}$

Converges $\Rightarrow$ $p/3 > 1$
$p > 3$

3. I. $\sum_{n=1}^\infty \frac{n}{2n^2+1}$ $\quad a_n = \frac{n}{2n^2+1} \quad b_n = \frac{n}{n^2} = \frac{1}{n} \rightarrow$ divergent harmonic

$\boxed{B}$

$\lim_{n\to\infty} \left| \frac{n}{2n^2+1} \cdot \frac{n^2}{n} \right| = \frac{1}{2} \rightarrow$ FINITE & POSITIVE, $\therefore$ a_n & b_n both diverge

II. $\sum_{n=1}^\infty n e^{-n^2} = \sum_{n=1}^\infty \frac{n}{e^{n^2}}$

$\int_1^\infty x e^{-x^2}\,dx = \lim_{b\to\infty} \int -\frac{1}{2} e^u\,du = \lim_{b\to\infty} -\frac{1}{2}\left[e^{-x^2}\right]_1^b$

$u = -x^2$
$du = -2x\,dx$

$= \lim_{b\to\infty} -\frac{1}{2}\left[e^{-b^2} - e^{-1}\right]$

$= \lim_{b\to\infty} -\frac{1}{2}\left[\frac{1}{e^{b^2}} - \frac{1}{e}\right] = \frac{1}{2e}$

By integral test, $\sum$ also converges.

III. $\sum_{n=2}^\infty \frac{1}{x\ln x}$

$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int \frac{1}{u}\,du = \lim_{b\to\infty} \left[\ln|\ln x|\right]_2^b$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \left[\ln|\ln b| - \ln|\ln 2|\right] = \infty$

By integral test, $\sum$ also diverges.

4. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^p+1}$ $\quad a_n = \frac{n^{1/2}}{n^p+1}$ $\quad b_n = \frac{n^{1/2}}{n^p} = \frac{1}{n^{p-1/2}}$

$\boxed{D}$

For b_n to converge (meaning a_n also converges)

$$p - \tfrac{1}{2} > 1 \Rightarrow p > \tfrac{3}{2}$$

5. $1 + (\sqrt{2})^k + (\sqrt{3})^k + (\sqrt{4})^k = \sum_{n=1}^{\infty} (\sqrt{n})^k = \sum_{n=1}^{\infty} n^{k/2} = \sum_{n=1}^{\infty} \frac{1}{n^{-k/2}}$

$\boxed{A}$

Converges $\Rightarrow \frac{-k}{2} > 1 \Rightarrow k < -2$

6. (a) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^3} \rightarrow$ Convergent p-series since $p > 1$

(b) $1 + \frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} + \frac{1}{4^{2/3}} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \rightarrow$ Divergent p-series since $p < 1$

Comparison Tests :

1\. I. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+3}$ $a_n = \frac{1}{n^2+n+3}$ $b_n = \frac{1}{n^2}$ → convergent p-series

$\boxed{D}$

$$\frac{1}{n^2+n+3} < \frac{1}{n^2} \Rightarrow n^2 < n^2+n+3$$

$$0 < n+3$$

$$-3 < n$$

Since $b_n > a_n$ for $n > -3$, and b_n converges, a_n must also converge by direct comparison.

II. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+2}$ $\frac{\cos^2 n}{n^2+2} \le \frac{1}{n^2+2} < \frac{1}{n^2}$ → convergent p-series

$-1 \le \cos n \le 1$

$0 \le \cos^2 n \le 1$

$$\frac{1}{n^2+2} < \frac{1}{n^2} \Rightarrow n^2 < n^2+2$$

$$0 < 2$$

Since $\frac{1}{n^2} > \frac{1}{n^2+2} \ge \frac{\cos^2 n}{n^2+2}$ and $\frac{1}{n^2}$ converges, both smaller series must also converge.

III. $\sum_{n=1}^{\infty} \frac{1+4^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \left(\frac{4}{3}\right)^n$ → The sum of two geometric series. $\left(\frac{1}{3}\right)^n$ converges, but since $\left(\frac{4}{3}\right)^n$ diverges, the overall series diverges.

