Limits and Continuity

Rate of Change:

1. $A_{ROC} = \dfrac{f(40)-f(30)}{40-30} = 0.935$ $\boxed{C}$

2. $A_{ROC} = \dfrac{y(10)-y(0)}{10-0} = 73$ $\boxed{B}$

3. (a) $A_{\substack{\cancel{\text{VELOCITY}}\\ \text{ACCELERATION}}} = \dfrac{f(90)-f(60)}{90-60} = \dfrac{35-74}{30} = \dfrac{-39}{30}\ ft/sec^2$

NOTE: WITH THE INFORMATION PROVIDED YOU CAN'T FIND AVERAGE VELOCITY. YOU CAN FIND AVERAGE ACCELERATION.

(b) $f'(40) \approx \dfrac{f(50)-f(30)}{50-30} = \dfrac{85-67}{20} = \dfrac{18}{20} = \dfrac{9}{10}\ ft/sec^2$

The Limit of a Function and One Sided Limits:

1. $\lim\limits_{x\to \pi/6} \sec^2 x = (\sec \pi/6)^2 = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$ $\boxed{C}$

2. $\lim\limits_{x\to 1} f(x)$ $\boxed{D}$

$\lim\limits_{x\to 1} x^2+3 = 4$

3. $|x-1| = \begin{cases} x-1, & x>1 \\ -(x-1), & x<1 \end{cases}$

$\lim\limits_{x\to 1^-} \dfrac{-(x-1)}{1-x} = 1$

$\lim\limits_{x\to 1^+} \dfrac{x-1}{1-x} = -1$

$\lim\limits_{x\to 1} \dfrac{|x-1|}{1-x} = \text{DNE}$ $\boxed{D}$

4. I. $\lim\limits_{x\to 0} \Rightarrow 3-0^2 \neq 2-0$

II. $\lim\limits_{x\to 2} \Rightarrow 2-2=\sqrt{2-2}$

$\boxed{B}$

III. $\lim\limits_{x\to [illegible]} f(x) = \lim\limits_{x\to 6} f(x) \Rightarrow 2-1 \neq \sqrt{6-2}$

5. $\lim\limits_{x\to -1} \cos\left(f(x)\right) = \cos\left[\lim\limits_{x\to -1} f(x)\right] = \cos(0) = 1$

6. $\lim\limits_{x\to 2^-} f(x) = 2$

7. $\lim\limits_{x\to 2^+} f(x) = 4$

8. $\lim\limits_{x\to 2} f(x) = \text{DNE}$

9. $f(2) = 3$

10. $\lim\limits_{x\to 5^-} \arctan\left(f(x)\right) = \arctan\left[\lim\limits_{x\to 5^-} f(x)\right] = \arctan(1) = \pi/4$

11. $\lim\limits_{x\to 5^+} \left[x \cdot f(x)\right] = \lim\limits_{x\to 5^+} x \cdot \lim\limits_{x\to 5^+} f(x) = 5 \cdot 2 = 10$

Calculating Limits Using the Limit Laws:

1. $\lim\limits_{x \to \pi/3} \dfrac{\sin(\pi/3 - x)}{\pi/3 - x} = 1$ Let $t = \pi/3 - x$. As $x \to \pi/3$, $t \to 0$ $\lim\limits_{t \to 0} \dfrac{\sin(t)}{t} = 1$ $\boxed{D}$

2. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{\sin 2x} \cdot \dfrac{2x}{2x} \cdot \dfrac{3x}{3x} = \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x} \cdot \dfrac{2x}{\sin 2x} \cdot \dfrac{3x}{2x} = \dfrac{3}{2}$ $\boxed{C}$

3. $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{x} \cdot \dfrac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2} = \lim\limits_{x \to 0} \dfrac{4+x-4}{x(\sqrt{4+x} + 2)} = \lim\limits_{x \to 0} \dfrac{x}{x(\sqrt{4+x} + 2)}$

