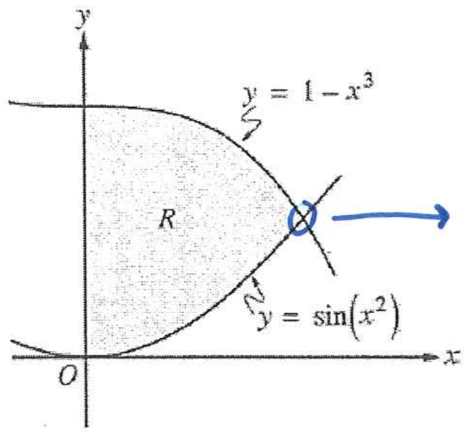


(A, B)

$1-x^3 = \sin(x^2) \Rightarrow (0.764972, 0.552352)$

1. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = 1 - x^3$ and $y = \sin(x^2)$, as shown in the figure above.

(a) Find the area of R.

$$\text{Area} = \int_0^A \left[(1-x^3) - \sin(x^2)\right] dx = 0.533 \quad (0.534)$$

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin(x^2)$ intersect. Find k and determine whether this line divides R into two regions of equal area. Show the work that leads to your conclusion. $K = B$

$y = 1 - x^3 \Rightarrow x = \sqrt[3]{1-y}$ $\quad y = \sin(x^2) \Rightarrow x = \sqrt{\arcsin(y)}$

$$\int_0^A (1 - x^3 - k)\, dx = 0.257$$

$$\int_0^A (k - \sin(x^2))\, dx = 0.277$$

The regions have unequal areas

OR

$$\int_0^K \sqrt{\arcsin y}\; dy = 0.277$$

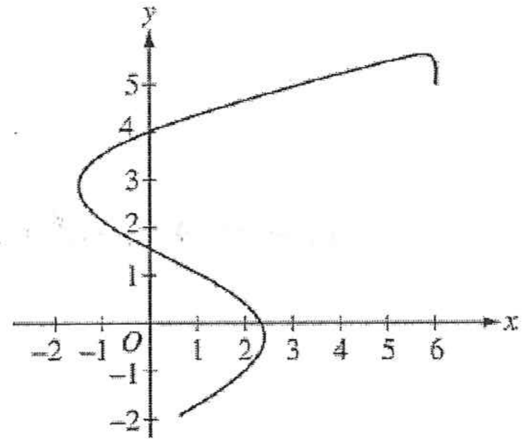
$$\int_K^1 \sqrt[3]{1-y}\; dy = 0.257$$

Unequal areas.

(c) Find the volume of the solid generated when R is revolved about the line $y = -3$.

$$\text{Volume} = \pi \int_0^A \left[(1 - x^3 - -3)^2 - (\sin(x^2) - -3)^2\right] dx = 11.841 \quad (11.840)$$

Free Response Section 1: Calculator Active



2. A planetary rover travels on a flat surface. The path of the rover for the time interval $0 \le t \le 2$ hours is shown in the rectangular coordinate system above. The rover starts at the point with coordinates $(6, 5)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position of the rover change at rates given by

$$x'(t) = -12\sin\left(2t^2\right)$$
$$y'(t) = 10\cos\left(1 + \sqrt{t}\right),$$

where $x(t)$ and $y(t)$ are measured in meters and t is measured in hours.

(a) Find the acceleration vector of the rover at time $t = 1$. Find the speed of the rover at time $t = 1$.

$a(1) = \langle x''(1), y''(1) \rangle$
$= \langle 19.975, -4.546 \rangle$

Speed $= \sqrt{[x'(1)]^2 + [y'(1)]^2} = 11.678$

(b) Find the total distance that the rover travels over the time interval $0 \le t \le 1$.

Distance $= \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = 6.704$
(6.703)

(c) Find the y-coordinate of the position of the rover at time $t = 1$.