2\. I. $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\frac{1}{n!} \le \frac{1}{n^2}$ for $n \ge 4$ $\sum \frac{1}{n^2}$ → Convergent p-series

$\therefore \sum \frac{1}{n!}$ converges by direct comparison

$\boxed{C}$

II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$ $a_n = \frac{1}{\sqrt{n}+2}$ $b_n = \frac{1}{\sqrt{n}}$ → divergent p-series

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}+2} \cdot \frac{\sqrt{n}}{1} \right) = 1$$ → finite & positive, so both series diverge

III. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $a_n = \sin\left(\frac{1}{n}\right)$ $b_n = \frac{1}{n}$ → Divergent Harmonic

$$\lim_{n\to\infty} \left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \right) = \lim_{t\to 0} \underbrace{\left(\frac{\sin(t)}{t} \right)}_{\text{Special Trig limit}} = 1$$ → finite & positive, so a_n & b_n both diverge.

$t = \frac{1}{n}$ $\lim_{n\to\infty} t = 0$

3. I. $\sum \frac{n^{3/2}}{3n^3+7}$ $\quad a_n = \frac{n^{3/2}}{3n^3+7}$ $\quad b_n = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3/2}}$ → convergent p-series

$$\lim_{n\to\infty}\left(\frac{n^{3/2}}{3n^3+7}\cdot\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty}\left(\frac{n^3}{3n^3+7}\right) = \frac{1}{3}$$ → finite & positive, so both converge

$\boxed{D}$

II. $\sum \frac{1}{\sqrt[3]{n^4+1}}$ $\quad a_n = \frac{1}{\sqrt[3]{n^4+1}}$ $\quad b_n = \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{4/3}}$ → convergent p-series

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[3]{n^4+1}}\cdot\frac{\sqrt[3]{n^4}}{1}\right) = \lim_{n\to\infty}\left(\sqrt[3]{\frac{n^4}{n^4+1}}\right) = 1$$ → both converge

III. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\quad \frac{n!}{n^n} < \frac{1}{n^2}$ for $n \ge 5$

$\frac{1}{n^2}$ → convergent p-series, so $\frac{n!}{n^n}$ must also converge

4. $$\lim_{n\to\infty}\left(\frac{1}{2^n-1}\cdot\frac{2^n}{1}\right) = 1$$

$\boxed{B}$

$$\lim_{n\to\infty}\left(\frac{n}{2^n}\cdot\frac{2^n}{1}\right) = \infty$$

$$\lim_{n\to\infty}\left(\frac{2n}{2^{n+1}\sqrt{n^2+1}}\cdot\frac{2^n}{1}\right) = \lim_{n\to\infty}\left(\frac{2n}{2\sqrt{n^2+1}}\right) = 1$$

$$\lim_{n\to\infty}\left(\frac{2n^2-3n}{2^n(n^2+n-100)}\cdot\frac{2^n}{1}\right) = \lim_{n\to\infty}\left(\frac{2n^2-3n}{n^2+n-100}\right) = 2$$

5. (a) $\sum_{n=1}^{\infty} \frac{\cos(2n)}{1+(1.6)^n}$ $\quad$ Since $|\cos(2n)| \le 1$

$$\frac{\cos(2n)}{1+(1.6)^n} \le \frac{1}{1+(1.6)^n} < \frac{1}{(1.6)^n}$$

$$\frac{1}{(1.6)^n} = \left(\frac{1}{1.6}\right)^n = \left(\frac{5}{8}\right)^n$$ → Convergent Geometric Series

$\therefore \sum_{n=1}^{\infty} \frac{\cos(2n)}{1+(1.6)^n}$ converges by direct comparison

5. (b) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$

$\displaystyle\lim_{n\to\infty} \frac{4^n}{2^n+3^n} = \infty \rightarrow$ Diverges by n^{th} term test

NOTE: You can also take a derivative, $\frac{d}{dn}\left[\frac{4^n}{2^n+3^n}\right]$, and show the derivative is positive, therefore the terms are increasing.

Alternating Series and Error Bound:

1. I. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n}$

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0$

$\downarrow$

$\frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

$\sqrt{n} < \sqrt{n+1}$

$n < n+1$

$0 < 1$

Convergent by Alternating Series Test

$\boxed{D}$

II. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

$\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$

$\frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$

$\ln(n) < \ln(n+1)$

$\ln(n) - \ln(n+1) < 0$

$\ln\left(\frac{n}{n+1}\right) < 0$

$\frac{n}{n+1} < e \rightarrow$ True for all n

Convergent by Alternating Series Test

III. $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi) = -1+1-1+1-1 \ldots$

$\searrow \displaystyle\lim_{n\to\infty} \cos(n\pi) \neq 0$ Diverges by n^{th} term test

