Differentiation

Definition of Derivatives and the Power Rule

1. $\lim_{h \to 0} \frac{\sqrt[3]{8+h}-2}{h} =$

(A) $\frac{1}{12}$ (B) $\frac{1}{4}$ (C) $\frac{\sqrt[3]{2}}{2}$ (D) $\sqrt[3]{2}$ (E) 2

2. $\lim_{h \to 0} \frac{(2+h)^5 - 32}{h}$ is

(A) $f'(5)$, where $f(x) = x^2$

(B) $f'(2)$, where $f(x) = x^5$

(C) $f'(5)$, where $f(x) = 2^x$

(D) $f'(2)$, where $f(x) = 2^x$

$$f(x) = \begin{cases} 1-2x, & \text{if } x \le 1 \\ -x^2, & \text{if } x > 1 \end{cases}$$

3. Let f be the function given above. Which of the following must be true?

I. $\lim_{x \to 1} f(x)$ exists.

II. f is continuous at $x = 1$.

III. f is differentiable at $x = 1$.

(A) I only

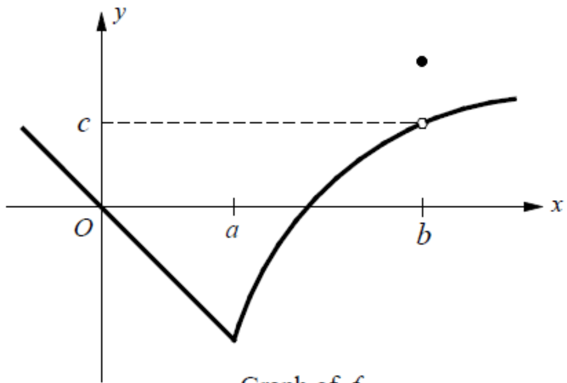
(B) I and II only

(C) II and III only

(D) I, II, and III

4. What is the instantaneous rate of change at $x=-1$ of the function $f(x)=-\sqrt[3]{x^2}$?

(A) $-\frac{2}{3}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$



Graph of f

5. The graph of a function f is shown in the figure above. Which of the following statements must be false?

(A) $f(x)$ is defined for $0 \le x \le b$.

(B) $f(b)$ exists.

(C) $f'(b)$ exists.

(D) $\lim_{x \to a^-} f'(x)$ exists.

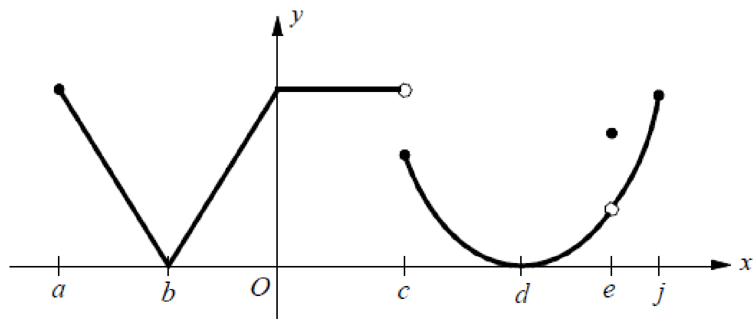
6. If f is a differentiable function, then $f'(1)$ is given by which of the following?

I. $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

II. $\lim_{x \to 1} \frac{f(x)-f(1)}{x-1}$

III. $\lim_{x \to 0} \frac{f(x+h)-f(x)}{h}$

(A) I only (B) II only (C) I and II only (D) I and III only



Graph of f

7. The graph of a function f is shown in the figure above. At how many points in the interval $a < x < j$ is f' not defined?

(A) 3 (B) 4 (C) 5 (D) 6

Free Response Questions

8. Let f be the function defined by $f(x) = \begin{cases} mx^2 - 2 & \text{if } x \le 1 \\ k\sqrt{x} & \text{if } x > 1 \end{cases}$. If f is differentiable at $x = 1$, what are the values of k and m?

9. Let f be a function that is differentiable throughout its domain and that has the following properties.

(1) $f(x+y) = f(x) + x^3 y - xy^3 - f(y)$

(2) $\lim_{x \to 0} \frac{f(x)}{x} = 1$

Use the definition of the derivative to show that $f'(x) = x^3 - 1$.

