Differentiation

Definition of Derivatives and the Power Rule :

1. $\lim\limits_{x \to 0} \frac{\sqrt[3]{8+h} - 2}{h} \Rightarrow f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$

$f'(8) = \frac{1}{3(8)^{2/3}} = \frac{1}{12}$

$\boxed{A}$

2. $\lim\limits_{h \to 0} \frac{(2+h)^5 - 32}{h} \Rightarrow f(x) = x^5$

$f'(x) = 5x^4$

$\boxed{B}$

3. I. $\lim\limits_{x \to 1^-} 1-2x = \lim\limits_{x \to 1^+} -x^2$

$-1 = -1$

II. $f(1) = -1 = \lim\limits_{x \to 1} f(x)$

III. $\lim\limits_{x \to 1^-} f'(x) = \lim\limits_{x \to 1^+} f'(x)$

$\lim\limits_{x \to 1^-} -2 = \lim\limits_{x \to 1^+} -2x$

$-2 = -2$

$\boxed{D}$

4. $f(x) = -\sqrt[3]{x^2} = -x^{2/3}$

$f'(x) = -\frac{2}{3} x^{-1/3} = \frac{-2}{3\sqrt[3]{x}}$

$f'(-1) = \frac{-2}{-3} = \frac{2}{3}$

$\boxed{D}$

5. A derivative can't exist at a point where a function is not continuous. $\boxed{C}$

6. I. Limit definition for $f'(x)$ at $x=1$ $\boxed{C}$

II. Alternate definition for $f'(x)$ at $x=1$.

III. Limit definition for $f'(x)$ at any x value.

7. f' is not defined at discontinuities and sharp points. $\boxed{B}$

8. For f to be differentiable, f must be continuous at $x=1$ and the one sided derivatives must be equal at $x=1$.

<u>Continuous</u>:

$$\lim_{x\to 1^-} mx^2-2 = \lim_{x\to 1^+} K\sqrt{x}$$

$$m-2 = K$$

$$m-2 = 4m$$

$$-\frac{2}{3} = m$$

<u>Differentiable</u>:

$$\lim_{x\to 1^-} 2mx = \lim_{x\to 1^-} \frac{K}{2\sqrt{x}}$$

$$2m = \frac{K}{2}$$

$$4m = K$$

$$-\frac{8}{3} = K$$

9. $\lim_{h\to 0} = \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[f(x)+x^3 \cdot h - xh^3 - f(h)] - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{x^3 \cdot h - xh^3 - f(h)}{h}$

$= \lim_{h\to 0} \dfrac{x^3 h - xh^3}{h} - \dfrac{f(h)}{h}$

$= \lim_{h\to 0} x^3 - xh^2 - \dfrac{f(h)}{h}$

$f'(x) = x^3 - 1$

10. (a) $\lim_{x\to 0^-} f'(x) = 1$

(b) $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} x+2 = 2$

(c) No, because $\lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x)$. This causes the graph of f to have a sharp point at $x=0$.

(d) Continuous $\Rightarrow$ $\lim_{x\to 0^-} x+2 = \lim_{x\to 0^+} a(x+b)^2$

$2 = ab^2 \longrightarrow a = \dfrac{2}{b^2}$

Differentiable $\Rightarrow$ $\lim_{x\to 0^-} 1 = \lim_{x\to 0^+} 2a(x+b)$ $\qquad a = \dfrac{2}{4^2}$

$1 = 2ab$ $\qquad a = \dfrac{1}{8}$

$1 = 2\left(\dfrac{2}{b^2}\right)\cdot b$

$1 = \dfrac{4}{b}$

$b = 4$

Product, Quotient, & Higher Order:

1. $f(x) = (x^3 - 2x + 5)(x^{-2} + x^{-1})$

$\boxed{A}$

$f'(x) = (x^3 - 2x + 5)(-2x^{-3} - x^{-2}) + (3x^2 - 2)(x^{-2} + x^{-1})$

$f'(1) = (4)(-3) + (1)(2) = -10$

2. $f(x) = \dfrac{x^{1/2} - 1}{x^{1/2} + 1}$

$$f'(x) = \frac{(x^{1/2}+1)(\frac{1}{2}x^{-1/2}) - (x^{1/2}-1)(\frac{1}{2}x^{-1/2})}{(x^{1/2}+1)^2} = \frac{\frac{1}{2} + \frac{1}{2}x^{-1/2} - \frac{1}{2} + \frac{1}{2}x^{-1/2}}{(x^{1/2}+1)^2}$$

