AP Calculus

Parametric, Vector, and Polar

Parametric Derivatives

Multiple Choice Questions

1. If $x = te^t$ and $y = t + e^t$, then $\frac{dy}{dx}$ at $t = 0$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

2. If $x = \tan t$ and $y = \sin t$, then $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{6}$ is

(A) $-\frac{9}{11}$ (B) $-\frac{27}{32}$ (C) $\frac{13}{16}$ (D) $\frac{7}{8}$

3. A curve C is defined by the parametric equations $x = t^3 - 3$ and $y = 2t^2$. Which of the following is the equation for the line tangent to the graph of C at the point $(5,8)$?

(A) $y = \frac{1}{3}x + \frac{8}{3}$ (B) $y = 2x - \frac{8}{3}$ (C) $y = \frac{2}{3}x + \frac{14}{3}$ (D) $y = 3x + 8$

4. If $x = \cos t$ and $y = 2\sin^2 t$, then $\frac{dy}{dx}$ at $t = 1$ is

(A) $-2\cos 1$ (B) $-4\cos 1$ (C) $-2\tan 1$ (D) $-2\sin 1$

5. For what value(s) of t does the curve defined by the parametric equations $x = \frac{5t}{1+t^3}$ and $y = \frac{2t^2}{1+t^3}$ have a horizontal tangent?

(A) 0 only

(B) $\sqrt[3]{2}$ only

(C) 0 and 4 only

(D) 0 and $\sqrt[3]{2}$ only

6. A point (x, y) is moving along a curve $y = f(x)$. At the instant when the slope of the curve is $\frac{3}{4}$, the x-coordinate of the point is decreasing at the rate of $\frac{2}{5}$ units per second. The rate of change, in units per second, of the y-coordinate of the point is

(A) $-\frac{15}{8}$ (B) $-\frac{3}{5}$ (C) $-\frac{3}{10}$ (D) $\frac{3}{10}$

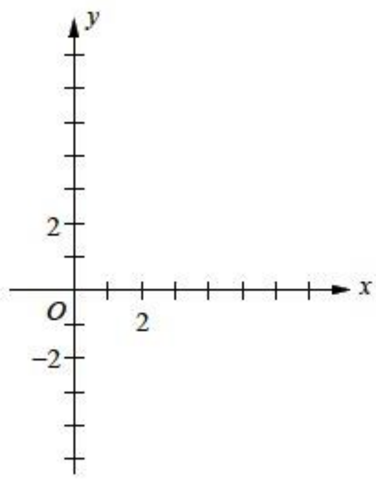
7. An object moving along a curve in the xy-plane is in position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 2 - \sin(t^2)$. At time $t = 3$, the object is at position $(2, 7)$. What is the x-coordinate of the position of the object at time $t = 6$?

(A) 8.135 (B) 9.762 (C) 10.375 (D) 11.308

Free Response Questions

8. A particle moves in the xy- plane so that its position at any time t, $0 \le t \le 2\pi$, is given by $x(t) = t - \sin t$ and $y(t) = t\cos t$.

   (a) Sketch the path of the particle in the xy- plane below. Indicate the direction of motion along the path.



   (b) At what time t, $0 \le t \le 2\pi$, does $y(t)$ attain its minimum value? What is the position $(x(t), y(t))$ of the particle at this time?

   (c) Write an equation for the line tangent to the curve at time $t = \pi$.

9. A particle moving along the curve is defined by the equation $y = x^3 - 4x^2 + 4$. The x-coordinate of the particle, $x(t)$, satisfies the equation $\dfrac{dx}{dt} = \dfrac{t}{\sqrt{t^2+9}}$, for $t \ge 0$ with initial condition $x(0) = 1$.

   (a) Find $x(t)$ in terms of t.

   (b) Find $\dfrac{dy}{dt}$ in terms of t.

   (c) Find the location of the particle at time $t = 4$.

   (d) Write an equation for the line tangent to the curve at time $t = 4$.

Arc Length (Distance Traveled for Parametric Motion) in Parametric Form

Multiple Choice Questions

1. The position of a particle at any time $t \geq 0$ is given by $x = t - t^2$ and $y = \frac{4}{3}t^{3/2}$. What is the total distance traveled by the particle from $t = 1$ to $t = 3$?

   (A) 7.165 (B) 8.268 (C) 9.431 (D) 10.346

2. The position of particle at any time $t \geq 0$ is given by $x(t) = a(\cos t + t\sin t)$ and $y(t) = a(\sin t - t\cos t)$. What is the total distance traveled by the particle from $t = 0$ to $t = \pi$?