2. I. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = 1 \neq 0 \rightarrow$ Diverges by n^{th} term test

II. $\sum_{n=1}^{\infty} \sin\left(\frac{2n-1}{2}\right)\pi$ $\lim_{n\to\infty} \sin\left(\frac{2n-1}{2}\right)\pi$

Since $\lim_{n\to\infty} \frac{2n-1}{2} = \infty$, the series oscillates forever $\rightarrow$ Diverges

$\boxed{C}$

III. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n^2+1}$ $\lim_{n\to\infty} \frac{2n}{n^2+1} = 0$

$$\frac{d}{dn}\left[\frac{2n}{n^2+1}\right] = \frac{(n^2+1)2 - 2n(2n)}{(n^2+1)^2}$$
$$= \frac{2n^2+2-4n^2}{(n^2+1)^2}$$
$$= \frac{2-2n^2}{(n^2+1)^2}$$

$\rightarrow$ Derivative is negative for $n > 1$, therefore terms are getting smaller

Converges by alt series test

3. $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{\sqrt{n}}$

$\downarrow$

k must be odd in order for this to converge by alt series test

$\sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^k+1} \rightarrow a_n = \frac{n^2\sqrt{n}}{n^k+1}$ $b_n = \frac{n^{5/2}}{n^k} = \frac{1}{n^{k-5/2}}$

$\boxed{C}$

$k - 5/2 > 1$

$k > 7/2$

$\downarrow$

For this to converge by limit comparison to a p-series.

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 1 - \frac{1}{8} + \frac{1}{27} - \dots$

$\boxed{B}$

alternating series remainder is less than first omitted term

$\frac{1}{n^3} < \frac{1}{500}$

$n^3 > 500$

$n > 7.937$

$n = 8 \rightarrow a_8 = \frac{1}{512}$

$|S - S_7| < a_8 = \frac{1}{512}$

5. I. $\sum_{n=2}^{\infty} (-1)^n \sqrt[n]{3}$ $\lim_{n\to\infty} 3^{\frac{1}{n}} \neq 0$ $\therefore$ Diverges by n^{th} term test.

$y = \lim_{n\to\infty} 3^{\frac{1}{n}}$

$\ln y = \lim_{n\to\infty} \frac{1}{n} \cdot \ln 3$

$\ln y = 0 \Rightarrow y = e^0 = 1$

$\boxed{D}$

II. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{\pi^n} = \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{\pi}\right)^n \rightarrow$ Convergent Geometric

III. $\sum_{n=1}^{\infty} \left(\arctan(n+1) - \arctan(n)\right)$

$\lim_{n\to\infty} \arctan(n+1) - \arctan(n)$

$\frac{\pi}{2} - \frac{\pi}{2} = 0$

$\frac{d}{dn}\left[\arctan(n+1) - \arctan(n)\right]$

$\frac{1}{1+(n+1)^2} - \frac{1}{1+n^2}$

$\frac{(1+n^2) - (1+(n+1)^2)}{(1+n^2)(1+(n+1)^2)}$

$\frac{-(2n+1)}{(1+n^2)(1+(n+1)^2)} \rightarrow$ Always negative for $n \geq 1$

Converges by Alt series test.

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n^2}$

$\sum_{n=1}^{\infty} \frac{n+3}{n^2}$ $a_n = \frac{n+3}{n^2}$ $b_n = \frac{n}{n^2} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{n+3}{n^2} \cdot \frac{n}{1} = 1$

$\downarrow$

Diverges by limit comparison

$\therefore$ Not absolutely convergent

$\lim_{n\to\infty} \frac{n+3}{n^2} = 0$

$\frac{n+4}{(n+1)^2} < \frac{n+3}{n^2}$

$n^2(n+4) < (n+3)(n+1)^2$

$n^3 + 4n^2 < n^3 + 5n^2 + 7n + 3$

$4n^2 < 5n^2 + 7n + 3$

$0 < n^2 + 7n + 3$

$\downarrow$

True for $n \geq 1$

$\boxed{A}$

7. A $\rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow$ Divergent p-series

B $\rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \rightarrow$ convergent p-series $\Rightarrow$ Since $\sum |a_n|$ converges, $\sum a_n$ also converges

$\boxed{B}$

C $\rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2 - \sqrt{n}} \rightarrow$ Diverges by limit comparison to $\frac{1}{n}$

D $\rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2+1}{n^3} \rightarrow$ Diverges by limit comparison to $\frac{1}{n}$

8. $|s - s_3| \leq m \rightarrow \underline{m = \text{first omitted term}}$

$\boxed{C}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+3} = \underbrace{\frac{1}{4} - \frac{1}{7} + \frac{1}{12}}_{S_3} - \frac{1}{19} + \dots$$