$\boxed{C}$

$$= \frac{x^{-1/2}}{(x^{1/2}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$$

3. $\dfrac{d}{dx}\left[\dfrac{g(x)}{x^2}\right] = \dfrac{x^2 \cdot g'(x) - g(x) \cdot 2x}{x^4}$

$\boxed{B}$

$$\left.\frac{d}{dx}\right|_{x=2} = \frac{4(-1) - (3) \cdot 4}{16} = \frac{-16}{16} = -1$$

4. $f(x) = \dfrac{x}{x - \frac{a}{x}} = \dfrac{x}{\frac{x^2 - a}{x}} = \dfrac{x^2}{x^2 - a}$

$$f'(x) = \frac{(x^2-a)2x - x^2(2x)}{(x^2-a)^2} = \frac{2x^3 - 2ax - 2x^3}{(x^2-a)^2} = \frac{-2ax}{(x^2-a)^2}$$

$\boxed{B}$

$$f'(1) = \frac{-2a}{(1-a)^2} = \frac{1}{2} \Rightarrow -4a = (1-a)^2$$

$-4a = 1 - 2a + a^2$

$0 = a^2 + 2a + 1$

$0 = (a+1)^2$

$a = -1$

5. $y = 4x^{1/2} - 16x^{1/4}$

$y' = 2x^{-1/2} - 4x^{-3/4}$

$y'' = -x^{-3/2} + 3x^{-7/4} = \frac{-1}{x^{3/2}} + \frac{3}{x^{7/4}} = \frac{-1}{x^{3/2}\cdot x^{7/4}}\cdot x^{7/4} + \frac{3}{x^{7/4}\cdot x^{3/2}}\cdot x^{3/2}$

$= \frac{-x^{7/4} + 3x^{3/2}}{x}$

$= \frac{x^{3/2}(-x^{1/4}+3)}{x}$

$= \frac{-x^{1/4}+3}{x^{7/4}} = \frac{-\sqrt[4]{x}+3}{x\cdot\sqrt[4]{x^3}}$

$\boxed{C}$

6. $y = x^2\cdot f(x)$

$y' = x^2\cdot f'(x) + 2xf(x)$

$y'' = x^2\cdot f''(x) + 2xf'(x) + 2xf'(x) + 2f(x) = x^2f''(x) + 4xf'(x) + 2f(x)$

$\boxed{D}$

7. $f(x) = \frac{1}{2}x^6 - 10x^3 + 12x$

$f'(x) = 3x^5 - 30x^2 + 12$

$f''(x) = 15x^4 - 60x$

$f'''(x) = 60x^3 - 60 \rightarrow f'''(x) = 0 \Rightarrow 60(x^3-1) = 0$

$x = 1$

$f(1) = \frac{1}{2} - 10 + 12 = \frac{5}{2}$

$\boxed{D}$

8. $h(x) = x \cdot f(x) \cdot g(x)$

$h'(x) = x\,f(x) \cdot g'(x) + x g(x) \cdot f'(x) + f(x) g(x)$

$h'(1) = (1)(-2)(\frac{1}{2}) + (1)(3)(1) + (-2)(3)$

$= -1 + 3 - 6 = -4$

9. $g(x) = \dfrac{x}{\sqrt{x} - 1}$

$$g'(x) = \frac{(x^{1/2} - 1) - x\left(\frac{1}{2}x^{-1/2}\right)}{(\sqrt{x} - 1)^2} = \frac{x^{1/2} - 1 - \frac{1}{2}x^{1/2}}{(\sqrt{x} - 1)^2} = \frac{\frac{1}{2}x^{1/2} - 1}{(\sqrt{x} - 1)^2} = \frac{x^{1/2} - 2}{2(\sqrt{x} - 1)^2}$$

$$g''(x) = \frac{2(x^{1/2} - 1)^2\left(\frac{1}{2}x^{-1/2}\right) - (x^{1/2} - 2)\left(4(x^{1/2} - 1) \cdot \frac{1}{2}x^{-1/2}\right)}{4(\sqrt{x} - 1)^4}$$