   (A) $\frac{1}{2}\pi a$ (B) πa^2 (C) $\frac{1}{2}\pi^2 a$ (D) $\frac{1}{2}\pi^2 a^2$

3. The length of the path described by the parametric equations $x = \sin t + \ln(\cos t)$ and $y = \cos t$, for $\frac{\pi}{6} \leq t \leq \frac{\pi}{3}$, is given by

   (A) $\int_{\pi/6}^{\pi/3} \sqrt{\cos^2 t + 2\sin t + 2}\, dt$

   (B) $\int_{\pi/6}^{\pi/3} \sqrt{\sin^2 t + 2\cos t + 2}\, dt$

   (C) $\int_{\pi/6}^{\pi/3} \sqrt{\cot^2 t + 2\cos t}\, dt$

   (D) $\int_{\pi/6}^{\pi/3} \sqrt{\sec^2 t - 2\sin t}\, dt$

4. A particle moving in the xy- plane has velocity vector given by $v(t) = \langle e^t - t,\ t\sin t \rangle$ for time $t \geq 0$. What is the magnitude of the displacement of the particle between time $t = 0$ and $t = 2$?

   (A) 4.722 (B) 4.757 (C) 4.933 (D) 5.109

Free Response Questions

5. A particle moving along a curve in the xy-plane is at position $(x(t),\, y(t))$ at any time t, where $\frac{dx}{dt} = 2\sin(t^2)$ and $\frac{dy}{dt} = \cos(t^3)$. At time $t = 1$, the object is at position $(3, 2)$.

(a) Write an equation for the line tangent to the curve at $(3, 2)$.

(b) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) Find the magnitude of the displacement of the particle between $t = 1$ and $t = 3$.

Vectors

Multiple Choice Questions

1. If a particle moves in the *xy*-plane so that at time $t > 0$ its position vector is $(t^3 - 1, \ln\sqrt{t^2+1})$, then at time $t = 1$, its velocity vector is

(A) $(0, \frac{1}{2})$ (B) $(1, \frac{1}{2})$ (C) $(3, \frac{1}{2})$ (D) $(3, \frac{1}{4})$

2. A particle moves in the *xy*-plane so that at any time t its coordinates are $x = t^3 - t^2$ and $y = t + \ln t$. At time $t = 2$, its acceleration vector is

(A) $(4, \frac{1}{2})$ (B) $(6, \frac{1}{4})$ (C) $(8, \frac{3}{4})$ (D) $(10, -\frac{1}{4})$

3. A particle moves in the *xy*-plane so that its position at time $t > 0$ is given by $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$. What is the speed of the particle when $t = 2$?

(A) $\sqrt{2}e$ (B) $\sqrt{2}e^2$ (C) $2e$ (D) $2e^2$

4. If f is a vector-valued function defined by $f(t) = \left(\ln(\sin t),\ t^2 + e^{-t}\right)$, then the acceleration vector is

(A) $(-\csc^2 t,\ 2 + e^{-t})$

(B) $(\sec^2 t,\ 2 + e^{-t})$

(C) $(\csc^2 t,\ 2 - e^{-t})$

(D) $(-\csc^2 t \cdot \cot t,\ 2 + e^{-t})$

5. A particle moves on the curve $y = x + \sqrt{x}$ so that the *x*-component has velocity $x'(t) = \cos t$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(1, 0)$. At time $t = \frac{\pi}{2}$, the particle is at the point

(A) $(0, 0)$ (B) $(1,\ 2)$ (C) $(\frac{\pi}{2},\ \frac{\pi}{2} + \sqrt{\frac{\pi}{2}})$ (D) $(2,\ 2 + \sqrt{2})$

6. In the xy- plane, a particle moves along the curve defined by the equation $y = 2x^4 - x$ with a constant speed of 20 units per second. If $\frac{dy}{dt} > 0$, what is the value of $\frac{dx}{dt}$ when the particle is at the point $(1, 1)$

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

Free Response Questions

7. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where $\frac{dx}{dt} = 1 + \cos(e^t)$. and $\frac{dy}{dt} = e^{(2-t^2)}$ for $t \geq 0$.

(a) At what time t is the speed of the object 3 units per second?

(b) Find the acceleration vector at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 4$.

(d) Find the magnitude of the displacement of the object over the time interval $1 \leq t \leq 4$.

8. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$, with $\frac{dx}{dt} = t - \sin(e^t)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 1$, the value of $\frac{dy}{dt}$ is 3 and the object is at position $(1, 4)$.

