AP Calculus BC Name ______________________________

Determine the convergence or divergence of the following series and state the test used. If the series is ometric or telescoping, find the sum of the series.

1. $e - \frac{e^2}{5} + \frac{e^3}{25} - \frac{e^4}{125} + \cdots$

2. $\sum_{n=1}^{\infty} \frac{e^n}{(1+e^n)^2}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\arctan n}$

4. $\sum_{n=1}^{\infty} \frac{n^2}{2n^2-3}$

5. $\sum_{n=1}^{\infty} \frac{n}{(n+1)\,2^{n-1}}$

6. $\sum_{n=2}^{\infty} \frac{4^n}{3^n-3}$

$\sum_{n=0}^{\infty} \frac{2^{n+1}+3}{4^n}$

8. $\sum_{n=1}^{\infty} \frac{(-n)^n}{n^{3n}}$

9. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+\sqrt[3]{n}}{n^2+n^3}$

10. $\sum_{n=1}^{\infty} \ln(1+1/n)$

11. $\sum_{n=1}^{\infty} \left(\frac{-2}{n}\right)^{3n}$

12. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

13. $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

14. $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\cdots+\frac{1}{4n-3}+\cdots$

AP Calculus BC:

① $e - \frac{e^2}{5} + \frac{e^3}{25} - \frac{e^4}{125} + \dots = \sum_{n=0}^{\infty} e\left(-\frac{e}{5}\right)^n$ $\quad |r| = \frac{e}{5} < 1$

Convergent Geometric

$\sum_{n=0}^{\infty} e\left(-\frac{e}{5}\right)^n$ converges to $S = \frac{e}{1 - -\frac{e}{5}} = \frac{e}{\frac{5+e}{5}} = \frac{5e}{5+e} \approx 1.761$

② $\sum_{n=1}^{\infty} \frac{e^n}{(1+e^n)^2}$

$\frac{e^n}{(1+e^n)^2}$ is positive (since $e^n > 0$), continuous (since $1+e^n \neq 0$ for $n \geq 1$), and decreasing (since $\frac{d}{dn}\left[\frac{e^n}{(1+e^n)^2}\right]$ is negative for $n > 0$).

$$\frac{d}{dn}\left[e^n(1+e^n)^{-2}\right]$$
$$= e^n(1+e^n)^{-2} + e^n \cdot -2(1+e^n)^{-3} \cdot e^n$$
$$= e^n(1+e^n)^{-2} - 2e^{2n}(1+e^n)^{-3}$$
$$= e^n(1+e^n)^{-3}\left[1+e^n - 2e^n\right]$$
$$= \frac{-e^n\left[e^n - 1\right]}{(1+e^n)^3} < 0 \text{ when}$$
$e^n - 1 > 0$
$e^n > 1$
$n > 0$

$$\int_1^{\infty} \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{e^x}{(1+e^x)^2}\,dx$$

$u = 1+e^x$
$du = e^x dx$
$\int \frac{1}{u^2}\,du$
$-\frac{1}{u}$

$$= \lim_{b\to\infty}\left[-\frac{1}{1+e^x}\right]_1^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{1+e^b} + \frac{1}{1+e}\right] = 0 + \frac{1}{1+e} = \frac{1}{1+e}$$

$\therefore \sum_{n=1}^{\infty} \frac{e^n}{(1+e^n)^2}$ converges by the Integral Test

③ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\arctan n}$

$\lim_{n\to\infty} \frac{1}{\arctan n} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 0$ $\quad \therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\arctan n}$ Diverges by n^{th} term test

④ $\sum_{n=1}^{\infty} \frac{n^2}{2n^2-3}$

$\lim_{n\to\infty} \frac{n^2}{2n^2-3} = \lim_{n\to\infty} \frac{1}{2 - \frac{3}{n^2}} = \frac{1}{2} \neq 0$ $\quad \therefore \sum_{n=1}^{\infty} \frac{n^2}{2n^2-3}$ Diverges by n^{th} term test