$$= \frac{x^{-1/2}(x^{1/2} - 1)^2 - 2x^{-1/2}(x^{1/2} - 2)(x^{1/2} - 1)}{4(\sqrt{x} - 1)^4}$$

$$= \frac{x^{-1/2}(x^{1/2} - 1)\left[x^{1/2} - 1 - 2(x^{1/2} - 2)\right]}{4(\sqrt{x} - 1)^4}$$

$$= \frac{3 - x^{1/2}}{4\sqrt{x}(\sqrt{x} - 1)^3}$$

$$g''(4) = \frac{3 - \sqrt{4}}{4\sqrt{4}(\sqrt{4} - 1)^3} = \frac{1}{8(1)^3} = 1/8$$

Chain Rule & Composite Functions :

1. $f(x) = \left(x + x^{1/2}\right)^{1/2}$

$f'(x) = \frac{1}{2}\left(x + x^{1/2}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)$ $\boxed{D}$

$$= \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}} = \frac{\frac{2\sqrt{x}+1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}} = \frac{2\sqrt{x}+1}{4\sqrt{x}\cdot\sqrt{x+\sqrt{x}}} = \frac{2\sqrt{x}+1}{4\sqrt{x^2+x\sqrt{x}}}$$

2. $f(x) = \left(x^2 - 3x\right)^{3/2}$ $\boxed{D}$

$f'(x) = \frac{3}{2}\left(x^2-3x\right)^{1/2}(2x-3)$

$f'(4) = \frac{3}{2}\sqrt{16-12}\,(8-3) = \frac{3}{2}(2)(5) = 15$

3. $\frac{d}{dx}\left[\frac{f}{g\cdot h}\right] = \frac{g\cdot h\cdot f' - f\left[g\cdot h' + h\cdot g'\right]}{[g\cdot h]^2} = \frac{ghf' - fgh' - fhg'}{g^2\cdot h^2}$ $\boxed{C}$

4. $f(x) = \left(3 - \sqrt{x}\right)^{-1}$

$f'(x) = -\left(3-\sqrt{x}\right)^{-2}\cdot -\frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-1/2}\left(3-\sqrt{x}\right)^{-2}$ $\boxed{A}$

$f''(x) = \frac{1}{2}x^{-1/2}\cdot -2\left(3-\sqrt{x}\right)^{-3}\cdot\frac{-1}{2}x^{-1/2} + \left(3-\sqrt{x}\right)^{-2}\cdot -\frac{1}{4}x^{-3/2}$

$$= \frac{1}{2x\left(3-\sqrt{x}\right)^3} - \frac{1}{4\sqrt{x^3}\left(3-\sqrt{x}\right)^2}$$

$f''(4) = \frac{1}{8(1)^3} - \frac{1}{32(1)^2} = \frac{3}{32}$

5. $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(1) = 1$

6. $h(x) = x \cdot f(x^2)$

$h'(x) = x \cdot f'(x^2) \cdot 2x + f(x^2)$

$= 2x^2 \cdot f'(x^2) + f(x^2)$

$h'(2) = 13$

7. $h(x) = \dfrac{f(x)}{\sqrt{g(x)}}$

$h'(3) = \dfrac{13}{16}$

$h'(x) = \dfrac{\sqrt{g(x)} \cdot f'(x) - f(x) \cdot \frac{1}{2}[g(x)]^{-1/2} \cdot g'(x)}{g(x)}$

8. $h(x) = [f(2x)]^2$

$h'(2) = 20$

$h'(x) = 2[f(2x)] \cdot f'(2x) \cdot 2$

$= 4 \cdot f(2x) \cdot f'(2x)$

9. $h(x) = (x^9 + f(x))^{-2}$

$h'(1) = -\dfrac{5}{16}$

$h'(x) = -2(x^9 + f(x))^{-3}(9x^8 + f'(x))$

10. (a) $\frac{d}{dx}\left[f(g(x)) = 2x\right] \Rightarrow f'(g(x)) \cdot g'(x) = 2$

$$g'(x) = \frac{2}{f'(g(x))}$$

(b) $f'(x) = 1 + [f(x)]^2$

$f'(g(x)) = 1 + [f(g(x))]^2$

$= 1 + [2x]^2$

$= 1 + 4x^2$

$$g'(x) = \frac{2}{f'(g(x))} = \frac{2}{1+4x^2}$$

Derivatives of Trigonometric Functions:

