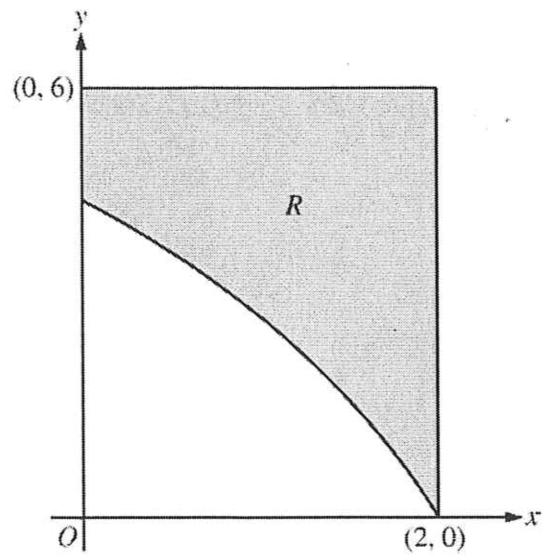
CALCULUS AB – PRACTICE EXAM
SECTION II, PART A
Calculator Active

Name Answers



1: Correct Limits

1. In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

(a) Find the area of R.

$$\int_0^2 6 - 4\ln(3-x)\,dx = 6.816$$

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

$$\pi \int_0^2 [8 - 4\ln(3-x)]^2 - [8-6]^2\,dx = 168.179$$

(53.533π)

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

$$\int_0^2 [6-4\ln(3-x)]^2\,dx = 26.266$$

$$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$$

(d) Set up but do not integrate an integral expression to find the value of a vertical line $x = k$ that divides the region R into two regions of equal area.

$$\int_0^k [6-4\ln(3-x)]\,dx = \int_k^2 [6-4\ln(3-x)]\,dx$$

OR

$$2\int_0^k [6-4\ln(3-x)]\,dx = \underbrace{\int_0^2 [6-4\ln(3-x)]\,dx}_{(6.816)}$$

~~equation~~

$$2: \begin{cases} 1: k \text{ as a limit} \\ 1: \text{answer} \end{cases}$$

2. The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t) = \frac{15t}{1 + 3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

$$\int_0^6 R(t)\,dt = 31.815 \text{ yd}^3$$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer w/ units} \end{cases}$

$$\int_0^6 2 + 5\sin\left(\frac{4\pi t}{25}\right) dt$$

(b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

$$Y(t) = 2500 + \int_0^t S(x) - R(x)\,dx$$

$$= 2500 + \int_0^t \frac{15x}{1+3x} - \left(2 + 5\sin\left(\frac{4\pi x}{25}\right)\right) dx$$

$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits} \\ 1: \text{answer} \end{cases}$

(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

$y(t)$ = Amount of sand

$y'(t) = S(t) - R(t)$

1: answer

$y'(4) = S(4) - R(4) = -1.908$ yd³/hr

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

Minimum at critical value or endpoint.

$y'(t) = 0 \Rightarrow S(t) - R(t) = 0 \Rightarrow t = 5.118$

t	$y(t)$
0	2500
5.118	2492.369
6	2493.277

$$3: \begin{cases} 1: y'(t) = 0 \\ 1: \text{critical value} \\ 1: \text{answer w/ justification} \end{cases}$$

The minimum amount of sand on the beach is 2492.369 yd³ occurring at $t = 5.117$ hours

CALCULUS AB – PRACTICE EXAM
SECTION II, PART B
Non – Calculator

Name Answers

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

$$v'(16) \approx \frac{v(20)-v(12)}{20-12} = \frac{240-200}{8} = \frac{40}{8} = 5 \text{ m/min}^2$$

1: { approx

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

$\int_0^{40} |v(t)|\, dt$ is the total distance, in meters, Johanna jogs over the time interval 0 minutes to 40 minutes.

$$\int_0^{40} |v(t)|\, dt \approx 12|v(12)| + 8|v(20)| + 4|v(24)| + 16|v(40)|$$
$$\approx 12(200) + 8(240) + 4(220) + 16(150)$$
$$\approx 2400 + 1920 + 880 + 2400$$
$$= 7600 \text{ meters}$$

