AP Calculus

Techniques of Integration

Integration Techniques

Multiple Choice Questions

1. $\int \frac{1+\sin x}{\cos^2 x}\,dx =$

(A) $\tan x - \sec x \tan x + C$

(B) $\tan x + \sec x + C$

(C) $\tan x + \sec^2 x + C$

(D) $\ln(1+\cos^2 x) + C$

2. $\int \frac{e^{2x}}{1+e^x}\,dx =$

(A) $e^{2x} + \ln(1+e^x) + C$

(B) $e^{2x} - \ln(1+e^x) + C$

(C) $2e^{2x} - \ln(1+e^x) + C$

(D) $e^x - \ln(1+e^x) + C$

3. $\int 2\tan x \ \ln(\cos x)\,dx =$

(A) $\cos x \ [\ln(\cos x)] + C$

(B) $\sin x [\ln(\cos x)] + C$

(C) $-[\ln(\cos x)]^2 + C$

(D) $[\ln(\sin x)]^2 + C$

4. $\int_2^3 \frac{1}{x^2-4x+5}\,dx =$

(A) $\frac{\pi}{4}$ (B) $1-\frac{\pi}{4}$ (C) $1+\frac{\pi}{6}$ (D) $1+\frac{\pi}{4}$

5. $\int \frac{2x}{x^2+2x+1}\,dx =$

(A) $-\operatorname{arccot} x-\frac{1}{x+1}+C$

(B) $\arctan x+\frac{1}{x+1}+C$

(C) $2\ln|x+1|-\frac{2}{(x+1)^2}+C$

(D) $2\ln|x+1|+\frac{2}{x+1}+C$

Free Response Questions

6. The region bounded by $y=\frac{\sin x}{\sqrt{\cos x}}$, $x=0$, $x=\frac{\pi}{4}$, and the x-axis is revolved around the x-axis. What is the volume of the resulting solid?

Trigonometric Substitutions

Trigonometric Substitution

1. For integrals involving $\sqrt{a^2-u^2}$, let $u = a\sin\theta$. Then

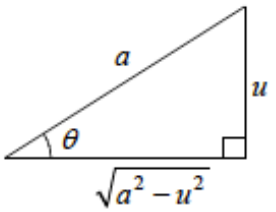
$$\sqrt{a^2-u^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

2. For integrals involving $\sqrt{a^2+u^2}$, let $u = a\tan\theta$. Then

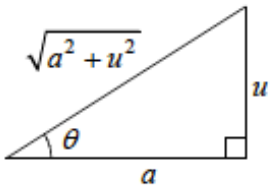
$$\sqrt{a^2+u^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$$

3. For integrals involving $\sqrt{u^2-a^2}$, let $u = a\sec\theta$. Then

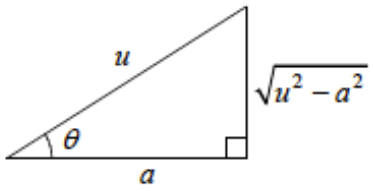
$$\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta-a^2+} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$$



$u = a\sin\theta$



$u = a\tan\theta$



$u = a\sec\theta$

Note: $\operatorname{arcsec} x = \arccos\frac{1}{x}$ $\qquad$ $\operatorname{arccsc} x = \arcsin\frac{1}{x}$

Multiple Choice Questions

1. If the substitution $x = 2\tan\theta$ is made in $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the resulting integral is

(A) $4\int \tan^2\theta\sec\theta\,d\theta$

(B) $4\int \tan^2\theta\sec^2\theta\,d\theta$

(C) $8\int \tan^3\theta\,d\theta$

(D) $8\int \tan^3\theta\sec\theta\,d\theta$

2. $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx =$

(A) $\frac{\pi}{18}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

3. $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx =$

(A) $-\frac{\sqrt{25-x^2}}{5x^2}+C$

(B) $-\frac{\sqrt{25-x^2}}{25}+C$

(C) $-\frac{\sqrt{25-x^2}}{25x}+C$

(D) $\frac{\sqrt{25-x^2}}{25x^2}+C$

4. If the substitution $x=\sec\theta$ is made in $\int \frac{\sqrt{x^2-1}}{x^4}\,dx$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, the resulting integral is