$|S - S_3| \leq \frac{1}{19}$

9. $1 - \frac{3}{2!} + \frac{9}{4!} = S_3$ $\qquad |S - S_3| \leq |a_4| = \frac{27}{6!} = \frac{\cancel{27}\,\cancel{9}\,3}{\cancel{6}\cdot 5\cdot 4\cdot \cancel{3}\cdot 2\cdot 1} = \frac{3}{80} < \frac{4}{80} = \frac{1}{20}$

<u>Ratio Test</u>

1. I. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ $\qquad \lim_{n\to\infty} \left| \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2} \right| = \infty$

$\boxed{C}$

II. $\sum_{n=1}^{\infty} \frac{n}{3^n}$ $\qquad \lim_{n\to\infty} \left| \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{3n} \right| = \frac{1}{3}$

III. $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ $\qquad \lim_{n\to\infty} \left| \frac{(n+1)\cdot 2^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n\cdot 2^n} \right| = \lim_{n\to\infty} \left| \frac{2(n+1)}{3n} \right| = \frac{2}{3}$

2\. I. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cdot n^n}{(n+1)^n \cdot (n+1)} \right|$

$$= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right| = e^{-1}$$

Let $y = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$\ln y = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} n \cdot \ln\left(\frac{n}{n+1}\right)$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{n^{-1}}$$

L'Hopitals $= \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)\left(\frac{n+1-n}{(n+1)^2}\right)}{-n^{-2}}$

$$= \lim_{n\to\infty} -\frac{(n+1)(n^2)}{n(n+1)^2}$$

$$= -1$$

$\ln y = -1 \Rightarrow y = e^{-1}$

$\boxed{D}$

II. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ $\lim_{n\to\infty} \left| \left[\frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \right] \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{n!\cdot n!} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| = \frac{1}{4}$$

III. $\sum_{n=1}^{\infty} \frac{n^9}{9^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)^9}{9^{n+1}} \cdot \frac{9^n}{n^9} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^9}{9n^9} \right| = \frac{1}{9}$

3\. (a) $\sum_{n=1}^{\infty} \frac{n!}{n\cdot 2^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)\cdot 2^{n+1}} \cdot \frac{n\cdot 2^n}{n!} \right|$

$\lim_{n\to\infty} \left| \frac{n}{2} \right| = \infty$ $\sum_{n=1}^{\infty} \frac{n!}{n\cdot 2^n}$ Diverges

3. (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(\cos x)^n}{2^n}$

$$\lim_{n\to\infty}\left|\frac{(\cos x)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(\cos x)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cos x}{2}\right| = \left|\frac{\cos x}{2}\right|$$

Since $|\cos x| \le 1$, $\left|\frac{\cos x}{2}\right| \le \frac{1}{2}$.

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{(\cos x)^n}{2^n}$ converges

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k \cdot k!}{(k+3)!}$

$$\lim_{n\to\infty}\left|\frac{3^{k+1}\cdot(k+1)!}{(k+4)!} \cdot \frac{(k+3)!}{3^k \cdot k!}\right|$$

$$\lim_{n\to\infty}\left|\frac{3(k+1)}{(k+4)}\right| = 3$$

$\displaystyle\sum_{k=1}^{\infty} \frac{3^k \cdot k!}{(k+3)!}$ Diverges

Power Series Convergence:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^3}$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{x^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^3\right| \cdot |x| = |x|$$

$\boxed{B}$

$|x| < 1 \Rightarrow -1 < x < 1 \Rightarrow -1 \le x \le 1$

$\boxed{x=-1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{-1}{n^3} \rightarrow$ convergent p-series

$\boxed{x=1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \rightarrow$ Absolutely convergent since $\sum |a_n|$ is a convergent p-series.

2. $\displaystyle\sum_{n=0}^{\infty} \frac{n(x-2)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{(n+1)(x-2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n(x-2)^n}\right| = \left|\frac{x-2}{3}\right| \cdot \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = \left|\frac{x-2}{3}\right|$$

$\boxed{A}$

$\left|\frac{x-2}{3}\right| < 1 \Rightarrow |x-2| < 3 \Rightarrow -1 < x < 5 \Rightarrow -1 < x < 5$

$\boxed{x=-1}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n \rightarrow$ Diverges by n^{th} term test

$\boxed{x=5}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(3)^n}{3^n} = \sum_{n=0}^{\infty} n$

3. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$ $\lim_{n\to\infty} \left| \frac{x^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{x^{n+1}} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{1}{n+2} \right| = 0$

Always < 1, so radius of convergence is ∞.