(a) Find the x-coordinate of the position of the object at time $t = 5$.

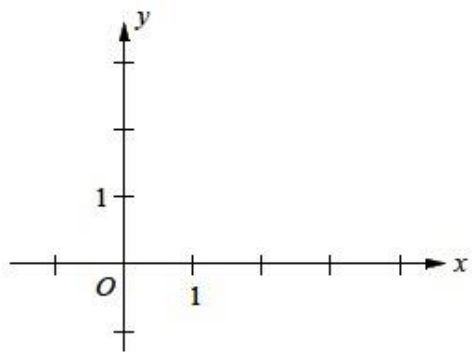
(b) Write an equation for the line tangent to the curve at the point $(x(1), y(1))$.

(c) Find the speed of the object at time $t = 1$.

(d) Suppose the line tangent to the curve at $(x(t), y(t))$ has a slope of $(t - 2)$ for $t \geq 0$. Find the acceleration vector of the object at time $t = 3$.

9. The position of a particle moving in the xy-plane is given by the parametric equations $x(t) = t - \sin(\pi t)$ and $y(t) = 1 - \cos(\pi t)$ for $0 \le t \le 2$.

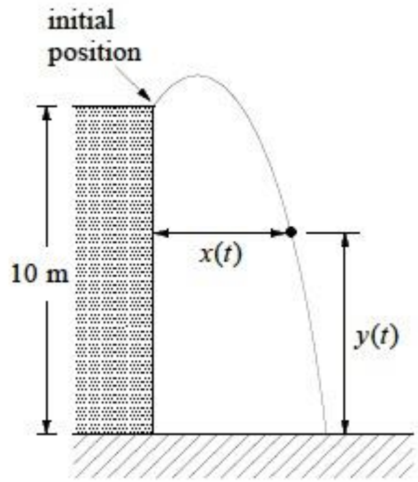
(a) On the axis provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 2$. Indicate the direction of the particle along its path.



(b) Find the position of the particle when $t = 1$.

(c) Find the velocity vector for the particle at any time t.

(d) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance traveled of the particle from $t = 0$ to $t = 2$.



Note: Figure not drawn to scale.

10. An object is thrown upward into the air 10 meters above the ground. The figure above shows the initial position of the object and the position at a later time. At time t seconds after the object is thrown upward, the horizontal distance from the initial position is given by $x(t)$ meters, and the vertical distance from the ground is given by $y(t)$ meters, where $\frac{dx}{dt} = 1.4$ and $\frac{dy}{dt} = 4.2 - 9.8t$, for $t \geq 0$.

(a) Find the time t when the object reaches its maximum height.

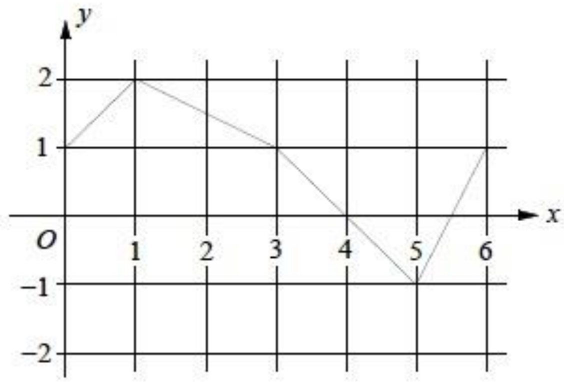
(b) Find the maximum vertical distance from the ground to the object.

(c) Find the time t when the object hit the ground.

(d) Find the total distance traveled by the object from time $t = 0$ until the object hit the ground.

(e) Find the magnitude of the displacement of the object from time $t = 0$ until the object hit the ground.

(f) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the object and the ground at the instance the object hit the ground.



11. At time t, the position of particle moving in the xy- plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = e^{\sqrt{x}} - \cos(x^2)$. The graph of y consisting of four line segments, is shown in the figure above. At time $t = 0$, the particle is at position $(2,1)$.

(a) Find the position of the particle at $t = 2$.

(b) Find the slope of the line tangent to the path of the particle at $t = 2$.

(c) Find the magnitude of the velocity vector at $t = 2$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 3$.