(A) $\int \sec^2\theta\tan\theta\,d\theta+C$

(B) $\int \sec\theta\tan^2\theta\,d\theta+C$

(C) $\int \sin\theta\cos^2\theta\,d\theta+C$

(D) $\int \sin^2\theta\cos\theta\,d\theta+C$

4. If the substitution $x = \sec\theta$ is made in $\int \frac{\sqrt{x^2-1}}{x^4}\,dx$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the resulting integral is

(A) $\int \sec^2\theta\tan\theta\,d\theta + C$

(B) $\int \sec\theta\tan^2\theta\,d\theta + C$

(C) $\int \sin\theta\cos^2\theta\,d\theta + C$

(D) $\int \sin^2\theta\cos\theta\,d\theta + C$

5. The average value of $f(x) = \frac{4}{\sqrt{9+x^2}}$ on the interval $[0,4]$ is

(A) $\ln 2$ (B) $\ln(\sqrt{2}-1)$ (C) $\ln 3$ (D) $\ln(\sqrt{2}+1)$

Free Response Questions

6. Let f be the function given by $f(x) = (9-x^2)^{3/2}$ on the closed interval $[0,3]$.

(a) Find the average value of f on the closed interval $[0,3]$.

(b) Substitute $x = 3\sin\theta$ for f. Set up, but do not integrate, an integral expression in terms of θ for the average value of f on the closed interval $[0,3]$.

Partial Fractions

Multiple Choice Questions

1. $\int \frac{dx}{x^2+x-6} =$

(A) $\frac{1}{5}\ln\left|\frac{x-1}{x+6}\right|+C$

(B) $\frac{1}{5}\ln\left|\frac{x+3}{x-2}\right|+C$

(C) $\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|+C$

(D) $\frac{1}{5}\ln\left|(x-2)(x+3)\right|+C$

2. $\int_4^7 \frac{5}{(x-2)(2x+1)}\,dx =$

(A) $\ln\frac{9}{10}$ (B) $\ln\frac{10}{9}$ (C) $\ln\frac{3}{2}$ (D) $\ln\frac{9}{4}$

3. $\int \frac{x}{x^2+5x+6}\,dx =$

(A) $-2\ln\left|x+2\right|+3\ln\left|(x+3)\right|+C$

(B) $2\ln\left|x+2\right|+3\ln\left|(x+3)\right|+C$

(C) $2\ln\left|(x+3)\right|-3\ln\left|x+2\right|+C$

(D) $-2\ln\left|(x+3)\right|-3\ln\left|x+2\right|+C$

4. $\int \frac{2e^{2x}}{(e^x-1)(e^x+1)}\,dx =$

(A) $\ln\left|e^x(e^{2x}-1)\right|+C$

(B) $\ln\left|2e^x(e^{2x}-1)\right|+C$

(C) $\ln\left|\frac{1}{e^{2x}-1}\right|+C$

(D) $\ln\left|(e^x-1)(e^x+1)\right|+C$

Free Response Questions

5. Let f be the function given by $f(\theta)=\int \frac{\sin\theta}{\cos\theta(\cos\theta-1)}\,d\theta$.

(a) Substitute $x=\cos\theta$ and write an integral expression for f in terms of x.

(b) Use the method of partial fractions to find $f(\theta)$.