10. Let f be the function defined by

$$f(x) = \begin{cases} x+2 & \text{for } x \le 0 \\ \dfrac{1}{2}(x+2)^2 & \text{for } x > 0. \end{cases}$$

(a) Find the left-hand derivative of f at $x = 0$.

(b) Find the right-hand derivative of f at $x = 0$.

(c) Is the function f differentiable at $x = 0$? Explain why or why not.

(d) Suppose the function g is defined by

$$g(x) = \begin{cases} x+2 & \text{for } x \le 0 \\ a(x+b)^2 & \text{for } x > 0, \end{cases}$$

where a and b are constants. If g is differentiable at $x = 0$, what are the values of a and b?

Products, Quotients, and Higher Derivatives

1. If $f(x)=(x^3-2x+5)(x^{-2}+x^{-1})$, then $f'(1)=$

(A) -10 (B) -6 (C) $-\dfrac{9}{2}$ (D) $\dfrac{7}{2}$

2. If $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$ then $f'(x)=$

(A) $\dfrac{\sqrt{x}}{(\sqrt{x}+1)^2}$

(B) $\dfrac{x}{(\sqrt{x}+1)^2}$

(C) $\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

(D) $\dfrac{\sqrt{x}-1}{\sqrt{x}(\sqrt{x}+1)^2}$

3. If $g(2)=3$ and $g'(2)=-1$, what is the value of $\dfrac{d}{dx}\left(\dfrac{g(x)}{x^2}\right)$ at $x=2$?

(A) -3 (B) -1 (C) 0 (D) 2

4. If $f(x)=\dfrac{x}{x-\dfrac{a}{x}}$ and $f'(1)=\dfrac{1}{2}$, what is the value of a?

(A) $-\dfrac{5}{2}$ (B) -1 (C) $\dfrac{1}{2}$ (D) 2

5. If $y=4\sqrt{x}-16\sqrt[4]{x}$, then $y''=$

(A) $\sqrt[4]{x}-3$ (B) $-3\sqrt{x}+3$ (C) $\dfrac{-\sqrt[4]{x}+3}{x\sqrt[4]{x^3}}$ (D) $\dfrac{\sqrt{x}-3}{x\sqrt[4]{x}}$

6. If $y = x^2 \cdot f(x)$, then $y'' =$

(A) $x^2 f''(x) + x f'(x) + 2f(x)$

(B) $x^2 f''(x) + x f'(x) + f(x)$

(C) $x^2 f''(x) + 2x f'(x) + f(x)$

(D) $x^2 f''(x) + 4x f'(x) + 2f(x)$

7. Let $f(x) = \frac{1}{2}x^6 - 10x^3 + 12x$. What is the value of $f(x)$, when $f'''(x) = 0$?

(A) $-\frac{23}{4}$ (B) $-\frac{3}{2}$ (C) $\frac{1}{2}$ (D) $\frac{5}{2}$

Free Response Questions

8. Let $h(x) = x \cdot f(x) \cdot g(x)$. Find $h'(1)$, if $f(1) = -2$, $g(1) = 3$, $f'(1) = 1$, and $g'(1) = \frac{1}{2}$.

9. Let $g(x) = \frac{x}{\sqrt{x} - 1}$. Find $g''(4)$.

Chain Rule and Composite Functions

1. If $f(x)=\sqrt{x+\sqrt{x}}$, then $f'(x)=$

(A) $\dfrac{1}{2\sqrt{x+\sqrt{x}}}$ (B) $\dfrac{\sqrt{x}+1}{2\sqrt{x+\sqrt{x}}}$ (C) $\dfrac{2\sqrt{x}}{4\sqrt{x+\sqrt{x}}}$ (D) $\dfrac{2\sqrt{x}+1}{4\sqrt{x^2+x\sqrt{x}}}$