(5) $$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}} = \sum_{n=1}^{\infty} \frac{2n}{(n+1)2^n} \qquad a_n = \frac{2n}{(n+1)\cdot 2^n} \qquad b_n = \frac{n}{n\cdot 2^n} = \frac{1}{2^n}$$

$\downarrow$ Convergent Geometric since $r = \frac{1}{2} < 1$

$$\lim_{n\to\infty} \frac{2n}{(n+1)2^n} \cdot \frac{n\cdot 2^n}{n} = \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = 2$$

$\downarrow$ Finite & positive

$$\therefore \sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$ Converges by Limit Comparison

(6) $$\sum_{n=2}^{\infty} \frac{4^n}{3^n - 3} \qquad a_n = \frac{4^n}{3^n-3} \qquad b_n = \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$$ Divergent Geometric

Consider: $$\frac{4^n}{3^n-3} > \frac{4^n}{3^n} \Rightarrow 4^n \cdot 3^n > 4^n(3^n-3) \Rightarrow 3^n > 3^n - 3 \Rightarrow 0 > -3$$ always true

Since $\displaystyle\sum_{n=2}^{\infty} \frac{4^n}{3^n-3} > \sum_{n=2}^{\infty}\left(\frac{4}{3}\right)^n$ and $\displaystyle\sum_{n=2}^{\infty}\left(\frac{4}{3}\right)^n$ diverges,

$\displaystyle\sum_{n=2}^{\infty} \frac{4^n}{3^n-3}$ diverges by direct comparison.

(7) $$\sum_{n=0}^{\infty} \frac{2^{n+1}+3}{4^n} = \sum_{n=0}^{\infty} \frac{2^{n+1}}{4^n} + \sum_{n=0}^{\infty} \frac{3}{4^n} = \sum_{n=0}^{\infty} 2\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n \Rightarrow$$ Both convergent geometric.

$$S = \frac{2}{1-\frac{1}{2}} = 4 \qquad S = \frac{3}{1-\frac{1}{4}} = 4$$

$$\therefore \sum_{n=0}^{\infty} \frac{2^{n+1}+3}{4^n}$$ converges to 8.

⑧ $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{n^{3n}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^n}{n^{3n}} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n^3}\right)^n = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^2}\right)^n \Rightarrow$ Alternating Series

1) $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \frac{1}{n^{2n}} = 0$

2) $\displaystyle\left(\frac{1}{(n+1)^2}\right)^{n+1} \le \left(\frac{1}{n^2}\right)^n \Rightarrow \frac{1}{(n+1)^{2(n+1)}} \le \frac{1}{n^{2n}} \Rightarrow n^{2n} \le (n+1)^{2(n+1)} \Rightarrow n^n \le (n+1)^{n+1}$

$n^n \le (n+1)^n \cdot (n+1)$

$\displaystyle\frac{1}{n+1} \le \left(\frac{n+1}{n}\right)^n$

As $n \to \infty$ $\quad 0 \le e$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{n^{3n}}$ Converges by Alternating Series Test

⑨ $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + \sqrt[3]{n}}{n^2 + n^3}$ $\quad a_n = \dfrac{\sqrt{n} + \sqrt[3]{n}}{n^2+n^3}$ $\quad b_n = \dfrac{\sqrt{n}}{n^3} = \dfrac{1}{n^{5/2}} \to$ Convergent p-series

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n} + \sqrt[3]{n}}{n^2+n^3} \cdot \frac{n^3}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^{7/2} + n^{10/3}}{n^{5/2} + n^{7/2}} = \lim_{n\to\infty} \frac{1 + \frac{1}{n^{1/6}}}{\frac{1}{n} + 1} = 1$ Finite & positive

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + \sqrt[3]{n}}{n^2 + n^3}$ converges by Limit Comparison

⑩ $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n)$

$S_1 = \ln(2) - \ln(1) = \ln(2)$

$S_2 = \cancel{\ln(2)} + \ln(3) - \cancel{\ln(2)} = \ln(3)$

$S_3 = \cancel{\ln(3)} + \ln(4) - \cancel{\ln(3)} = \ln(4)$

$\vdots$

$S_n = \ln(n+1)$

$\displaystyle\lim_{n\to\infty} \ln(n+1) = \infty$

$\therefore \displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$ Diverges by Sequence of Partial Sums

