BC Only: Alternating Series Remainder Theorem

Given $\sum a_n$ is a convergent alternating series, the error associated with approximating the sum of the series by the first *n* terms is less than or equal to the first omitted term.

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = S \approx S_n = a_1 - a_2 + \cdots + (-1)^{n+1}a_n \qquad \text{Error} = |S - S_n| \le |a_{n+1}|$$

BC Only: Lagrange Remainder of a Taylor Polynomial

When approximating a function $f(x)$ using an nth degree Taylor polynomial, $P_n(x)$, the associated error, $R_n(x)$, is bounded by

$$|R_n(x)| = |f(x) - P_n(x)| \le \left|\frac{(x-c)^{n+1}}{(n+1)!} \cdot \max f^{(n+1)}(z)\right| \quad \text{where } c \le z \le x$$

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty} \left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 : { 1 : sets up ratio
1 : computes limit of ratio
1 : identifies interior of interval of convergence
1 : considers both endpoints
1 : analysis and interval of convergence

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 : { 1 : uses the third term as an error bound
1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 : { 1 : first three terms
1 : general term

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$ $P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.	$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$
(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$ $h(1.9) \approx P_3(1.9) = 67.988$	$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$
(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left\lvert h^{(4)}(x) \right\rvert = \frac{584}{9}$. Therefore, $\lvert h(1.9) - P_3(1.9) \rvert \le \frac{584}{9} \frac{\lvert 1.9-2 \rvert^4}{4!}$ $= 2.7037 \times 10^{-4}$ $< 3 \times 10^{-4}$	$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$

BC Only: Polar Coordinates

A. The polar coordinates $(r, \theta)$of a point are related to the rectangular coordinates (x, y) as follows

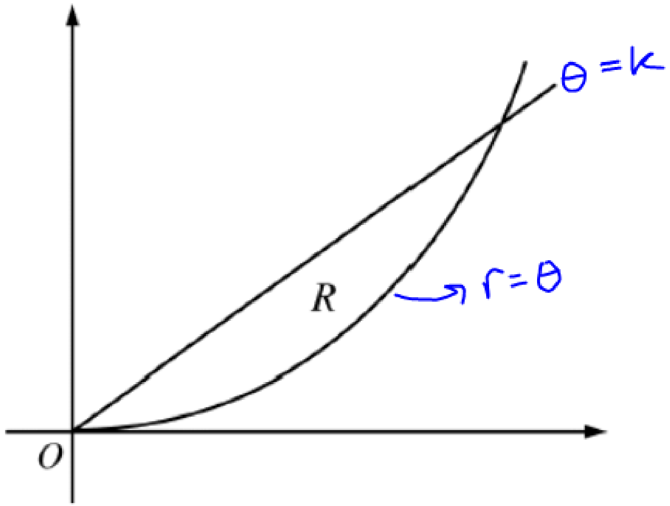
$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \tan\theta = \frac{x}{y}$$

B. If *f* is a differentiable function of θ (smooth curve), then the slope of the line tangent to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{f'(\theta)\sin\theta \ + \ f(\theta)\cos\theta}{f'(\theta)\cos\theta \ - \ f(\theta)\sin\theta}$$

C. If $r = f(\theta)$ is a smooth curve on the interval $[\alpha, \beta]$, where α and β are radial lines, then the area enclosed by the graph is

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta$$



Let R be the region in the first quadrant that is bounded by the polar curves $r = \theta$ and $\theta = k$, where k is a constant, $0 < k < \frac{\pi}{2}$, as shown in the figure above. What is the area of R in terms of k ?