$-\infty < x < \infty$

$\boxed{D}$

4. $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{2^n \sqrt{n}}$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1} \cdot \sqrt{n+1}} \cdot \frac{2^n \sqrt{n}}{x^n} \right| = \lim_{n\to\infty} \left| \sqrt{\frac{n}{n+1}} \right| \cdot \left| \frac{x}{2} \right| = \left| \frac{x}{2} \right|$

$\boxed{C}$

$\left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2 \Rightarrow -2 < x < 2 \Rightarrow -2 < x \leq 2$

$\boxed{x=-2}$ $\sum_{n=2}^{\infty} \frac{(-1)^n (-2)^n}{2^n \sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \rightarrow$ Divergent p-series

$\boxed{x=2}$ $\sum_{n=2}^{\infty} \frac{(-1)^n (2)^n}{2^n \sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \rightarrow$ Conditionally convergent by Alt series test.

5. $\sum_{n=1}^{\infty} n!(3x-2)^n$ $\lim_{n\to\infty} \left| \frac{(n+1)! \cdot (3x-2)^{n+1}}{n! \cdot (3x-2)^n} \right| = |3x-2| \cdot \lim_{n\to\infty} |n+1| = \infty$

Only convergent at the center

$|3x-2| = 3\left|x - \frac{2}{3}\right|$

$\boxed{C}$

6. $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^{n+1}$

$\lim_{n\to\infty} \left| \frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2) \cdot x^{n+2}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot x^{n+1}} \right|$

$\lim_{n\to\infty} \left| \frac{2n+2}{2n+1} \right| \cdot |x| = |x|$

$|x| < 1 \Rightarrow$ Radius of Convergence $= 1$

$-1 < x < 1$

$\boxed{x=-1}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Diverges by n^{th} term test

$\boxed{x=1}$ $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Diverges by n^{th} term test

$\Longrightarrow$ Interval of Convergence: $-1 < x < 1$

Representations of Functions as Power Series :

1. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \ldots$ $\boxed{B}$

$\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \ldots$

$\dfrac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + x^{12} \ldots$

2. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\boxed{C}$

$\dfrac{1}{2-x} = \dfrac{1/2}{1 - x/2} \Rightarrow \begin{matrix} a = 1/2 \\ r = x/2 \end{matrix} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \ldots$

3. $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{(x-2)^n}{n!} = (x-2) - \dfrac{(x-2)^2}{2!} + \dfrac{(x-2)^3}{3!} - \dfrac{(x-2)^4}{4!}$ $\boxed{D}$

$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \dfrac{n(x-2)^{n-1}}{n!} = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{(x-2)^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (-1)^{n+2} \cdot \dfrac{(x-2)^n}{n!}$

$\downarrow$
Same as $(-1)^n$

4. (a) $g(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \ldots + (-1)^n x^n + \ldots$

(b) $g(x^2) = 1 - x^2 + x^4 - x^6 + x^8 - \ldots + (-1)^n x^{2n} + \ldots$

(c) $h(x) = \int g(x^2)\, dx = \int \left(1 - x^2 + x^4 - x^6 + x^8 - \ldots + (-1)^n x^{2n}\right) dx$

$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \ldots + (-1)^n \dfrac{x^{2n+1}}{2n+1} + \ldots$

(d) $h(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ $\qquad h(1) = \sum_{n=0}^{\infty} (-1)^n \cdot \dfrac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots (-1)^n \cdot \dfrac{1}{2n+1}$

$\downarrow$
power series of arctan x

$h(1) = \arctan(1) = \pi/4$

5. (a) $f'(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2n+1)\cdot 2n\, x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)x^{2n-1}}{(2n-1)!} = -3x + \frac{5x^3}{3!} - \frac{7x^5}{5!} + \ldots + (-1)^n \frac{(2n+1)x^{2n-1}}{(2n-1)!}$

$f'(0) = 0$

$f''(x) = \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)(2n-1)\cdot x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)\cdot x^{2n-2}}{(2n-2)!} = -3 + \frac{5x^2}{2!} - \frac{7x^4}{4!} + \ldots + (-1)^n \frac{(2n+1)x^{2n-2}}{(2n-2)!}$

$f''(0) = -3$

f has a local maximum at $x=0$ since $f'(0)=0$ and $f''(0)<0$.