3: { 1: explanation, 1: Riemann Sum, 1: approx

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

$$B'(t) = 3t^2 - 12t$$

$$B'(5) = 3(5)^2 - 12(5) = 15 \text{ m/min}^2$$

$$2: \begin{cases} 1: B'(t) \\ 1: \text{answer} \end{cases}$$

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

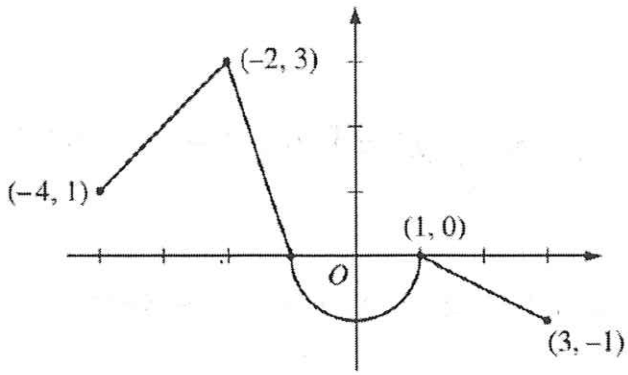
$$\frac{1}{10-0} \int_0^{10} t^3 - 6t^2 + 300 \, dt$$

$$\frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$$

$$\frac{1}{10} \left[\frac{10000}{4} - 2(1000) + 3000 - 0 \right]$$

$$350 \text{ m/min}$$

$$3: \begin{cases} 1: \text{integral} \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$$



Graph of f

4. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$$g(2) = \int_1^2 f(t)\,dt = -\tfrac{1}{2}(1)\left(\tfrac{1}{2}\right) = -\tfrac{1}{4}$$

$$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt = -\left[\tfrac{1}{2}(1)(3) + -\tfrac{1}{2}\pi(1)^2\right]$$

$$= \frac{\pi - 3}{2}$$

$$2: \begin{cases} 1: g(2) \\ 1: g(-2) \end{cases}$$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(x) = f(x)$

$g'(-3) = 2$

$$2: \begin{cases} 1: g'(-3) \\ 1: g''(-3) \end{cases}$$

$g''(x) = f'(x)$

$g''(-3) = 1$

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

Horizontal Tangent $\Rightarrow g'(x) = f(x) = 0 \Rightarrow x = -1, 1$

At $x = -1$, $g(x)$ has a relative maximum since $g'(x) = f(x)$ changes from positive to negative.

At $x = 1$, $g(x)$ does not have a relative extrema since $g'(x) = f(x)$ does not change sign.

$3: \begin{cases} 1: \text{uses } g'(x) = 0 \\ 1: x = -1, 1 \\ 1: \text{answers w/ justification} \end{cases}$

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

Point of inflection $\Rightarrow g''(x) = f'(x)$ changes sign.

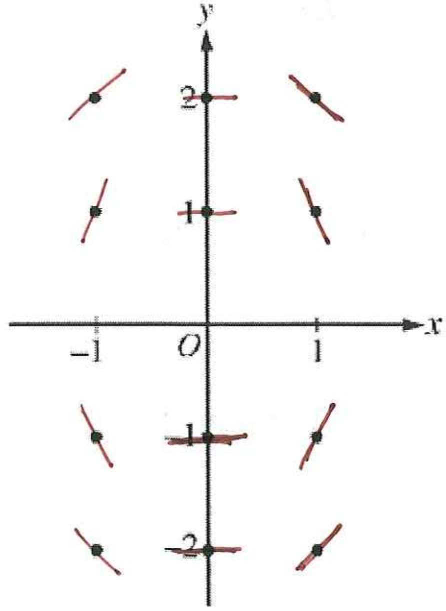
P.o.I. exist at $x = -2$ and $x = 1$ since $g''(x) = f'(x)$ changes from positive to negative.