Integration by Parts

Integration by Parts Formula

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du\ .$$

Multiple Choice Questions

1. $\int x\sin(2x)\,dx =$

(A) $-x\cos(2x)+\frac{1}{2}\sin(2x)+C$

(B) $\frac{x}{2}\cos(2x)-\frac{1}{4}\sin(2x)+C$

(C) $-\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+C$

(D) $\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+C$

2. $\int_0^2 xe^x\,dx =$

(A) e^2-1 (B) e^2+1 (C) $e-1$ (D) $e+1$

3. If $\int x^2\cos(3x)\,dx = f(x)-\frac{2}{3}\int x\sin(3x)\,dx$, then $f(x) =$

(A) $\frac{2}{3}x\sin(3x)$

(B) $\frac{1}{3}x^2\sin(3x)$

(C) $\frac{2}{3}x\cos(3x)$

(D) $\frac{1}{3}x\sin(3x)-\frac{2}{3}\cos(3x)$

4. $\int x^2 \ln x \, dx =$

(A) $\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

(B) $x^3 \ln x - \frac{x^3}{3} + C$

(C) $\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$

(D) $\frac{x(\ln x)^2}{2} - \frac{x^3}{3} + C$

5. $\int_0^{\pi/4} x \sec^2 x \, dx =$

(A) $\frac{\pi}{4} - \ln 2$ (B) $\frac{\pi}{4} + \ln 2$ (C) $\frac{\pi}{4} - \frac{\ln 2}{2}$ (D) $\frac{\pi}{4} + \frac{\ln 2}{2}$

6. $\int \sec^3 x \, dx =$

(A) $\frac{1}{4}\sec^4 x + C$

(B) $\frac{1}{2}\sec^2 x \tan x + \frac{1}{2}\ln|\sec x| + C$

(C) $\frac{1}{2}\sec^2 x \tan x + \frac{1}{2}\ln|\tan x| + C$

(D) $\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

7. $\int f(x)\cos(nx)\,dx =$

(A) $\frac{1}{n}f(x)\sin(nx)-\frac{1}{n}\int f'(x)\sin(nx)\,dx$

(B) $\frac{1}{n}f(x)\cos(nx)-\frac{1}{n}\int f'(x)\cos(nx)\,dx$

(C) $nf(x)\cos(nx)+\frac{1}{n}\int f'(x)\sin(nx)\,dx$

(D) $nf(x)\cos(nx)-\frac{1}{n}\int f'(x)\cos(nx)\,dx$

8. If $\int \arccos x\,dx = x\arccos x+\int f(x)\,dx$, then $f(x) =$

(A) $-x\sqrt{1-x^2}$ (B) $x\sqrt{1-x^2}$ (C) $-\frac{1}{\sqrt{1-x^2}}$ (D) $\frac{x}{\sqrt{1-x^2}}$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	−2	3	4	−1
3	2	−1	−3	5

9. The table above gives values of f, f', g, and g' for selected values of x.
If $\int_1^3 f(x)g'(x)\,dx = 8$, then $\int_1^3 f'(x)g(x)\,dx =$

(A) −4 (B) −1 (C) 5 (D) 8

Free Response Questions

10. Find the area of the region bounded by $y = \arcsin x$, $y = 0$, and $x = 1$. Show the work that leads to your answer.

Improper Integrals

1. $\int_2^{\infty} \frac{1}{\sqrt{x-1}}\,dx =$

(A) $-\infty$ (B) -2 (C) 1 (D) ∞

2. $\int_0^{\infty} \frac{1}{(x+3)(x+4)}\,dx =$

(A) $-\ln\frac{4}{3}$ (B) $-\ln\frac{3}{4}$ (C) 0 (D) $\ln 4$

3. $\int_0^{4} \frac{dx}{(x-1)^{2/3}} =$

(A) $3\sqrt[3]{3}$ (B) $3(1-\sqrt[3]{3})$ (C) $3(1+\sqrt[3]{3})$ (D) divergent

4. $\int_0^{\infty} x^2 e^{-x^3} =$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) divergent

5. $\int_0^{1} \frac{\ln x}{\sqrt{x}}\,dx =$

(A) -6 (B) -4 (C) -2 (D) divergent

6. If $\int_0^1 \frac{ke^{-\sqrt{x}}}{\sqrt{x}}\,dx = 1$, what is the value of k?

(A) $-\frac{1}{2}$ (B) $\frac{e}{2}$ (C) $\frac{1}{2}$ (D) There is no such value of k

Free Response Questions

7. Let f be the function given by $f(x) = \frac{x}{\sqrt{x^2+1}}\,dx$.

(a) Show that the improper integral $\int_1^\infty f(x)\,dx$ is divergent.

(b) Find the average value of f on the interval $[1,\infty)$.