2. If $f(x)=(x^2-3x)^{3/2}$, then $f'(4)=$

(A) $\dfrac{15}{2}$ (B) 9 (C) $\dfrac{21}{2}$ (D) 15

3. If f, g, and h are functions that is everywhere differentiable, then the derivative of $\dfrac{f}{g\cdot h}$ is

(A) $\dfrac{g\,h\,f'-f\,g'\,h'}{g\,h}$

(B) $\dfrac{g\,h\,f'-f\,g\,h'-f\,h\,g'}{g\,h}$

(C) $\dfrac{g\,h\,f'-f\,g\,h'-f\,g'h}{g^2h^2}$

(D) $\dfrac{g\,h\,f'-f\,g\,h'+f\,h\,g'}{g^2h^2}$

4. If $f(x)=(3-\sqrt{x})^{-1}$, then $f''(4)=$

(A) $\dfrac{3}{32}$ (B) $\dfrac{3}{16}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{9}{4}$

Free Response Questions

Questions 5-9 refer to the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	1	-1
2	-2	1	-1	3
3	1	4	2	3
4	5	2	1	-2

The table above gives values of f , f' , g , and g' at selected values of x .

5. Find $h'(1)$, if $h(x) = f(g(x))$.

6. Find $h'(2)$, if $h(x) = x\,f(x^2)$.

7. Find $h'(3)$, if $h(x) = \dfrac{f(x)}{\sqrt{g(x)}}$.

8. Find $h'(2)$, if $h(x) = \left[f(2x)\right]^2$.

9. Find $h'(1)$, if $h(x) = (x^9 + f(x))^{-2}$.

10. Let f and g be differentiable functions such that $f(g(x)) = 2x$ and $f'(x) = 1 + \left[f(x)\right]^2$.

(a) Show that $g'(x) = \dfrac{2}{f'(g(x))}$.

(b) Show that $g'(x) = \dfrac{2}{1+4x^2}$.

Derivatives of Trigonometric Functions

1. $\lim\limits_{h\to 0}\dfrac{\cos(\frac{\pi}{3}+h)-\frac{1}{2}}{h}=$

(A) $-\dfrac{1}{2}$ (B) $-\dfrac{\sqrt{3}}{2}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{\sqrt{3}}{2}$

2. $\lim\limits_{h\to 0}\dfrac{\sin 2(x+h)-\sin 2x}{h}=$

(A) $2\sin 2x$ (B) $-2\sin 2x$ (C) $2\cos 2x$ (D) $-2\cos 2x$

3. If $f(x)=\sin(\cos 2x)$, then $f'(\frac{\pi}{4})=$

(A) 0 (B) -1 (C) 1 (D) -2

4. If $y=a\sin x+b\cos x$, then $y+y''=$

(A) 0 (B) $2a\sin x$ (C) $2b\cos x$ (D) $-2a\sin x$

5. $\dfrac{d}{dx}\sec^2(\sqrt{x})=$

(A) $\dfrac{2\sec(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

(B) $\dfrac{2\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

(C) $\dfrac{\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

(D) $\dfrac{\sec(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

6. $\dfrac{d}{dx}\left[x^2 \cos 2x\right] =$

(A) $-2x\sin 2x$

(B) $2x(-x\sin 2x + \cos 2x)$

(C) $2x(x\sin 2x - \cos 2x)$

(D) $2x(x\sin 2x - \cos 2x)$

7. If $f(\theta) = \cos\pi - \dfrac{1}{2\cos\theta} + \dfrac{1}{3\tan\theta}$, then $f'(\dfrac{\pi}{6}) =$

(A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{4}{\sqrt{3}}$ (D) $2\sqrt{3}$

Free Response Questions

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	$-1/2$	$3/2$	4	$\sqrt{2}$
$\pi/4$	-2	1	2	3

8. The table above gives values of f , f' , g , and g' at selected values of x .

Find $h'(\dfrac{\pi}{4})$, if $h(x) = f(x)\cdot g(\tan x)$.

9. Find the value of the constants a and b for which the function

$f(x) = \begin{cases} \sin x, & x < \pi \\ ax + b, & x \geq \pi \end{cases}$ is differentiable at $x = \pi$.