Polar Derivatives

Multiple Choice Questions

1. If $r = \theta - 3\sin\theta$ then $\frac{dr}{d\theta}$ at (π, π) is

   (A) 2 (B) π (C) 4 (D) 2π

2. If $r = \frac{2}{1-\cos\theta}$ then $\frac{dr}{d\theta}$ at $(2, \frac{\pi}{2})$ is

   (A) -2 (B) -1 (C) 0 (D) 1

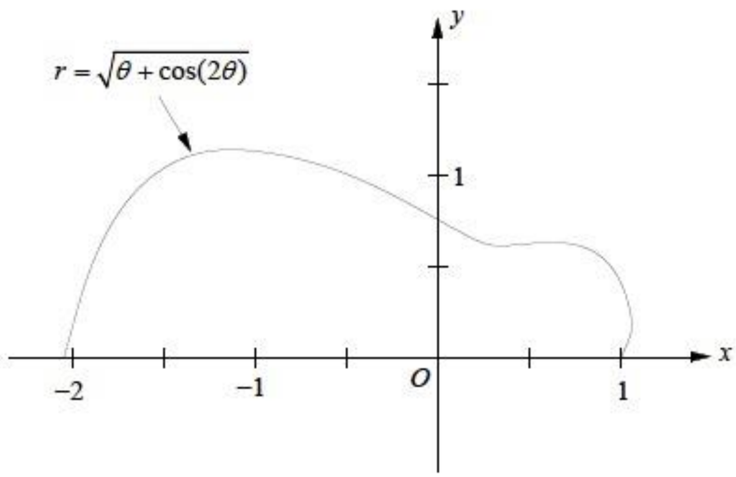
3. If $r = 3\sin\theta$ then $\frac{dy}{dx}$ at the point where $\theta = \frac{\pi}{3}$ is

   (A) -2 (B) $-\sqrt{3}$ (C) -1 (D) $\frac{\sqrt{3}}{3}$

4. The equation of the polar curve is given by $r = \frac{8}{1-\cos\theta}$. What is the angle θ that corresponds to the point on the curve with x-coordinate -3 ?

   (A) 1.248 (B) 1.356 (C) 1.596 (D) 2.214

Free Response Questions



5. The polar curve $r = \sqrt{\theta + \cos(2\theta)}$, for $0 \le \theta \le \pi$, is drawn in the figure above.

(a) Find $\frac{dr}{d\theta}$, the derivative of r with respect to θ.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate 0.5.

(c) For $\frac{\pi}{12} < \theta < \frac{5\pi}{12}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

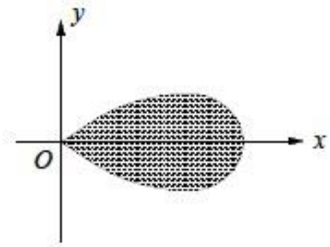
(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that correspond to the point on the curve in the first quadrant with the least distance from the origin. Justify your answer.

Polar Area

Multiple Choice Questions

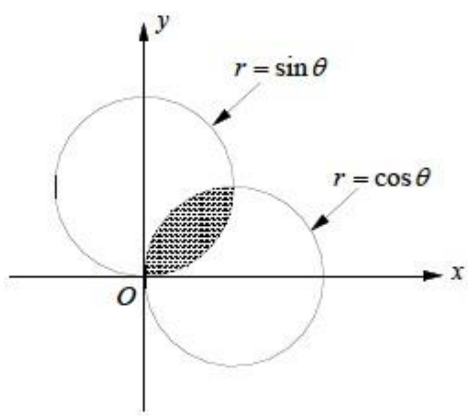
1. The area of the region enclosed by the polar curve $r^2 = 6\sin(2\theta)$ is

(A) 2 (B) 4 (C) 6 (D) 12



2. What is the area of the region enclosed by the loop of the graph of the polar curve $r = 2\cos(2\theta)$ shown in the figure above?

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{4}$ (D) π

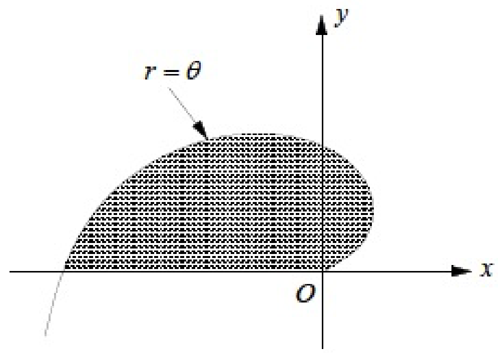


3. The area of the shaded region that lies inside the polar curves $r = \sin\theta$ and $r = \cos\theta$ is

(A) $\frac{1}{8}(\pi - 2)$ (B) $\frac{1}{4}(\pi - 2)$ (C) $\frac{1}{2}(\pi - 2)$ (D) $\frac{1}{8}(\pi - 1)$

4. The area of the region enclosed by the polar curve $r = 2 + \sin\theta$ is

(A) 3π (B) $\frac{7\pi}{2}$ (C) 4π (D) $\frac{9\pi}{2}$



5. The area of the shaded region bounded by the polar curve $r = \theta$ and the x-axis is

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^3}{6}$ (C) $\frac{\pi^3}{3}$ (D) $\frac{\pi^3}{2}$

6. Which of the following gives the area of the region inside the polar curve $r = 1 + \cos\theta$ and outside the polar curve $r = 2\cos\theta$?