P.O.I. exist at $x = 0$ since $g''(x) = f'(x)$ changes from negative to positive.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{explanation} \end{cases}$

5. Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{non zero slopes} \end{cases}$

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = -\frac{2(1)}{-1} = 2$$

$2: \begin{cases} 1: \text{tangent line} \\ 1: \text{approx} \end{cases}$

$$y + 1 = 2(x-1)$$

$$y = 2(x-1) - 1 = 2x - 3$$

$$f(1.1) \approx 2(0.1) - 1 = -0.8$$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

$$\frac{dy}{dx} = \frac{-2x}{y}$$

$$\int y\,dy = \int -2x\,dx$$

$$\frac{1}{2}y^2 = -x^2 + C$$

$$\frac{1}{2} = -1 + C$$

$$\frac{3}{2} = C$$

$$\frac{1}{2}y^2 = -x^2 + \frac{3}{2}$$

$$y^2 = -2x^2 + 3$$

$$y = \pm\sqrt{-2x^2+3}$$

$$y = -\sqrt{-2x^2+3}$$

5: { 1: seperation
1: antiderivatives
1: constant (+C)
1: initial condition
1: solves for y

6. Two particles move along the x-axis. For $0 \le t \le 6$, the position of particle P at time t is given by $p(t) = 2\cos\left(\frac{\pi}{4}t\right)$, while the position of particle R at time t is given by $r(t) = t^3 - 6t^2 + 9t + 3$.

(a) For $0 \le t \le 6$, find all times t during which particle R is moving to the right.

$$r'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-3)(t-1)$$

$r'(t)$

t	0 to 1	1 to 3	3 to 6
$r'(t)$	+	−	+

$2: \begin{cases} 1: r'(t) \\ 1: \text{answer} \end{cases}$

partical R moves right on intervals $(0,1)$ and $(3,6)$.

(b) For $0 \le t \le 6$, find all times t during which the two particles travel in opposite directions.

t	0 to 1	1 to 3	3 to 4	4 to 6
$r'(t)$	+	−	+	+
$p'(t)$	−	−	−	+

$$p'(t) = -2\sin\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4}$$

$$= -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right)$$

$$-\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right) = 0$$

$\frac{\pi}{4}t = \pi$ $\quad$ $\frac{\pi}{4}t = 2\pi$

$t = 4$ $\quad$ $\cancel{t = 8}$

$3: \begin{cases} 1: p'(t) \\ 1: \text{sign analysis} \\ 1: \text{answer} \end{cases}$

Particles travel in the same direction on the intervals $(0,1)$ and $(3,4)$

(c) Find the acceleration of particle P at time $t = 3$. Is particle P speeding up, slowing down, or doing neither at time $t = 3$? Explain your reasoning.

$$p'(t) = -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right)$$

$$p''(t) = -\frac{\pi}{2}\cos\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4}$$

$$= -\frac{\pi^2}{8}\cos\left(\frac{\pi}{4}t\right)$$

$$2: \begin{cases} 1: p''(3) \\ 1: \text{answer w/ reason} \end{cases}$$

$$p'(3) = -\frac{\pi}{2}\sin\left(\frac{3\pi}{4}\right) = -\frac{\pi}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{-\pi\sqrt{2}}{4} < 0$$

$$p''(3) = -\frac{\pi^2}{8}\cos\left(\frac{3\pi}{4}\right) = \frac{-\pi^2}{8}\left(-\frac{\sqrt{2}}{2}\right) = \frac{\pi^2\sqrt{2}}{16} > 0$$

The particle P is slowing down at $t=3$ since $p'(3) < 0$ and $p''(3) > 0$.

(d) Write, but do not evaluate, an expression for the average distance between the two particles on the interval $1 \le t \le 3$.

Distance between : $|p(t) - r(t)|$

$$\text{Avg Distance} = \frac{1}{3-1}\int_1^3 |p(t) - r(t)|\,dt$$

$$= \frac{1}{2}\int_1^3 |p(t) - r(t)|\,dt.$$

$$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits \& constant} \end{cases}$$