(11) $\sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)^{3n} = \sum_{n=1}^{\infty}(-1)^{3n}\cdot\left(\frac{2}{n}\right)^{3n} = \sum_{n=1}^{\infty}(-1)^{n}\left(\frac{8}{n^3}\right)^{n}$

1) $\lim_{n\to\infty}\left(\frac{8}{n^3}\right)^{n}$ $\quad 0^{\infty}$ Indeterminant

$$y = \lim_{n\to\infty}\left(\frac{8}{n^3}\right)^{n}$$

$$\ln y = \lim_{n\to\infty} n\ln\left(\frac{8}{n^3}\right) = \infty\cdot\ln(0) = \infty\cdot -\infty = -\infty$$

$$\ln y = -\infty$$

$$y = e^{-\infty} = 0$$

2) $\left[\left(\frac{2}{n+1}\right)^{n+1}\right]^{3} \leq \left[\left(\frac{2}{n}\right)^{n}\right]^{3}$

$$\frac{2^{n+1}}{(n+1)^{n+1}} \leq \frac{2^n}{n^n}$$

$$n^n\cdot 2^{n+1} \leq 2^n\cdot(n+1)^{n+1}$$

$$n^n\cdot 2^n\cdot 2 \leq 2^n(n+1)^n(n+1)$$

$$2n^n \leq (n+1)^n(n+1)$$

$$\frac{2}{n+1} \leq \left(\frac{n+1}{n}\right)^{n}$$

$$\frac{2}{n+1} \leq \left(1+\frac{1}{n}\right)^{n}$$

Consider: $\lim_{n\to\infty}\frac{2}{n+1} = 0$ & $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$

Therefore, at some point $\frac{2}{n+1} \leq \left(1+\frac{1}{n}\right)^{n}$ becomes true.

So, we can conclude $a_{n+1} \leq a_n$ for some value of n.

$\therefore \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)^{3n}$ converges by Alternating Series test.

(12) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = 1^{\infty} \text{ Indeterminant}$$

$$y = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\ln y = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+n^{-1})}{n^{-1}} \overset{\frac{0}{0} \text{ L'Hops}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+n^{-1}} \cdot -n^{-2}}{-n^{-2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$\ln y = 1$$
$$y = e^1 \neq 0$$

$\therefore \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ diverges by n^{th} term test.

(13) $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$

$$\frac{2}{(n+3)(n+1)} = \frac{A}{n+3} + \frac{B}{n+1}$$

$$2 = (A+B)n + (A+3B)$$

$$A+B=0$$
$$-(A+3B=2)$$
$$-2B=-2$$
$$\boxed{B=1}$$
$$\boxed{A=-1}$$

$$S_1 = \frac{1}{2} - \frac{1}{4}$$

$$S_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$$

$$S_3 = \frac{1}{2} \cancel{-\frac{1}{4}} + \frac{1}{3} - \frac{1}{5} \cancel{+\frac{1}{4}} - \frac{1}{6} = \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6}$$

$$S_4 = \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{5}} - \frac{1}{6} + \cancel{\frac{1}{5}} - \frac{1}{7} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$\therefore \sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ Converges (Telescoping) to $\frac{5}{6}$

(14) $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\dots+\frac{1}{4n-3}+\dots = \sum_{n=1}^{\infty} \frac{1}{4n-3}$

$\downarrow$

General Harmonic.

