Recall:

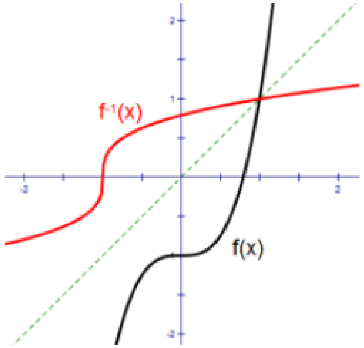
A function g is the **inverse function** of the function f if:

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

The function g is denoted by $f^{-1}(x)$, read as "*f inverse.*"

Properties of inverse functions:

1. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
2. The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).
   -Another way to say this is that f and f^{-1} are symmetric over the line $y = x$.



Continuity and Differentiability of Inverse Functions

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

$(f^{-1})'(x)$

The Derivative of an Inverse Function

Let f be a function that is differentiable on a given interval. If f has an inverse, then we can find the rule for the derivative of the inverse by using the chain rule.

Start with the fact that: $f\left(f^{-1}(x)\right) = x$

$$\frac{d}{dx}\left[f\left(f^{-1}(x)\right) = x\right]$$

$$f'\left(f^{-1}(x)\right) \cdot \frac{d}{dx}\left[f^{-1}(x)\right] = 1$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)}$$

A couple of approaches to calculating the derivative of an inverse function: $\frac{1}{f'(f^{-1}(x))}$

Method 1: Using the formula.

Example: Calculate $(f^{-1})'(2)$ for the function $f(x) = \sqrt{5x-1}$

$f'(x) = \frac{5}{2\sqrt{5x-1}}$

$f'(1) = \frac{5}{2\sqrt{4}} = \frac{5}{4}$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

$$= \frac{1}{f'(1)}$$

$$= \frac{4}{5}$$

Inverse:

$x = 2$ $\Longrightarrow$

Function:

$y = 2$

$\sqrt{5x-1} = 2$

$5x-1 = 4$

$5x = 5$

$x = 1$

$(2, 1) \Longleftarrow (1, 2)$

Method 2: Using the inverse function. (only works if the inverse is easily found)

Example: Calculate $(f^{-1})'(28)$ for the function $f(x) = 4x^3 - 4$. $\Rightarrow$ $y = 4x^3 - 4$

Inverse:

$$x = 4y^3 - 4$$

$$y = \sqrt[3]{\tfrac{1}{4}x + 1}$$

$$\downarrow$$

$$f^{-1}(x)$$

$$\frac{d}{dx}\left[f^{-1}(x) = \sqrt[3]{\tfrac{1}{4}x + 1}\right]$$

$$(f^{-1})'(x) = \frac{1}{3}\left(\frac{1}{4}x+1\right)^{-2/3} \cdot \frac{1}{4}$$

$$= \frac{1}{12\sqrt[3]{\left(\frac{1}{4}x+1\right)^2}}$$

$$(f^{-1})'(28) = \frac{1}{12\sqrt[3]{64}} = \frac{1}{48}$$

Alternate form of inverse derivative:

Consider calculating the inverse of $f(x) = 2x^2 + 3x + 1$.

$y = 2x^2 + 3x + 1$

Inverse:

$x = 2y^2 + 3y + 1$

We can use implicit differentiation to calculate $(f^{-1})'(x)$ for the function $f(x) = 2x^2 + 3x + 1$.

Inverse:

$$\frac{d}{dx}\left[x = 2y^2 + 3y + 1\right]$$

$$1 = 4y \cdot \frac{dy}{dx} + 3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{4y+3}$$

$f(y) = 2y^2 + 3y + 1$

$f'(y) = 4y + 3$

Approaching the derivative of the inverse using implicit differentiation is a way to illustrate the alternate form of the derivative of the inverse of a function.

If $y = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. Therefore:

$$(f^{-1})'(x) = \frac{1}{f'(y)}$$

y value of f^{-1}

or

x value of f

Example: Find the equation of the tangent line of the inverse of $f(x) = x^2 + 2x - 1$, with a restricted domain of $[-1, \infty)$, when $x = 2$.

$f(y) = y^2 + 2y - 1$

$f'(y) = 2y + 2$

Inverse:

$x = 2 \implies$

Function:

$y = 2$

$x^2 + 2x - 1 = 2$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = 1, \cancel{-3}$

$(2, 1) \Longleftarrow (1, 2)$

$$(f^{-1})'(x) = \frac{1}{2y+2}$$

$$(f^{-1})'(2) = \frac{1}{2(1)+2}$$

$$= \frac{1}{4}$$

Example: Selected values of a function $g(x)$ and its derivative $g'(x)$ are shown on the table below.

a) Find $(g^{-1})'(1)$

b) Find $(g^{-1})'(-3)$

x	-3	-1	1	4
$g(x)$	5	1	0	-3
$g'(x)$	-4	$-\frac{1}{5}$	$-\frac{1}{6}$	-2

(a) $(g^{-1})'(1) = \dfrac{1}{g'(g^{-1}(1))}$

$g^{-1}(1) \to x=1,\ y=-1$

$g \to y=1,\ x=-1$

$= \dfrac{1}{g'(-1)}$

$= -5$

(b) $(g^{-1})'(-3) = \dfrac{1}{g'(g^{-1}(-3))}$

$= \dfrac{1}{g'(4)}$

$= \dfrac{1}{-2}$

2007 Question 3 (calculator)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$ $h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$ Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.	$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$
(b) $\frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{3 - 1} = -5$ Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.	$2 : \begin{cases} 1 : \frac{h(3) - h(1)}{3 - 1} \\ 1 : \text{conclusion, using MVT} \end{cases}$
(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$	$2 : \begin{cases} 1 : \text{apply chain rule} \\ 1 : \text{answer} \end{cases}$
(d) $g(1) = 2$, so $g^{-1}(2) = 1$. $\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$ An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.	$3 : \begin{cases} 1 : g^{-1}(2) \\ 1 : \left(g^{-1}\right)'(2) \\ 1 : \text{tangent line equation} \end{cases}$

Non Calculator

$f'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$

$f^{-1}(1) = 0$

Let $f(x) = (2x+1)^3$ and let g be the inverse function of f. Given that $f(0) = 1$, what is the value of $g'(1)$?

(A) $-\frac{2}{27}$ (B) $\frac{1}{54}$ (C) $\frac{1}{27}$ (D) $\frac{1}{6}$ (E) 6

$g'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)}$

Calculator

$f(-2) = 4$

$f^{-1}(4) = -2$

x	$f(x)$	$g(x)$	$f'(x)$
–4	0	–9	5
–2	4	–7	4
0	6	–4	2
2	7	–3	1
4	10	–2	3

The table above gives values of the differentiable functions f and g, and f', the derivative of f, at selected values of x. If $g(x) = f^{-1}(x)$, what is the value of $g'(4)$?

(A) $-\frac{1}{3}$ (B $-\frac{1}{4}$ (C) $-\frac{3}{100}$ (D $\frac{1}{4}$ (E) $\frac{1}{3}$

$g'(4) = \dfrac{1}{f'(f^{-1}(4))} = \dfrac{1}{f'(-2)}$