(b) $s_3 = 1 - \frac{3}{2!} + \frac{5}{4!}$

$|s - s_3| \le |a_4| = \frac{7}{6!} = \frac{7}{720} < \frac{7.2}{720} = \frac{1}{100}$

(c) $g(x) = \int_0^x f(t)\,dt = \int_0^x \left[1 - \frac{3t^2}{2!} + \frac{5t^4}{4!} - \frac{7t^6}{6!} + \ldots + (-1)^n \frac{(2n+1)t^{2n}}{(2n)!} \right] dt$

$= \left[t - \frac{3t^3}{3\cdot 2!} + \frac{5t^5}{5\cdot 4!} - \frac{7t^7}{7\cdot 6!} + \ldots + (-1)^n \frac{(2n+1)\,t^{2n+1}}{(2n+1)(2n)!} \right]_0^x$

$= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n)!}$

$\frac{g(x)}{x} = \frac{x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n)!}}{x}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!}$

Taylor Polynomial and Lagrange Error Bound :

1. $f^{(4)}(0) \cdot \dfrac{x^4}{4!} = \dfrac{2}{9} \cdot x^4$ $\boxed{D}$

$f^{(4)}(0) = \dfrac{2}{9} \cdot 4! = \dfrac{16}{3}$

2. $f'''(0) \cdot \dfrac{x^3}{3!} = 0 \Rightarrow f'''(0) = 0$ $\boxed{C}$

3. $P_3(x) = f(1) + f'(1)(x-1) + f''(1) \cdot \dfrac{(x-1)^2}{2!} + f'''(1) \cdot \dfrac{(x-1)^3}{3!}$ $\boxed{A}$

$P_3(x) = 2 + -3(x-1) + 4\dfrac{(x-1)^2}{2!} + -9\dfrac{(x-1)^3}{3!}$

$= 2 - 3(x-1) + 2(x-1)^2 - \frac{3}{2}(x-1)^3$

4. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$ $\boxed{B}$

$xe^x = x\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}\right) = x + x^2 + \dfrac{x^3}{2!} + \dfrac{x^4}{3!}$

5. $f(x) = \sec x$ $\quad f(\pi/4) = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

$f'(x) = \sec x \tan x$ $\quad f'(\pi/4) = \sqrt{2} \cdot 1 = \sqrt{2}$ $\boxed{C}$

$f''(x) = \sec x \cdot \sec^2 x + \tan^2 x \cdot \sec x$ $\quad f''(\pi/4) = 2\sqrt{2} + \sqrt{2}$

$= 3\sqrt{2}$

$$P_2(x) = \sqrt{2} + \sqrt{2}(x - \pi/4) + \frac{3\sqrt{2}(x-\pi/4)^2}{2!}$$

6. $P_2(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2}$

$P_2(x) = \frac{1}{3} + f'(0)x + \frac{4}{3}\cdot\frac{x^2}{2} = \frac{1}{3} + f'(0)x + \frac{2}{3}x^2$

$\boxed{B}$

$P_2(\frac{1}{2}) = \frac{1}{8} = \frac{1}{3} + \frac{1}{2}f'(0) + \frac{2}{3}(\frac{1}{4})$

$\frac{1}{8} = \frac{1}{3} + \frac{1}{2}f'(0) + \frac{1}{6}$

$-\frac{3}{8} = \frac{1}{2}f'(0) \Rightarrow f'(0) = \frac{-3}{4}$

7. (a) $f(2) = 3$

$f'''(2)\cdot\frac{(x-2)^3}{3!} = -12(x-2)^3$

$f'''(2) = -12\cdot 3! = -72$

(b) $\frac{d}{dx}[P(x)] = f'(x) = 0 - 2 + 10(x-2) - 36(x-2)^2 + 12(x-2)^3$

$P_3(x) = -2 + 10(x-2) - 36(x-2)^2 + 12(x-2)^3$

$P_3(2.1) \approx f'(2.1) = -1.348$

(c) $g(x) = \int_2^x f(t)\,dt = \int_2^x \left[3 - 2(t-2) + 5(t-2)^2 - 12(t-2)^3 + 3(t-2)^4\right]dt$ $\rightarrow u = t-2$, $du = dt$

$= \int_0^{x-2}\left[3 - 2u + 5u^2 - 12u^3 + 3u^4\right]du$

$= \left[3u - u^2 + \frac{5}{3}u^3 - 3u^4 + \frac{3}{5}u^5\right]_0^{x-2}$

$= 3(x-2) - (x-2)^2 + \frac{5}{3}(x-2)^3 - 3(x-2)^4 + \frac{3}{5}(x-2)^5$

$P_4(x) = 3(x-2) - (x-2)^2 + \frac{5}{3}(x-2)^3 - 3(x-2)^4$

(d) No, $f(1)$ can't be determined since the taylor polynomial being considered is centered at $x=2$. All of the terms present provide information about f and the derivatives of f in relation to $x=2$, not $x=1$.