$y(1) = 5 + \int_0^1 y'(t)\, dt = 4.057$
(4.056)

(d) The rover receives a signal at each point where the line tangent to its path has slope $\frac{1}{2}$. At what times t, for $0 \le t \le 2$, does the rover receive a signal?

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \Rightarrow \frac{10\cos(1+\sqrt{t})}{-12\sin(2t^2)} = \frac{1}{2} \Rightarrow t = 1.072$

Free Response Section 2: Non Calculator

t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

3. The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in gigaliters (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \le t \le 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

(a) Use the tangent line approximation to W at time $t = 30$ to predict the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

Tangent: $y - 125 = 0.5(t - 30)$

$$W(32) \approx 0.5(32-30) + 125 = 126$$

(b) Use a left Riemann sum, with the three subintervals indicated by the data in the table, to approximate $\int_0^{30} W'(t)\,dt$. Use this approximation to estimate the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 0$. Show the computations that lead to your answer.

$$\int_0^{30} W'(t)dt \approx 10 \cdot W'(0) + 12 \cdot W'(10) + 8 \cdot W'(22) = 22.4$$

$$\int_0^{30} W'(t)dt = W(30) - W(0) \Rightarrow W(0) = W(30) - \int_0^{30} W'(t)dt$$

$$W(0) = 125 - 22.4 = 102.6$$

(c) Explain why there must be at least one time t, other than $t = 10$, such that $W'(t) = 0.7$ GL/day.

W' differentiable $\Rightarrow$ W' continuous

$W'(30) = 0.5 < 0.7$

$W'(22) = 1.0 > 0.7$

By IVT, there must be at least one t, on $22 \le t \le 30$, such that $W'(t) = 0.7$.

(d) The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometers per day, with respect to t when $t = 30$ days.

$$\frac{dA}{dt} = (0.3) \cdot \frac{2}{3} W^{-1/3} \cdot \frac{dW}{dt} = \frac{1}{5\sqrt[3]{W}} \cdot \frac{dW}{dt}$$

$$\left.\frac{dA}{dt}\right|_{t=30} = \frac{1}{5\sqrt[3]{125}} \cdot \frac{1}{2} = \frac{1}{50}$$

5. Let f be the function satisfying $f'(x) = 4x - 2xf(x)$ for all real numbers x, with $f(0) = 5$ and $\lim_{x\to\infty} f(x) = 2$.

(a) Find the value of $\int_0^\infty (4x - 2xf(x))\, dx$. Show the work that leads to your answer.

$$\int_0^\infty (4x - 2x\, f(x))\, dx = \int_0^\infty f'(x)\, dx$$

$$= \lim_{b\to\infty} \int_0^b f'(x)\, dx = \lim_{b\to\infty} \left[f(x)\right]_0^b$$

$$= \lim_{b\to\infty} f(b) - f(0)$$

$$= 2 - 5 = -3$$

(b) Use Euler's method to approximate $f(-1)$, starting at $x = 0$, with two steps of equal size.

$$f(\tfrac{-1}{2}) = 5 + \tfrac{-1}{2}\left[4(0) - 2(0)\cdot f(0)\right] = 5 + \tfrac{-1}{2}\cdot 0 = 5$$

$$f(-1) = 5 + \tfrac{-1}{2}\left[4(\tfrac{-1}{2}) - 2(\tfrac{-1}{2})\cdot f(\tfrac{-1}{2})\right] = 5 + \tfrac{-1}{2}\cdot 3 = \tfrac{7}{2}$$

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 4x - 2xy$ with the initial condition $f(0) = 5$.