1. $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{3}+h\right) - \frac{1}{2}}{h}$ $\rightarrow$ Definition of the derivative for $f(x) = \cos x$ @ $x = \pi/3$

$f'(x) = -\sin x \quad f'(\pi/3) = -\sqrt{3}/2$

$\boxed{B}$

2. $\lim_{h\to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$ $\rightarrow$ Definition of the derivative for $f(x) = \sin(2x)$

$f'(x) = \cos(2x) \cdot 2$

$\boxed{C}$

3. $f(x) = \sin(\cos(2x))$

$f'(x) = \cos(\cos(2x)) \cdot -\sin(2x) \cdot 2$

$f'(\pi/4) = -2\cos(0) \cdot 1 = -2$

$\boxed{D}$

4. $y = a\sin x + b\cos x$ $\qquad y + y'' = a\sin x + b\cos x - a\sin x - b\cos x = 0$

$y' = a\cos x - b\sin x$

$y'' = -a\sin x - b\cos x$

$\boxed{A}$

5. $\frac{d}{dx} \sec^2(\sqrt{x}) = \frac{d}{dx}\left[\sec(x^{1/2})\right]^2 = 2\sec(x^{1/2}) \cdot \sec(x^{1/2})\tan(x^{1/2}) \cdot \frac{1}{2}x^{-1/2}$

$= \dfrac{\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

$\boxed{C}$

6. $\frac{d}{dx}\left[x^2\cos 2x\right] = x^2 \cdot -\sin(2x) \cdot 2 + \cos(2x) \cdot 2x$

$= -2x^2\sin(2x) + 2x\cos(2x)$

$= 2x\left[-x\sin(2x) + \cos(2x)\right]$

$\boxed{B}$

7. $f(\theta) = \cos\pi - \frac{1}{2}\sec\theta + \frac{1}{3}\cot\theta$

$f'(\theta) = -\frac{1}{2}\sec\theta\tan\theta + \frac{1}{3} \cdot -\csc^2\theta$

$f'(\pi/6) = -\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{3}\left(2\right)^2 = \frac{-5}{3}$

CORRECT ANSWER NOT PRESENT

8. $h(x) = f(x) \cdot g(\tan x)$

$h'(x) = f(x) \cdot g'(\tan x) \cdot \sec^2 x + g(\tan x) \cdot f'(x)$

$h'(\pi/4) = (-2) \cdot (\sqrt{2}) \cdot \left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{3}{2}\right) \cdot (2) = -4\sqrt{2} + 3$

9. $\lim_{x \to \pi^-} \sin x = \lim_{x \to \pi^+} ax+b$ and $\lim_{x \to \pi^-} \cos x = \lim_{x \to \pi^+} a$

$0 = a\pi + b$ $\quad\quad -1 = a$

$0 = -\pi + b$

$\pi = b$

Derivatives of Exponential & Logarithmic:

1. $\lim_{h \to 0} \frac{\frac{1}{2}[\ln(e+h) - 1]}{h}$ → Definition of the derivative for $f(x) = \frac{1}{2} \ln x = \ln \sqrt{x}$ at $x = e$ $\quad \boxed{C}$

2. $f(x) = e^{\tan x}$ $\quad \boxed{C}$

$f'(x) = e^{\tan x} \cdot \sec^2 x$

$f'(\pi/4) = e^1 \cdot \left(\frac{2}{\sqrt{2}}\right)^2 = 2e$

3. $y = \ln(\cos x)$ $\boxed{A}$

$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$

4. $y = x^x \Rightarrow \ln y = x \ln x$ $\boxed{B}$

$\frac{1}{y} \cdot y' = x \cdot \frac{1}{x} + \ln x$

$\frac{1}{y} \cdot y' = 1 + \ln x$

$y' = y(1 + \ln x) = x^x(1 + \ln x)$

5. $y = e^{(x^2+1)^{1/2}}$ $\boxed{D}$

$y' = e^{(x^2+1)^{1/2}} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

6. $y = (\sin x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(\sin x)$ $\boxed{D}$

$\frac{1}{y} \cdot y' = \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \cos x + \ln(\sin x) \cdot \frac{-1}{x^2}$