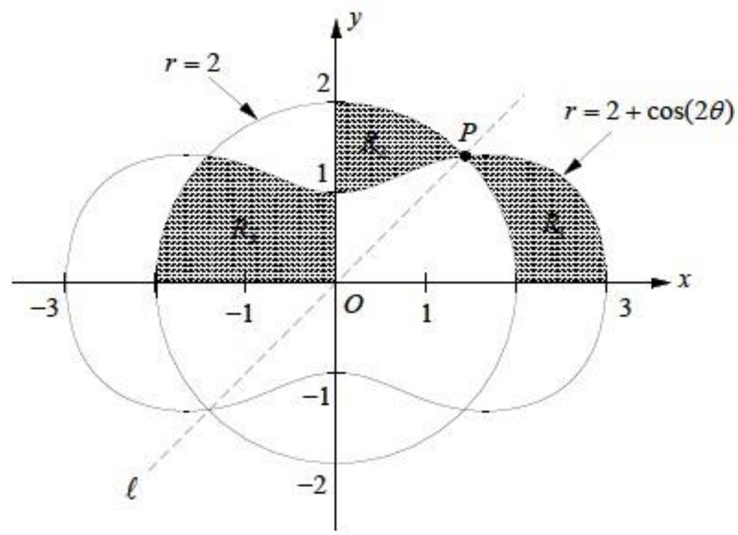
(A) $\frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\, d\theta$

(B) $\int_0^{2\pi} (1+2\cos\theta)^2\, d\theta$

(C) $\int_0^{2\pi} (1+\cos\theta)^2\, d\theta - \pi$

(D) $\frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta - \pi$

Free Response Questions



7. The figure above shows the graphs of the polar curves $r = 2 + \cos(2\theta)$ and $r = 2$. Let R_1 be the shaded region in the first quadrant bounded by the two curves and the x-axis, and R_2 be the shaded region in the first quadrant bounded by the two curves and the y-axis. The graphs intersect at point P in the first quadrant.

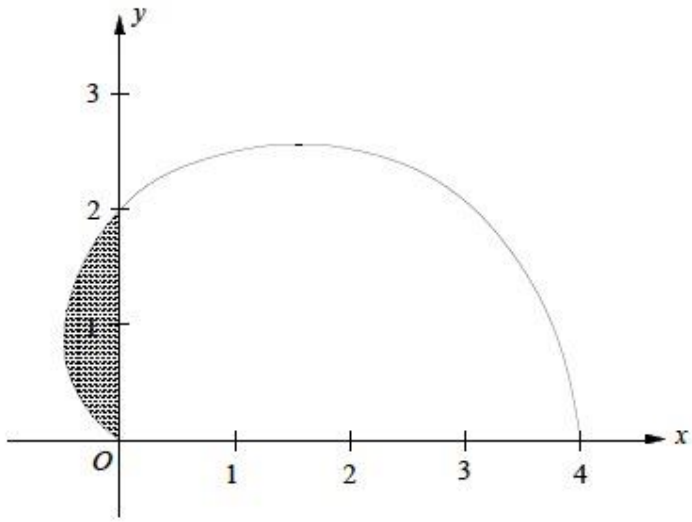
   (a) Find the polar coordinates of point P and write the polar equation for the line ℓ.

   (b) Set up, but do not integrate, an integral expression that represents the area of R_1.

   (c) Set up, but do not integrate, an integral expression that represents the area of R_2.

   (d) Let R_3 be the shaded region in the second quadrant bounded by the two curves and the coordinate axis. Find the area of R_3.

   (e) The distance between the two curves changes for $0 < \theta < \dfrac{\pi}{4}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \dfrac{\pi}{6}$.



8. The graph of the polar curve $r = 2 + 2\cos(\theta)$ for $0 \le \theta \le \pi$ is shown above.

(a) Write an integral expression for the area of the shaded region.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \frac{\pi}{2}$.