$a_n = \frac{1}{4n-3} \quad b_n = \frac{1}{n} \rightarrow$ divergent harmonic

$$\lim_{n\to\infty} \frac{1}{4n-3} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{4n-3} = \frac{1}{4} \rightarrow \text{finite \& positive}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{4n-3}$ Diverges by Limit Comparison

(15) $$\sum_{n=1}^{\infty} \frac{2^{n+1}+9^{n/2}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty} \frac{9^{n/2}}{5^n} = \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$$

$$= \sum_{n=0}^{\infty} 2\left(\frac{2}{5}\right)^{n+1} + \sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \frac{3}{5}\left(\frac{3}{5}\right)^n$$

Both convergent geometric.

$$\sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{2}{5}\right)^n = \frac{\frac{4}{5}}{1-\frac{2}{5}} = \frac{4}{3} \qquad \sum_{n=0}^{\infty} \frac{3}{5}\left(\frac{3}{5}\right)^n = \frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{3}{2}$$

$\therefore \sum_{n=1}^{\infty} \frac{2^{n+1}+9^{n/2}}{5^n}$ is a composition of convergent geometric series.

This series converges to $\frac{4}{3}+\frac{3}{2} = \frac{17}{6}$

(16) $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

$a_n = \frac{\ln n}{n+1}$ $\quad b_n = \frac{1}{n+1} \rightarrow$ Divergent General Harmonic

$$\frac{\ln n}{n+1} \geq \frac{1}{n+1}$$

$$(n+1)\ln n \geq n+1$$

$$\ln n \geq 1$$

$$n \geq e$$

$\therefore \sum_{n=2}^{\infty} \frac{\ln n}{n+1}$ Diverges by Direct Comparison

(17) $\sum_{n=2}^{\infty} \frac{n}{\ln n^3} = \sum_{n=2}^{\infty} \frac{n}{3\ln n}$

$$\lim_{n\to\infty} \frac{n}{3\ln n} = \frac{\infty}{\infty}$$

L'Hops $\lim_{n\to\infty} \frac{1}{3\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3} = \infty \neq 0$

$\therefore \sum_{n=2}^{\infty} \frac{n}{\ln n^3}$ Diverges by n^{th} term test

(18) $\frac{5}{1\times 3} + \frac{5}{2\times 4} + \frac{5}{3\times 5} + \dots = \sum_{n=1}^{\infty} \frac{5}{n(n+2)} = \sum_{n=1}^{\infty} \frac{5/2}{n} - \frac{5/2}{n+2}$

$$\frac{5}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$5 = (A+B)n + 2A$$

$$A+B=0$$

$$2A=5$$

$$\boxed{A = 5/2}$$

$$\boxed{B = -5/2}$$

$$S_1 = \frac{5}{2} - \frac{5}{6}$$

$$S_2 = \frac{5}{2} - \frac{5}{6} + \frac{5}{4} - \frac{5}{8}$$

$$S_3 = \frac{5}{2} - \cancel{\frac{5}{6}} + \frac{5}{4} - \frac{5}{8} + \cancel{\frac{5}{6}} - \frac{5}{10} = \frac{5}{2} + \frac{5}{4} - \frac{5}{8} - \frac{5}{10}$$

$$S_4 = \frac{5}{2} + \frac{5}{4} - \cancel{\frac{5}{8}} - \frac{5}{10} + \cancel{\frac{5}{8}} - \frac{5}{12} = \frac{5}{2} + \frac{5}{4} - \frac{5}{10} - \frac{5}{12}$$

$$\vdots$$

$$S_n = \frac{5}{2} + \frac{5}{4} - \frac{5}{2n+2} - \frac{5}{2n+4} \qquad \lim_{n\to\infty} S_n = \frac{5}{2} + \frac{5}{4} = \frac{15}{4}$$

$\therefore \sum_{n=1}^{\infty} \frac{5}{n(n+2)}$ Converges (Telescoping) to $\frac{15}{4}$

(19) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{e^{1/n}}{n}$

1) $\displaystyle\lim_{n\to\infty} \frac{e^{1/n}}{n} = \frac{e^0}{\infty} = \frac{1}{\infty} = 0$

2) $\displaystyle\frac{e^{\frac{1}{n+1}}}{n+1} \le \frac{e^{\frac{1}{n}}}{n}$

$\displaystyle\frac{e^{\frac{1}{n+1}}}{e^{\frac{1}{n}}} \le \frac{n+1}{n}$

$\displaystyle e^{\frac{1}{n+1}-\frac{1}{n}} \le \frac{n+1}{n}$

$\displaystyle e^{\frac{-1}{n^2+n}} \le 1+\frac{1}{n}$

$\displaystyle\frac{-1}{n^2+n} \le \ln\left(1+\frac{1}{n}\right) \longrightarrow$ For $n \ge 1$ this must be true since $\frac{-1}{n^2+n} < 0$ & $\ln\left(1+\frac{1}{n}\right) > 0$.