$\frac{1}{y} \cdot y' = \frac{\cot x}{x} - \frac{\ln(\sin x)}{x^2}$

$y' = y\left[\frac{x\cot x - \ln(\sin x)}{x^2}\right]$

$y' = (\sin x)^{1/x}\left[\frac{x\cot x - \ln(\sin x)}{x^2}\right]$

7. $f(x) = \ln\left[\sec(\ln x)\right]$

$$f'(x) = \frac{1}{\sec(\ln x)} \cdot \sec(\ln x) \cdot \tan(\ln x) \cdot \frac{1}{x}$$

$$= \frac{\tan(\ln x)}{x}$$

$$f'(e) = \frac{\tan(1)}{e}$$

$\boxed{C}$

8. $y = x^{\ln\sqrt{x}} \Longrightarrow \ln y = \frac{1}{2}\ln x \cdot \ln x$

$$\frac{1}{y} \cdot y' = \frac{1}{2}\ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2} \cdot \frac{1}{x}$$

$$\frac{1}{y} \cdot y' = \frac{\ln x}{2x} + \frac{\ln x}{2x}$$

$$\frac{1}{y} \cdot y' = \frac{\ln x}{x}$$

$$y' = y \cdot \frac{\ln x}{x} = x^{\ln\sqrt{x}} \cdot \frac{\ln x}{x}$$

$\boxed{B}$

9. $f(x) = xe^x$

$f'(x) = xe^x + e^x = (x+1)e^x$

$f''(x) = xe^x + e^x + e^x = (x+2)e^x$

$\vdots$

$f^{(10)}(x) = (x+10)e^x$

$n = 10$

10. $h(x) = e^{f(x)}$

$h'(x) = e^{f(x)} \cdot f'(x)$

$h''(x) = e^{f(x)} \cdot f''(x) + f'(x) \cdot e^{f(x)} \cdot f'(x)$

$h''(x) = e^{f(x)}\left[f''(x) + [f'(x)]^2 \right] = e^{f(x)}\left[1+x^2\right]$

$f''(x) + [f'(x)]^2 = 1+x^2 \Rightarrow f'(x) = x$

Tangent Lines & Normal Lines :

1. $y = x\sqrt{3+x^2}$ $\boxed{C}$

$\frac{dy}{dx} = x \cdot \frac{1}{2}(3+x^2)^{-1/2} \cdot 2x + (3+x^2)^{1/2} = \frac{x^2}{\sqrt{3+x^2}} + \sqrt{3+x^2}$ $\quad y-2 = \frac{5}{2}(x-1)$

$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{1}{2} + 2 = \frac{5}{2}$

2. $f(x) = x^2 - x$ $\boxed{C}$

$f'(x) = 2x-1$ $\quad 2x-1 = 3$, $x = 2$ $\quad f(2) = 2$ $\quad y-2 = 3(x-2)$

3. $\frac{dy}{dx} = 2x + x^{-2} \Rightarrow y = x^2 - x^{-1} + C \Rightarrow C = 3$ $\boxed{B}$

Note: This question is in the wrong topic. Should be with integration

4. $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$ $\qquad \frac{dy}{dx}\Big|_{(\pi/6,\ \frac{1}{\sqrt{3}})} = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$

Tangent Line:

$y - \frac{1}{\sqrt{3}} = \frac{4}{3}\left(x - \frac{\pi}{6}\right)$

Normal Line (Perpendicular):

$y - \frac{1}{\sqrt{3}} = -\frac{3}{4}\left(x - \frac{\pi}{6}\right)$

$\boxed{C}$

5. $2x + 3y = 4 \Rightarrow y = -\frac{2}{3}x + \frac{4}{3} \Rightarrow f'(-1) = \frac{3}{2}$

$\boxed{D}$

6. $f(x) = x^4 - x$

$f'(x) = 4x^3 - 1$

$4x^3 - 1 = -\frac{1}{2}$

$4x^3 = \frac{1}{2}$

$x^3 = 1/8 \Rightarrow x = \frac{1}{2}$

$f(\frac{1}{2}) = -\frac{7}{16}$

$2x - y = k \Rightarrow y = 2x - k$

$-\frac{7}{16} = 2(\frac{1}{2}) - k$

$k = \frac{23}{16}$

$\boxed{A}$

7. (a) $y' = 1 - \frac{1}{60}x$

$p(x, y) \Rightarrow p(x, x - \frac{x^2}{120})$

Slope of l $= \dfrac{(x - \frac{x^2}{120}) - (0)}{x - -15} = \dfrac{x - \frac{x^2}{120}}{x + 15} = 1 - \frac{1}{60}x$