$= \lim\limits_{x \to 0} \dfrac{1}{\sqrt{4+x} + 2} = \dfrac{1}{4}$ $\boxed{B}$

4. $\lim\limits_{x \to 1} \dfrac{\sqrt{3+x} - 2}{x^3 - 1} \cdot \dfrac{\sqrt{3+x} + 2}{\sqrt{3+x} + 2} = \lim\limits_{x \to 1} \dfrac{3+x-4}{(x-1)(x^2+x+1)(\sqrt{3+x} + 2)}$ $\boxed{A}$

$= \lim\limits_{x \to 1} \dfrac{x-1}{(x-1)(x^2+x+1)(\sqrt{3+x} + 2)} = \dfrac{1}{3 \cdot 4} = \dfrac{1}{12}$

5. $\lim\limits_{\theta \to 0} \dfrac{\theta + \theta\cos\theta}{\sin\theta\cos\theta} = \lim\limits_{\theta \to 0} \dfrac{\theta(1+\cos\theta)}{\sin\theta \cdot \cos\theta} = \lim\limits_{\theta \to 0} \dfrac{\theta}{\sin\theta} \cdot \dfrac{1+\cos\theta}{\cos\theta} = 1 \cdot \dfrac{2}{1} = 2$ $\boxed{D}$

6. $\lim\limits_{x \to 0} \dfrac{\tan 3x}{x} = \lim\limits_{x \to 0} \dfrac{\sin 3x}{\cos 3x} \cdot \dfrac{1}{x} \cdot \dfrac{3}{3} = \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x} \cdot \dfrac{3}{\cos 3x} = 1 \cdot \dfrac{3}{1} = 3$ $\boxed{D}$

7. $\lim\limits_{x \to 3} \dfrac{\frac{1}{x} - \frac{1}{3}}{x-3} = \lim\limits_{x \to 3} \dfrac{\frac{3-x}{3x}}{x-3} = \lim\limits_{x \to 3} \dfrac{-(x-3)}{3x(x-3)} = \lim\limits_{x \to 3} \dfrac{-1}{3x} = \dfrac{-1}{9}$ $\boxed{A}$

8. $\lim_{x\to 0} \dfrac{\sqrt{2+ax}-\sqrt{2}}{x} = \sqrt{2}$

$$\lim_{x\to 0} \frac{\sqrt{2+ax}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+ax}+\sqrt{2}}{\sqrt{2+ax}+\sqrt{2}} = \sqrt{2}$$

$$\lim_{x\to 0} \frac{2+ax-2}{x\left(\sqrt{2+ax}+\sqrt{2}\right)} = \sqrt{2}$$

$$\lim_{x\to 0} \frac{a}{\sqrt{2+ax}+\sqrt{2}} = \sqrt{2}$$

$$\frac{a}{2\sqrt{2}} = \sqrt{2} \Rightarrow a = 4$$

9. $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}$$

$$\lim_{h\to 0} \frac{2(x+h)+1-(2x+1)}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}$$

$$\lim_{h\to 0} \frac{2h}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)} = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

10. $\lim_{x\to 0} \dfrac{f(x)-g(x)}{\sqrt{g(x)+7}} = \dfrac{2+3}{\sqrt{-3+7}} = \dfrac{5}{2}$

11. $\lim_{x\to \sqrt{3}} \dfrac{1}{x^2+g(x)} = \dfrac{1}{5}$

$$\lim_{x\to \sqrt{3}} x^2+g(x) = 5$$

$$\lim_{x\to \sqrt{3}} g(x)+3 = 5 \Rightarrow \lim_{x\to \sqrt{3}} g(x) = 2$$

Continuity and IVT:

1. $\lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a)}{(x-a)} = \lim_{x\to a} x+a = 4$ $\boxed{D}$

$$\Downarrow$$

$$2a = 4 \Rightarrow a = 2$$

2. $a = 2$ since there exists a hole at $x=2$ and both sides approach the same point $\boxed{C}$

