AP Calculus BC Name ANSWERS

Taylor's Theorem Practice

. Find the fourth degree Taylor polynomial, centered at c = 2 for $f(x) = x\ln x$. Use the Taylor polynomial to approximate $f(2.1)$. Apply Taylor's Theorem to determine the maximum possible error associated with this approximation.

$f(x) = x\ln x \qquad f(2) = 2\ln 2$

$f'(x) = x\cdot\frac{1}{x} + \ln x = 1+\ln x \qquad f'(2) = 1+\ln 2$

$f''(x) = \frac{1}{x} \qquad f''(2) = \frac{1}{2}$

$f'''(x) = -\frac{1}{x^2} \qquad f'''(2) = -\frac{1}{4}$

$f^{(4)}(x) = \frac{2}{x^3} \qquad f^{(4)}(2) = \frac{1}{4}$

$f^{(5)}(x) = \frac{-6}{x^4}$

$P_4(x) = (2\ln 2) + (1+\ln 2)(x-2) + \frac{1}{2}\cdot\frac{(x-2)^2}{2!} - \frac{1}{4}\cdot\frac{(x-2)^3}{3!} + \frac{1}{4}\cdot\frac{(x-2)^4}{4!}$

$= \ln 4 + (1+\ln 2)(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{24}(x-2)^3 + \frac{1}{96}(x-2)^4$

$P_4(2.1) = 1.558068454$

Taylor's:

$|R_4(2.1)| \le \left|\frac{(2.1-2)^5}{5!}\right| \cdot \max\left|\frac{-6}{x^4}\right|$

$2 < z < 2.1$

$\max f^{(5)}(x) \to @\ x=2$

$f^{(5)}(2) = \frac{3}{8}$

$|R_4(2.1)| \le \left|\frac{1}{12000000}\right| \cdot \left|\frac{-3}{8}\right|$

$\le \left|\frac{3}{96000000}\right|$

$|R_4(2.1)| \le 0.00000003125$

2. Find the fifth degree Taylor polynomial, centered at c = 0 for $f(x) = e^{2x}$. Use the Taylor polynomial to approximate $f(0.75)$. Apply Taylor's Theorem to determine the maximum possible error associated with this approximation.

$f(x) = e^{2x} \qquad f(0) = 1$

$f'(x) = 2e^{2x} \qquad f'(0) = 2$

$f''(x) = 4e^{2x} \qquad f''(0) = 4$

$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$

$f^{(4)}(x) = 16e^{2x} \qquad f^{(4)}(0) = 16$

$f^{(5)}(x) = 32e^{2x} \qquad f^{(5)}(0) = 32$

$f^{(6)}(x) = 64e^{2x}$

$P_5(x) = 1 + 2x + 4\frac{x^2}{2!} + 8\frac{x^3}{3!} + 16\frac{x^4}{4!} + 32\frac{x^5}{5!}$

$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5$

$P_5(0.75) = 4.46171875$

Taylor's:

$|R_5(0.75)| \le \left|\frac{(.75)^6}{6!}\right| \cdot \max\left|64e^{2x}\right|$

$0 < z < .75$

$\max f^{(6)}(x) \to @\ x = .75$

$f^{(6)}(.75) = 64e^{1.5}$

$|R_5(0.75)| \le \left|\frac{81}{327680}\right| \cdot \left|64e^{1.5}\right|$

$|R_5(0.75)| \le 0.070901721$

3. Find the third degree Taylor polynomial, centered at c = 0 for $f(x) = \arctan(x)$. Use the Taylor polynomial to approximate $f(0.4)$. Apply Taylor's Theorem to determine the maximum possible error associated with this approximation.

$f(x) = \arctan x$ $\quad f(0) = 0$

$f'(x) = (1+x^2)^{-1}$ $\quad f'(0) = 1$

$f''(x) = -2x(1+x^2)^{-2}$ $\quad f''(0) = 0$

$f'''(x) = 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2} = \frac{6x^2-2}{(1+x^2)^3}$ $\quad f'''(0) = -2$

$f^{(4)}(x) = -48x^3(1+x^2)^{-4} + 16x(1+x^2)^{-3} + 4(1+x^2)^{-3}$

$= \frac{-24x^3+24x}{(x^2+1)^4}$

$P_3(x) = 1 \cdot x - 2 \cdot \frac{x^3}{3!}$

$= x - \frac{1}{3}x^3$

$P_3(0.4) = .378\overline{6}$

Taylor's:

$|R_3(0.4)| \le \left|\frac{(0.4)^4}{4!}\right| \cdot \max\left|\frac{-24x^3+24x}{(x^2+1)^4}\right|$

$0 < z < 0.4$

max $f^{(4)}(x)$ @ $x = 0.4$

$f^{(4)}(0.4) = \frac{3150000}{707281}$

$|R_3(0.4)| \le \left|\frac{2}{1875}\right| \cdot \left|\frac{3150000}{707281}\right|$

$\le \left|\frac{3360}{707281}\right|$

$|R_3(0.4)| \le .004751$

4. Find the sixth degree Taylor polynomial, centered at c = 1 for $f(x) = \frac{1}{x}$. Use the Taylor polynomial to approximate $f(1.2)$. Apply Taylor's Theorem to determine the maximum possible error associated with this approximation.

$f(x) = x^{-1}$ $\quad f(1) = 1$

$f'(x) = -x^{-2}$ $\quad f'(1) = -1$

$f''(x) = 2x^{-3}$ $\quad f''(1) = 2$

$f'''(x) = -6x^{-4}$ $\quad f'''(1) = -6$

$f^{(4)}(x) = 24x^{-5}$ $\quad f^{(4)}(1) = 24$

$f^{(5)}(x) = -120x^{-6}$ $\quad f^{(5)}(1) = -120$

$f^{(6)}(x) = 720x^{-7}$ $\quad f^{(6)}(1) = 720$

$f^{(7)}(x) = -5040x^{-8}$

$P_6(x) = 1 - (x-1) + 2 \cdot \frac{(x-1)^2}{2!} - 6 \cdot \frac{(x-1)^3}{3!} + 24 \cdot \frac{(x-1)^4}{4!} - 120 \cdot \frac{(x-1)^5}{5!} + 720 \frac{(x-1)^6}{6!}$

$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5 + (x-1)^6$

$P_6(1.2) = .833344$

Taylor's:

$|R_6(1.2)| \le \left|\frac{(1.2-1)^7}{7!}\right| \cdot \max\left|\frac{-5040}{x^8}\right|$

$1 < z < 1.2$

max $f^{(7)}(x)$ @ $x = 1$

$f^{(7)}(1) = -5040$

$|R_6(1.2)| \le \left|\frac{1}{393750000}\right| \cdot |-5040|$

$\le \frac{1}{78125}$

$|R_6(1.2)| \le 0.0000128$