(A) $\frac{k^3}{6}$ (B) $\frac{k^3}{3}$ (C) $\frac{k^3}{2}$ (D) $\frac{k^2}{4}$ (E) $\frac{k^2}{2}$

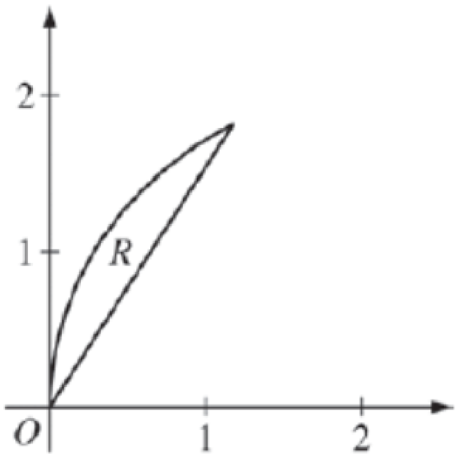
$$R = \frac{1}{2}\int_0^k \theta^2\, d\theta = \left[\frac{1}{6}\theta^3\right]_0^k = \frac{1}{6}k^3$$

What is the slope of the line tangent to the polar curve $r = \cos\theta$ at the point where $\theta = \frac{\pi}{6}$?

(A) $-\sqrt{3}$ (B) $-\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{\sqrt{3}}{2}$ (E) $\sqrt{3}$

$$\frac{dy}{dx} = \frac{-\sin\theta \cdot \sin\theta + \cos\theta \cdot \cos\theta}{-\sin\theta \cdot \cos\theta - \cos\theta \cdot \sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{-\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}{-\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}} = \frac{-\frac{1}{4} + \frac{3}{4}}{-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$



Let R be the region in the first quadrant that is bounded above by the polar curve $r = 4\cos\theta$ and below by the line $\theta = 1$, as shown in the figure above. What is the area of R ?

(A) 0.317 (B) 0.465 (C) 0.929 (D) 2.618 (E) 5.819

Calculator:

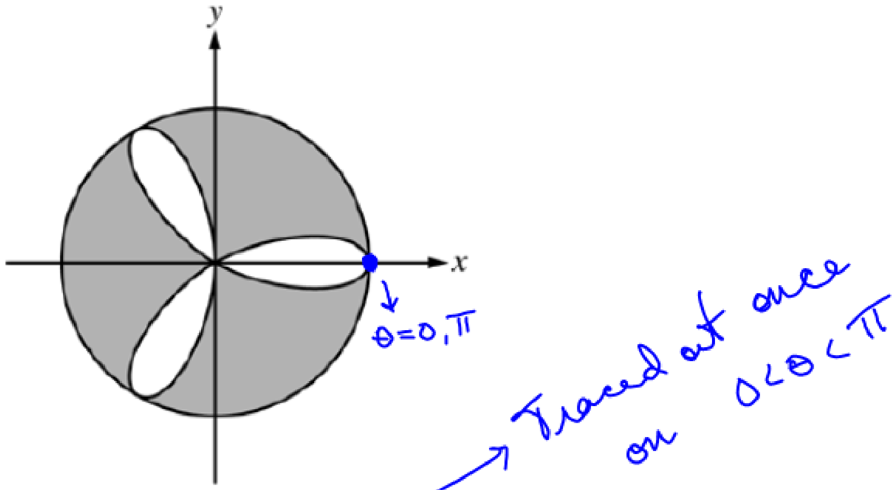
$$R = \frac{1}{2}\int_{1}^{\pi/2} [4\cos\theta]^2 \, d\theta$$

What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

(A) 2 (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{1}{2}$ (E) -2

$$\frac{dy}{dx} = \frac{2\cos\theta \cdot \sin\theta + (1+2\sin\theta)\cos\theta}{2\cos\theta \cdot \cos\theta - (1+2\sin\theta)\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{0+1}{2-0} = \frac{1}{2}$$



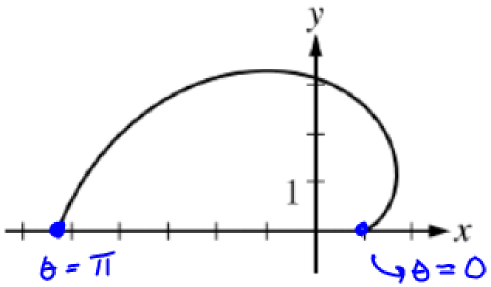
The figure above shows the graphs of the polar curves $r = 2\cos(3\theta)$ and $r = 2$. What is the sum of the areas of the shaded regions?