3. $\lim_{x\to 1} f(x) = a \Rightarrow \frac{1}{2\sqrt{2}} = a \Rightarrow a = \frac{\sqrt{2}}{4}$ $\boxed{B}$

$$\frac{\sqrt{3x-1}-\sqrt{2x}}{x-1} \cdot \frac{\sqrt{3x-1}+\sqrt{2x}}{\sqrt{3x-1}+\sqrt{2x}} = \frac{3x-1-2x}{(x-1)(\sqrt{3x-1}+\sqrt{2x})} = \frac{x-1}{(x-1)(\sqrt{3x-1}+\sqrt{2x})} = \frac{1}{\sqrt{3x-1}+\sqrt{2x}}$$

4. Since $f(-2)>0$ and $f(7)<0$, IVT guarantees some value such that $f(c)=0$, where $-2<c<7$ $\boxed{D}$

5. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x)$ and $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x)$

$\lim_{x\to 0^-} \frac{\pi \sin x}{x} = \lim_{x\to 0^+} a-bx$

$\pi = a$

$\lim_{x\to 1^-} a-bx = \lim_{x\to 1^+} \arctan x$

$a-b = \arctan 1$

$a-b = \pi/4$

$\pi - b = \pi/4$

$b = 3\pi/4$

6. $\lim\limits_{a\to 0} \dfrac{-1+\sqrt{1+a}}{a} \cdot \dfrac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}$

$$\lim_{a\to 0} \frac{1-(1+a)}{a(-1-\sqrt{1+a})} = \lim_{a\to 0} \frac{-a}{a(-1-\sqrt{1+a})} = \frac{-1}{-2} = \frac{1}{2}$$

7. $\lim\limits_{x\to 0} \dfrac{\sqrt{ax+9}-3}{x} = 1$

$$\lim_{x\to 0} \frac{\sqrt{ax+9}-3}{x} \cdot \frac{\sqrt{ax+9}+3}{\sqrt{ax+9}+3} = \lim_{x\to 0} \frac{ax+9-9}{x(\sqrt{ax+9}+3)} = \lim_{x\to 0} \frac{a}{\sqrt{ax+9}+3} = \frac{a}{6} = 1$$

$$\Downarrow$$

$$a = 6$$

Limits and Asymptotes:

1. $\lim\limits_{x\to\infty} \dfrac{3+2x^2-x^4}{3x^4-5} = \lim\limits_{x\to\infty} \dfrac{\frac{3+2x^2-x^4}{x^4}}{\frac{3x^4-5}{x^4}} = \lim\limits_{x\to\infty} \dfrac{\frac{3}{x^4}+\frac{2}{x^2}-1}{3-\frac{5}{x^4}} = \dfrac{-1}{3}$ $\boxed{B}$

2. $\lim\limits_{x\to-\infty} \dfrac{x^3+x-8}{2x^3+3x-1} = \lim\limits_{x\to-\infty} \dfrac{\frac{x^3+x-8}{(-x)^3}}{\frac{2x^3+3x-1}{(-x)^3}} = \lim\limits_{x\to-\infty} \dfrac{-1-\frac{1}{x^2}+\frac{8}{x^3}}{-2-\frac{3}{x^2}+\frac{1}{x^3}} = \dfrac{1}{2}$ $\boxed{C}$

3. $f(x) = \dfrac{x^2+5x+6}{x^2-x-12} = \dfrac{(x+2)(x+3)}{(x-4)(x+3)}$

$\downarrow$

$x=4$ is a vertical asymptote

$\lim\limits_{x\to\infty} \dfrac{x^2+5x+6}{x^2-x-12} = \lim\limits_{x\to\infty} \dfrac{1+\frac{5}{x}+\frac{6}{x^2}}{1-\frac{1}{x}-\frac{12}{x^2}} = 1$ $\boxed{C}$