$\therefore \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{e^{1/n}}{n}$ converges by Alternating Series Test.

(20) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ Positive, Continuous, and Decreasing $\longrightarrow \displaystyle\frac{d}{dn}\left[\frac{n}{n^2+1}\right] = \frac{n^2+1-n(2n)}{(n^2+1)^2}$

$\displaystyle = \frac{-n^2+1}{(n^2+1)^2}$

$\displaystyle = \frac{-(n^2-1)}{(n^2+1)^2} < 0$ for $n > 1$

$\displaystyle\int_1^{\infty} \frac{x}{x^2+1}\,dx$

$\displaystyle\lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+1}\,dx$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}\int \frac{1}{u}\,du$
$\frac{1}{2}\ln|u|$

$\displaystyle\lim_{b\to\infty} \left[\frac{1}{2}\ln|x^2+1|\right]_1^b$

$\displaystyle\lim_{b\to\infty} \left[\frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln|2|\right] = \infty - \frac{1}{2}\ln(2) = \infty$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ Diverges by Integral Test.

(21) $\displaystyle\sum_{n=1}^{\infty} \frac{8\arctan n}{n^2+1}$ Positive, Continuous, Decreasing $\rightarrow$ $\displaystyle\frac{d}{dn}\left[\frac{8\arctan n}{n^2+1}\right] = \frac{(n^2+1)\cdot\frac{8}{n^2+1} - 8\arctan n\,(2n)}{(n^2+1)^2}$

$$= \frac{8-16n\arctan n}{(n^2+1)^2}$$

$$= \frac{8(1-2n\arctan n)}{(n^2+1)^2} < 0$$

when $1-2n\arctan n < 0$

$1 < 2n\arctan n$

$\frac{1}{2} < n\arctan n$

$0.765 < n$

$$8\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx$$

$$\lim_{b\to\infty} 8\int_1^{b} \frac{\arctan x}{x^2+1}\,dx$$

$u = \arctan x$

$du = \frac{1}{x^2+1}\,dx$

$\int u\,du$

$\frac{1}{2}u^2$

$$\lim_{b\to\infty}\left[4(\arctan x)^2\right]_1^b = \lim_{b\to\infty}\left[4(\arctan b)^2 - 4(\arctan 1)^2\right]$$

$$= 4\cdot\left(\frac{\pi}{2}\right)^2 - 4\left(\frac{\pi}{4}\right)^2$$

$$= \pi^2 - \frac{1}{4}\pi^2$$

$$= \frac{3}{4}\pi^2$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{8\arctan n}{n^2+1}$ Converges by Integral Test

(22) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n-2}}$ $\quad a_n = \frac{1}{\sqrt{n-2}}$ $\quad b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}} \rightarrow$ Divergent P-series

$$\lim_{n\to\infty} \frac{1}{\sqrt{n-2}}\cdot\frac{\sqrt{n}}{1} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n-2}} = \lim_{n\to\infty}\sqrt{\frac{n}{n-2}} = \sqrt{1} = 1$$ Finite & Positive

$\therefore \displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n-2}}$ Diverges by Limit Comparison

(23) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ Positive, Continuous, Decreasing $\rightarrow \frac{d}{dn}\left[\frac{1}{n(\ln n)^3}\right] = \frac{d}{dn}\left[n^{-1}(\ln n)^{-3}\right]$