(A) 0.858 (B) 3.142 (C) 8.566 (D) 9.425 (E) 15.708

Calculator:

$$\pi(2)^2 - \frac{1}{2}\int_0^{\pi}\left[2\cos(3\theta)\right]^2 d\theta$$

$\pi(2)^2$ → Area of Circle $r=2$

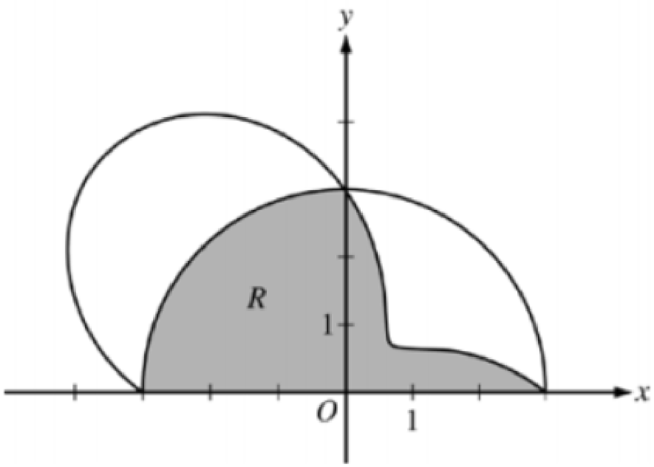


The graph above shows the polar curve $r = 2\theta + \cos\theta$ for $0 \le \theta \le \pi$. What is the area of the region bounded by the curve and the x-axis?

(A) 3.069 (B) 4.935 (C) 9.870 (D) 17.456 (E) 34.912

Calculator:

$$\frac{1}{2}\int_0^{\pi} [2\theta + \cos\theta]^2 \, d\theta$$



The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$.

Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

(a) Area $= \frac{9\pi}{4} + \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta$ $= 9.708$ (or 9.707)	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$
(b) $x = (3 - 2\sin(2\theta))\cos\theta$ $\left.\frac{dx}{d\theta}\right\|_{\theta=\pi/6} = -2.366$	$2 : \begin{cases} 1 : \text{expression for } x \\ 1 : \text{answer} \end{cases}$
(c) The distance between the two curves is $D = 3 - (3 - 2\sin(2\theta)) = 2\sin(2\theta)$. $\left.\frac{dD}{d\theta}\right\|_{\theta=\pi/3} = -2$	$2 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{answer} \end{cases}$
(d) $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot 3$ $\left.\frac{dr}{dt}\right\|_{\theta=\pi/6} = (-2)(3) = -6$	$2 : \begin{cases} 1 : \text{chain rule with respect to } t \\ 1 : \text{answer} \end{cases}$

The polar curve r is given by $r(\theta) = 3\theta + \sin\theta$, where $0 \le \theta \le 2\pi$.

(a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.

(b) For $\frac{\pi}{2} \le \theta \le \pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

(c) A particle is traveling along the polar curve r so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

(a) Area $= \frac{1}{2}\int_{\pi/2}^{\pi} (r(\theta))^2 \, d\theta = 47.513$	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$
(b) $-3 = r(\theta)\cos\theta = (3\theta + \sin\theta)\cos\theta$ $\theta = 2.01692$ $y = r(\theta)\sin(\theta) = 6.272$	$3 : \begin{cases} 1 : \text{equation} \\ 1 : \text{value of } \theta \\ 1 : y\text{-coordinate} \end{cases}$
(c) $y = r(\theta)\sin\theta = (3\theta + \sin\theta)\sin\theta$ $\left.\frac{dy}{dt}\right\|_{\theta=2\pi/3} = \left[\frac{dy}{d\theta} \cdot \frac{d\theta}{dt}\right]_{\theta=2\pi/3} = -2.819$ The y-coordinate of the particle is decreasing at a rate of 2.819.	$3 : \begin{cases} 1 : \text{uses chain rule} \\ 1 : \text{answer} \\ 1 : \text{interpretation} \end{cases}$