Derivatives of Exponential and Logarithmic Functions

1. $\lim\limits_{h\to 0}\dfrac{\frac{1}{2}\left[\ln(e+h)-1\right]}{h}$ is

   (A) $f'(1)$, where $f(x)=\ln\sqrt{x}$

   (B) $f'(1)$, where $f(x)=\ln\sqrt{x+e}$

   (C) $f'(e)$, where $f(x)=\ln\sqrt{x}$

   (D) $f'(e)$, where $f(x)=\ln(\frac{x}{2})$

2. If $f(x)=e^{\tan x}$, then $f'(\frac{\pi}{4})=$

   (A) $\frac{e}{2}$ (B) e (C) $2e$ (D) $\frac{e^2}{2}$

3. If $y=\ln(\cos x)$, then $y'=$

   (A) $-\tan x$ (B) $\tan x$ (C) $-\cot x$ (D) $\csc x$

4. If $y=x^x$, then $y'=$

   (A) $x^x \ln x$ (B) $x^x(1+\ln x)$ (C) $x^x(x+\ln x)$ (D) $\frac{x^x \ln x}{x}$

5. If $y = e^{\sqrt{x^2+1}}$, then $y' =$

(A) $\sqrt{x^2+1}\, e^{\sqrt{x^2+1}}$

(B) $2x\sqrt{x^2+1}\, e^{\sqrt{x^2+1}}$

(C) $\dfrac{e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

(D) $\dfrac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

6. If $y = (\sin x)^{1/x}$, then $y' =$

(A) $(\sin x)^{\frac{1}{x}}\left[\dfrac{\ln(\sin x)}{x}\right]$

(B) $(\sin x)^{\frac{1}{x}}\left[\dfrac{x - \ln(\sin x)}{x^2}\right]$

(C) $(\sin x)^{\frac{1}{x}}\left[\dfrac{x\sin x - \ln(\sin x)}{x^2}\right]$

(D) $(\sin x)^{\frac{1}{x}}\left[\dfrac{x\cot x - \ln(\sin x)}{x^2}\right]$

7. If $f(x) = \ln\left[\sec(\ln x)\right]$, then $f'(e) =$

(A) $\dfrac{\cos 1}{e}$ (B) $\dfrac{\sin 1}{e}$ (C) $\dfrac{\tan 1}{e}$ (D) $\dfrac{\cot 1}{e}$

8. If $y = x^{\ln\sqrt{x}}$, then $y' =$

(A) $\dfrac{x^{\ln\sqrt{x}}\ln x}{2x}$

(B) $\dfrac{x^{\ln\sqrt{x}}\ln x}{x}$

(C) $\dfrac{2x^{\ln\sqrt{x}}\ln x}{x}$

(D) $\dfrac{x^{\ln\sqrt{x}}(1+\ln x)}{x}$

Free Response Questions

9. Let $f(x) = xe^x$ and $f^{(n)}(x)$ be the nth derivative of f with respect to x. If $f^{(10)}(x) = (x+n)\,e^x$, what is the value of n?

10. Let f and h be twice differentiable functions such that $h(x) = e^{f(x)}$. If $h''(x) = e^{f(x)}\left[1+x^2\right]$, then $f'(x) =$

Tangent Lines and Normal Lines

1. The equation of the line tangent to the graph of $y = x\sqrt{3+x^2}$ at the point $(1,2)$ is

(A) $y = \frac{3}{2}x - \frac{1}{2}$ (B) $y = 2x + \frac{1}{2}$ (C) $y = \frac{5}{2}x - \frac{1}{2}$ (D) $y = \frac{5}{2}x + \frac{1}{2}$

2. Which of the following is an equation of the line tangent to the graph of $f(x) = x^2 - x$ at the point where $f'(x) = 3$?