8. (a) $f(x) = \sin(2x) + \cos(2x)$ $f(0) = 1$

$f'(x) = 2\cos(2x) - 2\sin(2x)$ $f'(0) = 2$

$f''(x) = -4\sin(2x) - 4\cos(2x)$ $f''(0) = -4$

$f'''(x) = -8\cos(2x) + 8\sin(2x)$ $f'''(0) = -8$

$$P_3(x) = 1 + 2\cdot x + -4\cdot\frac{x^2}{2!} + -8\cdot\frac{x^3}{3!}$$

$$P_3(x) = 1 + 2x - 2x^2 - \frac{4}{3}x^3$$

(b) Derivatives come in patters every 4 derivatives

$$f^{(n)}(x) = \begin{cases} 2^n(\sin(2x)+\cos(2x)), & \text{if } n/4 \text{ has no remainder} \\ 2^n(\cos(2x)-\sin(2x)), & \text{if } n/4 \text{ has remainder of } 1 \\ (-2)^n(\sin(2x)+\cos(2x)), & \text{if } n/4 \text{ has remainder of } 2 \\ (-2)^n(\cos(2x)-\sin(2x)), & \text{if } n/4 \text{ has remainder of } 3 \end{cases}$$

For coefficient of x^{19}, we need $f^{(19)}(x)$.

$\frac{19}{4} = 4\frac{3}{4} \Rightarrow$ Remainder of 3.

$f^{(19)}(x) = (-2)^{19}(\cos(2x) - \sin(2x))$

$f^{(19)}(0) = (-2)^{19}$

Coefficient of x^{19} is $\frac{(-2)^{19}}{19!}$.

(c) $\left|f(\frac{1}{5}) - P(\frac{1}{5})\right| \le \left|\frac{x^4}{4!}\right| \cdot \left|\max f^{(4)}(z)\right|$

$\downarrow$ 3rd Degree Polynomial from (a)

$\downarrow$ $0 < z < \frac{1}{5}$

$f^{(4)}(z) = 2^4(\sin(2x) + \cos(2x)) \rightarrow$ Since maximum of both sine and cosine is 1, $f^{(4)}(z) < 2^5 = 32$

• Use 32 as max $f^{(4)}(z)$

$$\left|f(\tfrac{1}{5}) - P(\tfrac{1}{5})\right| \le \left|\frac{(\frac{1}{5})^4}{4!}\right| \cdot |32| = \frac{4}{1875} < \frac{1}{100}$$

(d) $$h(x) = \int_0^x f(t)\,dt = \int_0^x \left[1 + 2t - 2t^2 - \frac{4}{3}t^3\right]dt = \left[t + t^2 - \frac{2}{3}t^3 - \frac{1}{3}t^4\right]_0^x$$

$$= x + x^2 - \frac{2}{3}x^3 - \frac{1}{3}x^4$$

$$P_3(x) = x + x^2 - \frac{2}{3}x^3$$

Taylor and Maclaurin Series:

1. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ $\boxed{B}$

$$\frac{\arctan x}{x} = \frac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots$$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ $\boxed{A}$

$$e^{-2x} = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} + \dots \qquad \frac{-8}{3!} = \frac{-4}{3}$$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ $\boxed{D}$

$$x\sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots$$

$$x\sin x - x^2 = -\frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots$$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ $\boxed{C}$

$$x - \sin x = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\frac{x - \sin x}{x^2} = \frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} + \dots$$

5. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\boxed{C}$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$x^2 e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n!}$$

6. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \arctan x$ $\boxed{B}$

$$\arctan x = e^{-x} \Rightarrow x = 0.607$$

7. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$ $\boxed{D}$

$$\cos^2 x = \cos x \cdot \cos x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right)$$

$$= 1 + \left(\frac{-x^2}{2} + \frac{-x^2}{2}\right) + \left(\frac{x^4}{24} + \frac{x^4}{24} + \frac{x^4}{4}\right) + \dots$$

$$= 1 - x^2 + \frac{1}{3}x^4$$

8. $\tan(x^2) = x^2 + \frac{1}{3}x^6 + \frac{2}{15}x^{10}$ $\boxed{A}$

$$f(x) = \int f'(x)\,dx = \int \left(x^2 + \frac{1}{3}x^6 + \frac{2}{15}x^{10}\right)dx = \frac{1}{3}x^3 + \frac{1}{21}x^7 + \frac{2}{165}x^{11} + C$$

9. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ $\boxed{C}$

$$\cos x \cdot f(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\left(\frac{x}{2} - \frac{2x^2}{3} + x^3 + \dots\right)$$

$$= \frac{1}{2}x + \left(-\frac{2}{3}x^2\right) + \left(x^3 - \frac{1}{4}x^3\right) + \dots$$

$$= \frac{1}{2}x - \frac{2}{3}x^2 + \frac{3}{4}x^3$$

10. $f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$ $\boxed{B}$

$$f'(x) = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \dots \qquad f'(0) = 0$$

$$f''(x) = \frac{-2}{3!} + \frac{12x^2}{5!} - \frac{30x^4}{7!} + \dots \qquad f''(0) = \frac{-2}{3!} < 0$$

11. (a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^n}{n!}$

(b) $1 - x - f(x) = 1 - x - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^n}{n!}\right]$

$= 1 - x - 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + (-1)^{n+1} \frac{x^n}{n!}$

$= \frac{-x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + (-1)^{n+1} \frac{x^n}{n!}$

$g(x) = \frac{1 - x - f(x)}{x} = \frac{\frac{-x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + (-1)^{n+1} \frac{x^n}{n!}}{x}$

$= \frac{-x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} + \dots + (-1)^{n+1} \frac{x^{n-1}}{n!}$

OR

$= \frac{-x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} + \dots + (-1)^n \frac{x^n}{(n+1)!}$

(c) $g'(x) = \frac{-1}{2!} + \frac{2x}{3!} - \frac{3x^2}{4!} + \frac{4x^3}{5!} + \dots + (-1)^{n+1} \frac{(n-1)x^{n-2}}{n!} + (-1)^n \frac{n x^{n-1}}{(n+1)!}$

$g'(-1) = -\frac{1}{2!} - \frac{2}{3!} - \frac{3}{4!} - \frac{4}{5!} + \dots = \sum_{n=1}^{\infty} \frac{-n}{(n+1)!}$

$g'(x) = \frac{x\left(-1 - f'(x)\right) - \left(1 - x - f(x)\right)}{x^2} = \frac{x(-1 + e^{-x}) - (1 - x - e^{-x})}{x^2} = \frac{-1 + xe^{-x} + e^{-x}}{x^2}$

$g'(-1) = \frac{-1 - e + e}{1} = -1$

$$\sum_{n=1}^{\infty} \frac{-n}{(n+1)!} = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$$

12. (a) $f'(x) = \frac{1}{2!} - \frac{3x^2}{4!} + \frac{5x^4}{6!} + \dots$

$f''(x) = \frac{-6x}{4!} + \frac{20x^3}{6!} + \dots$

$f'''(x) = \frac{-6}{4!} + \frac{60x^2}{6!} + \dots \Rightarrow f'''(0) = \frac{-6}{4!} = -\frac{1}{4}$

For $f^{(15)}(x)$, every term will have an x variable except for the term from $f(x)$ with x^{15}.

$$f(x) = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} + \dots + \frac{-x^{15}}{16!} + \dots + (-1)^{n+1}\frac{x^{2n-1}}{(2n)!}$$

$$f^{(15)}(x) = \frac{-15!}{16!} + \underbrace{\dots}_{\text{All of these terms contain } x} \Rightarrow f^{(15)}(0) = \frac{-15!}{16!} = \frac{-1}{16}$$

(b) $g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = |x|$$

$$|x| < 1 \Rightarrow -1 < x < 1$$

$\boxed{x=-1}$ $\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \rightarrow$ Divergent General Harmonic

$\boxed{x=1}$ $\sum_{n=0}^{\infty} (-1)^n \frac{(1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \rightarrow$ Convergent Alternating General Harmonic

I.O.C. $\Rightarrow -1 < x \leq 1$

(c) $y' = f'(x) + g'(x) \Rightarrow y'(0) = f'(0) + g'(0) \Rightarrow y'(0) = \frac{1}{2} - \frac{1}{2} = 0$

$y'' = f''(x) + g''(x) \Rightarrow y''(0) = f''(0) + g''(0) \Rightarrow y''(0) = 0 + \frac{2}{3} = \frac{2}{3}$

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.