$\dfrac{\frac{120x - x^2}{120}}{x + 15} = \dfrac{60 - x}{60}$

$\dfrac{120x - x^2}{120x + 1800} = \dfrac{60 - x}{60}$

$7200x - 60x^2 = -120x^2 + 5400x + 108000$

$60x^2 + 1800x - 108000 = 0$

$x^2 + 30x - 1800 = 0$

$(x + 60)(x - 30) = 0$

$\cancel{x = -60}$ $\quad x = 30$

$\downarrow$

Value in first quadrant

7. (b) $x = 30 \quad y = 30 - \frac{900}{120} = \frac{45}{2}$

$$y - \frac{45}{2} = \frac{1}{2}(x - 30)$$

$$y'(30) = 1 - \frac{30}{60} = \frac{1}{2}$$

(c) $Q \Rightarrow$ Horizontal Tangent

$$y' = 0 \Rightarrow 1 - \frac{x}{60} = 0 \Rightarrow x = 60 \Rightarrow Q(60, 30)$$

$$f(60) = 30$$

$R \Rightarrow$ Same x value as Q

$$\text{length } \overline{QR} = \frac{75}{2} - 30 = \frac{15}{2}$$

$$y - \frac{45}{2} = \frac{1}{2}(60 - 30) \Rightarrow y = \frac{75}{2}$$

Implicit Differentiation :

1. $3xy + x^2 - 2y^2 = 2$

$$3\left[x\frac{dy}{dx} + y\right] + 2x - 4y \cdot \frac{dy}{dx} = 0 \overset{@\,(1,1)}{\Longrightarrow} 3\left[\frac{dy}{dx} + 1\right] + 2 - 4 \cdot \frac{dy}{dx} = 0$$

$$3\frac{dy}{dx} + 3 + 2 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 5$$

$\boxed{A}$

2. $3x^4 - x^2 - y^2 = 0$

$$12x^3 - 2x - 2y \cdot \frac{dy}{dx} = 0 \overset{@\,(1,\sqrt{2})}{\Longrightarrow} 12(1)^3 - 2(1) - 2\sqrt{2} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{10}{2\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

$\boxed{C}$

3. $x^2y + 2xy^2 = 5x$

$x^2 \cdot \frac{dy}{dx} + y \cdot 2x + 2x \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 2 = 5$

$\boxed{B}$

$x^2 \frac{dy}{dx} + 2xy + 4xy \frac{dy}{dx} + 2y^2 = 5$

$\frac{dy}{dx}\left[x^2 + 4xy\right] = 5 - 2xy - 2y^2$

$\frac{dy}{dx} = \frac{5 - 2xy - 2y^2}{x^2 + 4xy}$

4. $xy + \tan(xy) = \pi$

$x \cdot \frac{dy}{dx} + y + \sec^2(xy) \cdot \left[x \cdot \frac{dy}{dx} + y\right] = 0$

$\boxed{D}$

$x \frac{dy}{dx} + y + x\sec^2(xy) \frac{dy}{dx} + y\sec^2(xy) = 0$

$\frac{dy}{dx}\left[x + x\sec^2(xy)\right] = -y - y\sec^2(xy)$

$\frac{dy}{dx} = \frac{-y\left(1 + \sec^2(xy)\right)}{x\left(1 + \sec^2(xy)\right)} = -\frac{y}{x}$

5. $3y^2 - x^3 - xy^2 = 7$

$6y \cdot \frac{dy}{dx} - 3x^2 - \left[x \cdot 2y \cdot \frac{dy}{dx} + y^2\right] = 0$

$y - 2 = \frac{7}{8}(x - 1)$

$6y \frac{dy}{dx} - 3x^2 - 2xy \frac{dy}{dx} - y^2 = 0$

$\boxed{D}$

$\frac{dy}{dx}\left[6y - 2xy\right] = 3x^2 + y^2$

$\frac{dy}{dx} = \frac{3x^2 + y^2}{6y - 2xy}$ $\quad \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3+4}{12-4} = \frac{7}{8}$