(A) $y = 3x - 2$

(B) $y = 3x + 2$

(C) $y = 3x - 4$

(D) $y = 3x + 4$

3. A curve has slope $2x + x^{-2}$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(1,3)$?

(A) $y = 2x^2 + \frac{1}{x}$

(B) $y = x^2 - \frac{1}{x} + 3$

(C) $y = x^2 + \frac{1}{x} + 1$

(D) $y = x^2 - \frac{2}{x^2} + 4$

4. An equation of the line normal to the graph of $y = \tan x$, at the point $(\frac{\pi}{6}, \frac{1}{\sqrt{3}})$ is

(A) $y - \frac{1}{\sqrt{3}} = -\frac{1}{4}(x - \frac{\pi}{6})$

(B) $y - \frac{1}{\sqrt{3}} = \frac{1}{4}(x - \frac{\pi}{6})$

(C) $y - \frac{1}{\sqrt{3}} = -\frac{3}{4}(x - \frac{\pi}{6})$

(D) $y - \frac{1}{\sqrt{3}} = \frac{3}{4}(x - \frac{\pi}{6})$

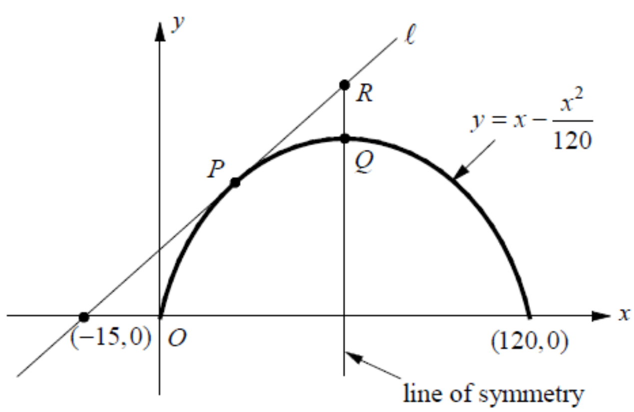
5. If $2x+3y=4$ is an equation of the line normal to the graph of f at the point $(-1,2)$, then $f'(-1)=$

(A) $-\frac{2}{3}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$

6. If $2x-y=k$ is an equation of the line normal to the graph of $f(x)=x^4-x$, then $k=$

(A) $\frac{23}{16}$ (B) $\frac{13}{18}$ (C) $\frac{15}{16}$ (D) $\frac{9}{8}$

Free Response Questions



7. Line ℓ is tangent to the graph of $y=x-\frac{x^2}{120}$ at the point P and intersects x-axis at $(-15,0)$ as shown in the figure above.

   (a) Find the x-coordinates of point P.

   (b) Write an equation for line ℓ.

   (c) If the line of symmetry for the curve $y=x-\frac{x^2}{120}$ intersects line ℓ at point R, what is the length of $\overline{QR}$?

Implicit Differentiation

1. If $3xy + x^2 - 2y^2 = 2$, then the value of $\frac{dy}{dx}$ at the point $(1,1)$ is

(A) 5 (B) $\frac{7}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{7}{2}$

2. If $3x^4 - x^2 - y^2 = 0$, then the value of $\frac{dy}{dx}$ at the point $(1, \sqrt{2})$ is

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{3\sqrt{2}}{2}$ (C) $\frac{5\sqrt{2}}{2}$ (D) $\frac{7\sqrt{2}}{2}$

3. If $x^2 y + 2xy^2 = 5x$, then $\frac{dy}{dx} =$

(A) $\frac{5 - 4xy - 4y}{x^2 + 4xy}$

(B) $\frac{5 - 2xy - 2y^2}{x^2 + 4xy}$

(C) $\frac{5 - 2xy - y^2}{x^2 + 2xy}$

(D) $\frac{5 - xy - 2y}{x^2 - 2xy}$

4. If $xy + \tan(xy) = \pi$, then $\frac{dy}{dx} =$

(A) $-y\sec^2(xy)$ (B) $-y\cos^2(xy)$ (C) $-x\sec^2(xy)$ (D) $-\frac{y}{x}$

5. An equation of the line tangent to the graph of $3y^2 - x^3 - xy^2 = 7$ at the point $(1, 2)$ is

(A) $y = \frac{3}{4}x - \frac{3}{8}$ (B) $y = \frac{3}{4}x + \frac{1}{2}$ (C) $y = -\frac{7}{8}x + \frac{3}{2}$ (D) $y = \frac{7}{8}x + \frac{9}{8}$

6. An equation of the line normal to the graph of $2x^2 + 3y^2 = 5$ at the point $(1,1)$ is

(A) $y = \frac{3}{2}x + 1$ (B) $y = \frac{3}{2}x - \frac{1}{2}$ (C) $y = -\frac{2}{3}x + \frac{5}{3}$ (D) $y = -\frac{2}{3}x + \frac{3}{2}$

7. If $x + \sin y = y + 3$, then $\frac{d^2y}{dx^2} =$

(A) $\frac{-\sin y}{(1-\cos y)^2}$ (B) $\frac{-\sin y}{(1+\cos y)^2}$ (C) $\frac{-\sin y}{(1-\cos y)^3}$ (D) $\frac{-\sin y}{(1+\cos y)^3}$

Free Response Questions

8. Consider the curve given by $x^3 - xy + y^2 = 3$.

(a) Find $\frac{dy}{dx}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is horizontal.