$= -n^{-2}(\ln n)^{-3} + n^{-1} \cdot -3(\ln n)^{-4} \cdot \frac{1}{n}$

$= -n^{-2}(\ln n)^{-3} - 3n^{-2}(\ln n)^{-4}$

$= -n^{-2}(\ln n)^{-4}\left[\ln n + 3\right]$

$= \frac{-\left[\ln n + 3\right]}{n^2(\ln n)^4} < 0$ for

$\ln n + 3 > 0$

$\ln n > -3$

$n > e^{-3}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

$$\lim_{b\to\infty} \int_2^b (\ln x)^{-3} \cdot \frac{1}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\int u^{-3}\,du$

$-\frac{1}{2}u^{-2}$

$$\lim_{b\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]_2^b = \lim_{b\to\infty}\left[\frac{-1}{2(\ln b)^2} + \frac{1}{2(\ln 2)^2}\right] = \frac{1}{2(\ln 2)^2}$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ Converges by Integral Test

(24) $\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$ $a_n = \frac{3}{\sqrt{n^2-4}}$ $b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n} \rightarrow$ Divergent Harmonic

$$\lim_{n\to\infty} \frac{3}{\sqrt{n^2-4}} \cdot \frac{\sqrt{n^2}}{1} = \lim_{n\to\infty} 3\sqrt{\frac{n^2}{n^2-4}} = \lim_{n\to\infty} 3\sqrt{\frac{1}{1-\frac{4}{n^2}}} = 3$$

$\downarrow$ Finite & Positive

$\therefore \sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$ Diverges by Limit Comparison

(25) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}+1}$

1) $\lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = \frac{1}{\infty+1} = 0$

2) $\frac{1}{\sqrt{n+1}+1} \leq \frac{1}{\sqrt{n}+1}$

$\sqrt{n}+1 \leq \sqrt{n+1}+1$

$\sqrt{n} \leq \sqrt{n+1}$

$n \leq n+1$

$0 \leq 1$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}+1}$ Converges by Alternating Series Test

$|R_n| \leq a_{n+1} < \frac{1}{1000}$

$\frac{1}{\sqrt{n+1}+1} < \frac{1}{1000}$

$1000 < \sqrt{n+1}+1$

$999 < \sqrt{n+1}$

$(999)^2 < n+1$

$(999)^2 - 1 < n$

$998000 < n$

You would need to use 998,001 terms.

(26) $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n}\right)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{2n+2}}{n^2}$

$\boxed{\text{Note: } (-1)^{2n+2} = 1}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2}$ → Not an alternating series (Convergent p-series)

(27) $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$

L'Hops

1) $\lim_{n\to\infty} n e^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{\infty} = 0$

2) $(n+1) e^{-(n+1)} \leq n e^{-n}$

$\frac{n+1}{e^{n+1}} \leq \frac{n}{e^n}$

$e^n(n+1) \leq n e^n \cdot e$

$n+1 \leq en$

$1 \leq en - n$

$1 \leq n(e-1)$

$\frac{1}{e-1} \leq n$

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$ Converges by Alternating Series Test

$|R_n| \leq a_{n+1} < \frac{1}{1000}$

$\frac{n+1}{e^{n+1}} < \frac{1}{1000}$

$(n+1)\cdot 1000 < e^{n+1}$ → Solve this w/ Calculator

$8.118 < n$

You would need to use 9 terms.

(28) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{8^n}$

1) $\lim_{n\to\infty} \frac{n}{8^n} = \lim_{n\to\infty} \frac{1}{\ln 8 \cdot 8^n} = \frac{1}{\infty} = 0$

2) $\frac{n+1}{8^{n+1}} \leq \frac{n}{8^n}$

$8^n(n+1) \leq n \cdot 8^n \cdot 8$

$n+1 \leq 8n$

$1 \leq 7n$

$\frac{1}{7} \leq n$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{8^n}$ Converges by Alternating Series Test

$|R_n| \leq a_{n+1} < \frac{1}{1000}$

$\frac{n+1}{8^{n+1}} < \frac{1}{1000}$

$1000(n+1) < 8 \cdot 8^n$

$125(n+1) < 8^n$

$2.987 < n$

You need to use 3 terms.