$$\frac{dy}{dx} = x(4-2y)$$

$$\frac{1}{4-2y}\, dy = x\, dx$$

$$-\tfrac{1}{2}\ln|4-2y| = \tfrac{1}{2}x^2 + C$$

$$-\tfrac{1}{2}\ln|-6| = C$$

$$-\tfrac{1}{2}\ln(6) = C$$

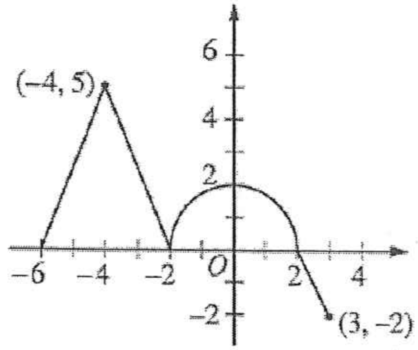
$$-\tfrac{1}{2}\ln|4-2y| = \tfrac{1}{2}x^2 - \tfrac{1}{2}\ln|6|$$

$$\ln|4-2y| = -x^2 + \ln(6)$$

$$4-2y = \pm e^{-x^2+\ln(6)} = \pm 6e^{-x^2}$$

$$y = 2 \pm 3e^{-x^2}$$

If $f(0) = 5 \Longrightarrow y = 2 + 3e^{-x^2}$



Graph of f

5. The graph of the continuous function f, consisting of three line segments and a semicircle, is shown above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$.

(a) Find $g(-6)$ and $g(3)$.

$g(-6) = \int_{-2}^{-6} f(t)dt = -\int_{-6}^{-2} f(t)dt = -\left[\frac{1}{2}\cdot 4\cdot 5\right] = -10$

$g(3) = \int_{-2}^{3} f(t)dt = \frac{1}{2}\pi(2)^2 + -\left[\frac{1}{2}\cdot 1\cdot 2\right] = 2\pi - 1$

(b) Find $g'(0)$.

$g'(x) = f(x)$

$g'(0) = f(0) = 2$

(c) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a horizontal tangent. Determine whether g has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

Horizontal Tangents $\Rightarrow g'(x) = f(x) = 0 \Rightarrow x = -2, 2$

At $x = -2$, g has no local extrema since $g'(x)$ does not change sign.

At $x = 2$, g has a local maxima since $g'(x)$ changes from positive to negative.

(d) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a point of inflection. Explain your reasoning.

$g''(x) = f'(x)$ $\quad g''(x) = 0 \Rightarrow x = 0$ $\quad g''(x) = \text{undefined} \Rightarrow x = -4, -2, 2$

POI at $x = -4, -2, 0$ since $g''(x)$ changes sign at these values.

At $x = -4$ & $x = 0$, $g''(x)$ changes from positive to negative.

At $x = -2$, $g''(x)$ changes from negative to positive.

Free Response Section 2: Non Calculator

6. The function f satisfies the equation

$$f'(x) = f(x) + x + 1$$

and $f(0) = 2$. The Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

(a) Write an equation for the line tangent to the curve $y = f(x)$ at the point where $x = 0$.

$f'(0) = f(0) + 0 + 1 = 3$

Tangent: $y - 2 = 3x$

(b) Find $f''(0)$ and find the second-degree Taylor polynomial for f about $x = 0$.

$f''(x) = f'(x) + 1$

$f''(0) = f'(0) + 1 = 4$

$P_2(x) = 2 + 3 \cdot x + 4 \cdot \frac{x^2}{2!} = 2 + 3x + 2x^2$

(c) Find the fourth-degree Taylor polynomial for f about $x = 0$.

$f'''(x) = f''(x) \Rightarrow f'''(0) = 4$

$f^{(4)}(x) = f'''(x) \Rightarrow f^{(4)}(0) = 4$

$P_4(x) = 2 + 3x + 4 \cdot \frac{x^2}{2!} + 4 \cdot \frac{x^3}{3!} + 4 \cdot \frac{x^4}{4!}$

$= 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4$

(d) Find $f^{(n)}(0)$, the nth derivative of f at $x = 0$, for $n \geq 2$. Use the Taylor series for f about $x = 0$ and the Taylor series for e^x about $x = 0$ to find a polynomial expression for $f(x) - 4e^x$.

$f^{(n)}(0) = 4$ for $n \geq 2$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$f(x) = 2 + 3x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \cdots$

$4e^x = 4 + 4x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \cdots$

$f(x) - 4e^x = -2 - x$