9. Consider the curve $x^2 + y^2 - xy = 7$.

(a) Find $\frac{dy}{dx}$.

(b) Find all points on the curve whose x-coordinate is 2, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

Derivatives of an Inverse Function

1. Let f and g be functions that are differentiable everywhere. If g is the inverse function of f and if $g(3)=4$ and $f'(4)=\frac{3}{2}$, then $g'(3)=$

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

2. If $f(-3)=2$ and $f'(-3)=\frac{3}{4}$, then $(f^{-1})'(2)=$

(A) $\frac{1}{2}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $-\frac{3}{4}$

3. If $f(x)=x^3-x+2$, then $(f^{-1})'(2)=$

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) 4 (D) 6

4. If $f(x)=\sin x$, then $(f^{-1})'(\frac{\sqrt{3}}{2})=$

(A) $\frac{1}{2}$ (B) $\frac{2\sqrt{3}}{3}$ (C) $\sqrt{3}$ (D) 2

5. If $f(x)=1+\ln x$, then $(f^{-1})'(2)=$

(A) $-\frac{1}{e}$ (B) $\frac{1}{e}$ (C) $-e$ (D) e

Free Response Questions

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	3	-2	2	6
0	-2	-1	0	-3
1	0	1	-1	2
2	-1	4	3	-1

6. The functions f and g are differentiable for all real numbers. The table above gives the values of the functions and their first derivatives at selected values of x.

(a) If f^{-1} is the inverse function of f, write an equation for the line tangent to the graph of $y = f^{-1}(x)$ at $x = -1$.

(b) Let h be the function given by $h(x) = f(g(x))$. Find $h(1)$ and $h'(1)$.

(c) Find $(h^{-1})'(3)$, if h^{-1} is the inverse function of h.

Derivatives of Inverse Trigonometric Functions

1. $\frac{d}{dx}(\arcsin x^2) =$

(A) $-\frac{2x}{\sqrt{1-x^2}}$ (B) $\frac{2x}{\sqrt{x^2-1}}$ (C) $\frac{2x}{\sqrt{x^4-1}}$ (D) $\frac{2x}{\sqrt{1-x^4}}$

2. If $f(x) = \arctan(e^{-x})$, then $f'(-1) =$

(A) $\frac{-e}{1+e}$ (B) $\frac{e}{1+e}$ (C) $\frac{-e}{1+e^2}$ (D) $\frac{-1}{1+e^2}$

3. If $f(x) = \arctan(\sin x)$, then $f'(\frac{\pi}{3}) =$

(A) $\frac{2}{7}$ (B) $\frac{1}{2}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{\sqrt{3}}{3}$

4. If $y = \cos(\sin^{-1} x)$, then $y' =$

(A) $-\frac{1}{\sqrt{1-x^2}}$ (B) $-\frac{x}{\sqrt{1-x^2}}$ (C) $\frac{2x}{\sqrt{1-x^2}}$ (D) $-\frac{2x}{\sqrt{x^2-1}}$

Free Response Questions

5. Let f be the function given by $f(x) = x^{\tan^{-1} x}$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$.

Approximating a Derivative

1. Some values of differentiable function f are shown in the table below.
   What is the approximation value of $f'(3.5)$?

x	3.0	3.3	3.8	4.2	4.9
$f(x)$	21.8	26.1	32.5	38.2	48.7

(A) 8 (B) 10 (C) 13 (D) 16

Free Response Questions

Month	1	2	3	4	5	6
Temperature	−8	0	25	50	72	88

2. The normal daily maximum temperature F for a certain city is shown in the table above.

   (a) Use data in the table to find the average rate of change in temperature from $t = 1$ to $t = 6$.

   (b) Use data in the table to estimate the rate of change in maximum temperature at $t = 4$.

   (c) The rate at which the maximum temperature changes for $1 \le t \le 6$ is modeled by $F(t) = 40 - 52\sin(\frac{\pi t}{6} - 5)$ degrees per minute. Find $F'(4)$ using the given model.