6. $2x^2+3y^2=5$

$$4x+6y\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-2x}{3y} \quad \left.\frac{dy}{dx}\right|_{(1,1)}=-\frac{2}{3}$$

$\boxed{B}$

$$y-1=\frac{3}{2}(x-1)$$

7. $x+\sin y=y+3$

$$1+\cos y\cdot\frac{dy}{dx}=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{1-\cos y}=(1-\cos y)^{-1}$$

$\boxed{C}$

$$\frac{d^2y}{dx^2}=-(1-\cos y)^{-2}\cdot\sin y\cdot\frac{dy}{dx}=\frac{-\sin y}{(1-\cos y)^2}\cdot\frac{1}{1-\cos y}=\frac{-\sin y}{(1-\cos y)^3}$$

8. (a) $x^3-xy+y^2=3$

$$3x^2-\left[x\frac{dy}{dx}+y\right]+2y\frac{dy}{dx}=0$$

$$3x^2-x\frac{dy}{dx}-y+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{y-3x^2}{2y-x} \text{ OR } \frac{3x^2-y}{x-2y}$$

(b) x-coordinate of 1 $\Rightarrow$ $1-y+y^2=3$

$$y^2-y-2=0$$

$$(y-2)(y+1)=0$$

$$y=2 \quad y=-1$$

$(1,2)$ & $(1,-1)$

$$\left.\frac{dy}{dx}\right|_{(1,2)}=\frac{2-3}{4-1}=\frac{-1}{3}$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)}=\frac{-1-3}{-2-1}=\frac{-4}{-3}=\frac{4}{3}$$

@ $(1,2) \Rightarrow y-2=-\frac{1}{3}(x-1)$

@ $(1,-1) \Rightarrow y+1=\frac{4}{3}(x-1)$

8. (c) Horizontal Tangent $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow y - 3x^2 = 0$

$$y = 3x^2$$

$$x^3 - xy + y^2 = 3 \Rightarrow x^3 - 3x^3 + 9x^4 = 3$$

$$9x^4 - 2x^3 - 3 = 0 \Rightarrow x = -0.710, 0.822$$

9. (a) $x^2 + y^2 - xy = 7$

$$2x + 2y\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\right] = 0$$

$$2x + 2y\frac{dy}{dx} - x\frac{dy}{dx} + -y = 0 \Longrightarrow \frac{dy}{dx} = \frac{y-2x}{2y-x} \text{ or } \frac{2x-y}{x-2y}$$

(b) x coordinate of 2 $\Rightarrow 4 + y^2 - 2y = 7$

$$y^2 - 2y - 3 = 0$$

$$(y-3)(y+1) = 0$$

$$y = 3 \quad y = -1$$

(2,3) & (2,-1)

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3-4}{6-2} = \frac{-1}{4}$$

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{-1-4}{-2-2} = \frac{5}{4}$$

@ (2,3) $\Rightarrow y - 3 = -1/4(x-2)$

@ (2,-1) $\Rightarrow y + 1 = 5/4(x-2)$

(c) Vertical Tangent $\Rightarrow \frac{dy}{dx}$ = undefined $\Rightarrow 2y - x = 0$

$$2y = x$$

$$y = \tfrac{1}{2}x$$

$$x^2 + y^2 - xy = 7 \Rightarrow x^2 + \tfrac{1}{4}x^2 - \tfrac{1}{2}x^2 = 7$$

$$\tfrac{3}{4}x^2 = 7$$

$$x^2 = \tfrac{28}{3} \Longrightarrow x = \pm\sqrt{\tfrac{28}{3}} = \pm\frac{2\sqrt{21}}{3}$$

Derivatives of Inverse Functions:

1. $g(x) = f^{-1}(x)$

$$g'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$\boxed{C}$

$$g'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(4)} = \frac{2}{3}$$

2. $$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(-3)} = \frac{4}{3}$$

$\boxed{B}$

If $f(-3) = 2 \rightarrow f^{-1}(2) = -3$

3. $f^{-1}(2) = \underline{1}$

$\downarrow$

$f(x) = x^3 - x + 2 = 2$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x = 0 \quad x = \pm 1$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{2}$$