$y=1$ is a horizontal asymptote

4. If $\lim\limits_{x\to\infty} f(x) = L$, $y = L$ is a horizontal asymptote $\boxed{A}$

5. $f(x) = \dfrac{-3(x^2-1)}{(x-1)^2} = \dfrac{-3(x+1)(x-1)}{(x-1)(x-1)}$

Vertical Asymptote at $x = 1$

$\lim\limits_{x\to\infty} \dfrac{-3(x+1)(x-1)}{(x-1)(x-1)}$ $\boxed{D}$

$\lim\limits_{x\to\infty} \dfrac{-3(x+1)}{x-1} = -3$

Horizontal Asymptote at $y = -3$

6. $\lim\limits_{x\to\infty} \dfrac{6+3e^x}{3-3e^x} = \lim\limits_{x\to\infty} \dfrac{3e^x}{-3e^x} = -1$ $\boxed{C}$

$\frac{\infty}{\infty}$ L'Hops

Horizontal Asymptote $y = -1$

$\lim\limits_{x\to-\infty} \dfrac{6+3e^x}{3-3e^x} = \dfrac{6+0}{3-0} = 2$

Horizontal Asymptote $y = 2$

7. (a) $f(x) = \dfrac{3x-1}{x^3-8} = \dfrac{3x-1}{(x-2)(x^2+2x+4)} \longrightarrow x \neq 2$

$\lim\limits_{x\to 2^+} \dfrac{3x-2}{(x-2)(x^2+2x+4)} = \infty$

$\lim\limits_{x\to 2^-} \dfrac{3x-2}{(x-2)(x^2+2x+4)} = -\infty$

Vertical Asymptote $x = 2$

NOTE: No Vertical Asymptote from x^2+2x+4 since $x^2+2x+4 \neq 0$ ANYWHERE

(b) $\lim\limits_{x\to\infty} \dfrac{3x-1}{x^3-8} = \lim\limits_{x\to\infty} \dfrac{\frac{3x-1}{x^3}}{\frac{x^3-8}{x^3}} = \lim\limits_{x\to\infty} \dfrac{\frac{3}{x^2} - \frac{1}{x^3}}{1 - \frac{8}{x^3}} = \dfrac{0}{1} = 0$

$\lim\limits_{x\to-\infty} \dfrac{\frac{3x-1}{(-x)^3}}{\frac{x^3-8}{(-x)^3}} = \lim\limits_{x\to\infty} \dfrac{\frac{-3}{x^2} + \frac{1}{x^3}}{-1 + \frac{8}{x^3}} = \dfrac{0}{-1} = 0$

Horizontal Asymptote $y = 0$

8. (a) $f(x) = \dfrac{\sin x}{x^2+2x} = \dfrac{\sin x}{x(x+2)} \longrightarrow x \neq 0, -2$

$$\lim_{x\to 0^+} \frac{\sin x}{x(x+2)} = \frac{1}{2}$$

$$\lim_{x\to 0^-} \frac{\sin x}{x(x+2)} = \frac{1}{2}$$

No Vertical Asymptote at $x=0$ (HOLE)

$$\lim_{x\to -2^+} \frac{\sin x}{x(x+2)} = \infty$$

$$\lim_{x\to -2^-} \frac{\sin x}{x(x+2)} = -\infty$$

Vertical Asymptote $x=-2$

(b) $$\lim_{x\to\infty} \frac{\sin x}{x^2+2x} = \lim_{x\to\infty} \frac{\frac{\sin x}{x^2}}{\frac{x^2+2x}{x^2}} = \lim_{x\to\infty} \frac{\frac{\sin x}{x^2}}{1+\frac{2}{x}} = \frac{0}{1} = 0$$

$$\lim_{x\to -\infty} \frac{\frac{\sin x}{(-x)^2}}{\frac{x^2+2x}{(-x)^2}} = \lim_{x\to\infty} \frac{\frac{\sin x}{x^2}}{1+\frac{2}{x}} = \frac{0}{1}$$

$\longrightarrow$ Horizontal Asymptote $y=0$

<u>NOTE:</u> Since $|\sin x| \leq 1$, $\dfrac{\sin x}{x^2}$ must approch 0.