$\boxed{A}$

$f'(x) = 3x^2 - 1$

4. $f(x) = \sin x$

$f^{-1}\left(\frac{\sqrt{3}}{2}\right) = \underline{\pi/3}$

$\downarrow$

$f(x) = \sin x = \frac{\sqrt{3}}{2}$

$x = \pi/3$

$$(f^{-1})'\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)} = \frac{1}{f'(\pi/3)} = \frac{1}{\cos(\pi/3)} = 2$$

$\boxed{D}$

$f'(x) = \cos x$

5. $f^{-1}(2) = \underline{e}$

$\downarrow$

$f(x) = 1 + \ln x = 2$

$\ln x = 1$

$x = e$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(e)} = e$$

$f'(x) = \frac{1}{x}$

$\boxed{D}$

6. (a) $f^{-1}(-1) = \underline{2}$

$\downarrow$

$f(x) = -1 \Rightarrow x = 2$

$$(f^{-1})'(-1) = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(2)} = \frac{1}{4}$$

$$y - 2 = \frac{1}{4}(x+1)$$

(b) $h(1) = f(g(1)) = f(-1) = 3$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(1) = f'(g(1)) \cdot g'(1) = f'(-1) \cdot 2 = -4$

(c) $h^{-1}(3) = \underline{1}$

$\downarrow$

$h(x) = 3 \Rightarrow x = 1$

$$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))} = \frac{1}{h'(1)}$$

$$= -\frac{1}{4}$$

Derivatives of Inverse Trig:

1. $\frac{d}{dx}\left[\arcsin x^2\right] = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$ $\boxed{D}$

2. $f(x) = \arctan\left(e^{-x}\right)$ $\boxed{C}$

$f'(x) = \frac{1}{1+(e^{-x})^2} \cdot e^{-x} \cdot -1 = \frac{-e^{-x}}{1+e^{-2x}}$

$f'(-1) = \frac{-e}{1+e^2}$

3. $f(x) = \arctan(\sin x)$ $\boxed{A}$

$f'(x) = \frac{1}{1+\sin^2 x} \cdot \cos x = \frac{\cos x}{1+\sin^2 x}$

$f'(\pi/3) = \frac{1/2}{1+\left(\sqrt{3}/2\right)^2} = \frac{1/2}{7/4} = \frac{2}{7}$

4. $y = \cos\left(\sin^{-1} x\right)$ $\boxed{B}$

$y' = -\sin\left(\sin^{-1} x\right) \cdot \frac{1}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$

5. (a) $f(x) = x^{\tan^{-1}x} \Rightarrow \ln[f(x)] = \tan^{-1}x \cdot \ln x$

$$\frac{1}{f(x)} \cdot f'(x) = \tan^{-1}x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{1+x^2}$$

$$f'(x) = f(x)\left[\frac{\tan^{-1}x}{x} + \frac{\ln x}{1+x^2}\right]$$

$$f'(x) = x^{\tan^{-1}x}\left[\frac{\tan^{-1}x}{x} + \frac{\ln x}{1+x^2}\right]$$

(b) $f'(1) = (1)^{\tan^{-1}(1)}\left[\frac{\tan^{-1}(1)}{1} + \frac{\ln 1}{1+1^2}\right] = \frac{\pi}{4}$

$f(1) = 1^{\tan^{-1}(1)} = 1$

$y - 1 = \frac{\pi}{4}(x-1)$

Approximating a Derivative:

1. $f'(3.5) \approx \frac{f(3.8) - f(3.3)}{3.8 - 3.3} = \frac{32.5 - 26.1}{0.5} = 12.8$

ANSWER CHOICE NOT PRESENT

2. (a) $A_{ROC} = \frac{F(6) - F(1)}{6-1} = \frac{88 - -8}{5} = 19.5$ °F/month

(b) $F'(4) \approx \frac{F(5) - F(3)}{5-3} = \frac{72 - 25}{2} = 11.75$ °F/month

(c) $F'(t) = -52\cos\left(\frac{\pi t}{6} - 5\right) \cdot \frac{\pi}{6} = \frac{-26\pi}{3}\cos\left(\frac{\pi t}{6} - 5\right)$

$$F'(4) = \frac{-26\pi}{3}\cos\left(\frac{2\pi}{3} - 5\right) = 26.472 \text{ °F/month}$$