• Nth Term	If the terms of a sequence do not converge to 0, then the series must diverge. **I.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. **II.** If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive.

$$\sum_{n=0}^{\infty} \frac{3n-1}{2n+1}$$

$$\lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$$

$\therefore\ \Sigma$ Diverges

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+2n}$$

$$\lim_{n\to\infty} \frac{n^2+1}{n^3+2n} = 0$$

$\therefore$ Inconclusive

• Sequence of Partial Sums	Given the series $\sum a_n = a_1 + a_2 + a_3 + \cdots$ The sequence of partial sums for the series is $S_1 = a_1 \quad S_2 = a_1 + a_2 \quad S_3 = a_1 + a_2 + a_3 \quad \ldots \quad S_n = a_1 + a_2 + a_3 + \cdots + a_n$ If $\lim_{n\to\infty} S_n = S$, then $\sum a_n$ converges to S.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$S_1 = \underbrace{\left(1 - \frac{1}{2}\right)}_{a_1}$$

$$S_2 = \underbrace{\left(1 - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)}_{a_2}$$

$$S_3 = \underbrace{\left(1 - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \underbrace{\left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)}_{a_3}$$

$\vdots$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = 1$$

$\therefore \Sigma$ converges to 1

$$\sum_{n=1}^{\infty} \frac{8}{4n^2 - 1} \longrightarrow \sum_{n=1}^{\infty} \frac{8}{(2n+1)(2n-1)}$$

$$\frac{8}{(2n+1)(2n-1)} = \frac{A}{2n+1} + \frac{B}{2n-1}$$

$\vdots$

$$S_1 = \frac{8}{3}$$

$$S_2 = \frac{8}{3} + \frac{8}{15} = \frac{48}{15} = \frac{16}{5}$$

$$S_3 = \frac{16}{5} + \frac{8}{35} = \frac{120}{35} = \frac{24}{7}$$

$\vdots$

$$S_n = \frac{8n}{2n+1}$$

$$\lim_{n\to\infty} S_n = 4$$

• P – Series	The form of a p – series is $$\sum \frac{1}{n^p}$$ **I.** If $p > 1$, then the series converges. **II.** If $p < 1$, then the series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$p = 3/2 > 1$

$\therefore \Sigma$ converge

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$$

$$\frac{1}{n^{2/3}}$$

$p = 2/3 < 1$

$\therefore \Sigma$ Diverges

• Geometric Series	A geometric series is any series of the form $$\sum_{n=0}^{\infty} ar^n$$ **I.** If $\lvert r\rvert < 1$, then the series converges to $\frac{a}{1-r}$ *Series must be indexed at *n = 0* **II.** If $\lvert r\rvert > 1$, then the series diverges.

$$\sum_{n=1}^{\infty} 10\left(\frac{3}{2}\right)^n$$

$r = \frac{3}{2} > 1$

$\therefore \Sigma$ divergent

$$\sum_{n=0}^{\infty} 2\left(-\frac{1}{3}\right)^n$$

$|r| = \left|-\frac{1}{3}\right| < 1$

$S = \frac{2}{1 - -\frac{1}{3}} = \frac{3}{2}$

$$\sum_{n=1}^{\infty} \frac{4^{n+2}}{3^{2n}} = \sum_{n=1}^{\infty} 16\left(\frac{4}{9}\right)^n$$

$\frac{4^n \cdot 4^2}{(3^2)^n}$

$= \sum_{n=0}^{\infty} 16\left(\frac{4}{9}\right)^{n+1}$

$= \sum_{n=0}^{\infty} \frac{64}{9}\left(\frac{4}{9}\right)^n$

$S = \frac{\frac{64}{9}}{1 - \frac{4}{9}} = \frac{64}{5}$

• Integral	If f is positive, continuous, and decreasing for $x \geq 1$, then $$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$ either both converge or both diverge. $a_n = f(n)$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$\lim_{b\to\infty} \left[\frac{1}{2}\ln|x^2+1|\right]_1^b$$

$$\lim_{b\to\infty} \left[\underbrace{\frac{1}{2}\ln|b^2+1|}_{\infty} - \frac{1}{2}\ln(2)\right] = \infty$$

$\therefore \sum$ Diverges

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\lim_{b\to\infty} \left[-\frac{1}{2(\ln x)^2}\right]_2^b$$

$$\lim_{b\to\infty} \left[\underbrace{-\frac{1}{2b^2}}_{0} + \frac{1}{2(\ln(2))^2}\right] = \frac{1}{2(\ln 2)^2}$$

• Alternating Series	A series, containing both positive terms, negative terms, and $a_n > 0$, of the form $$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$ The series' converge if both of the following conditions are met I. $a_{n+1} \le a_n$ for all n → Terms decrease II. $\boxed{\lim_{n\to\infty} a_n = 0}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = 0$$

Method 1:

$$\frac{\sqrt{n+1}}{n+2} \le \frac{\sqrt{n}}{n+1}$$

$$\left(\sqrt{\frac{n+1}{n}}\right)^2 \le \left(\frac{n+2}{n+1}\right)^2$$

$$\frac{n+1}{n} \le \frac{(n+2)^2}{(n+1)^2}$$

$$(n+1)^3 \le n(n+2)^2$$

$$\cancel{n^3} + 3n^2 + 3n + 1 \le \cancel{n^3} + 4n^2 + 4n$$

$$0 \le n^2 + n - 1$$

Method 2:

$$\frac{d}{dn}\left[\frac{\sqrt{n}}{n+1}\right]$$

$$\frac{(n+1)\cdot\frac{1}{2}n^{-1/2} - 1\,n^{1/2}}{(n+1)^2}$$

$$\frac{\frac{1}{2}n^{-1/2}\left[n+1-2n\right]}{(n+1)^2}$$

$$\frac{1-n}{2\sqrt{n}(n+1)^2}$$ → Negative for $n > 1$

• Direct Comparison	When comparing two series, if $a_n \leq b_n$ for all n, **I.** If $\sum a_n$ diverges, then $\sum b_n$ diverges. **II.** If $\sum b_n$ converges, then $\sum a_n$ converges. *The convergence or divergence of the series chosen for comparison should be known

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 1}$$

$$3^n + 1 > 3^n$$

$$\frac{1}{3^n+1} < \frac{1}{3^n}$$

Convergent Geometric

$\therefore$ Converges by Direct Comparison

$$\sum_{n=1}^{\infty} \frac{1}{n-1}$$

$$n - 1 < n$$

$$\frac{1}{n-1} > \frac{1}{n}$$

Divergent Harmonic

$\therefore$ Diverges by Direct Comp

• Limit Comparison	If $a_n > 0$ and $b_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. *The convergence or divergence of the series chosen for comparison should be known *When choosing a series to compare to, disregard all but the highest powers (growth factor) in the numerator and denominator

$$\sum_{n=1}^{\infty} \underbrace{\boxed{\frac{n2^n}{4n^3+1}}}_{a_n}$$

$$b_n = \frac{n2^n}{n^3} = \frac{2^n}{n^2}$$

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$$

Divergent by n^{th} term

$$\lim_{n\to\infty} \frac{\cancel{n2^n}}{4n^3+1} \cdot \frac{n^3}{\cancel{n2^n}} = \frac{1}{4}$$

Finite Positive

$\therefore \sum$ Diverges

$$\sum_{n=1}^{\infty} \underbrace{\boxed{\frac{1}{n\sqrt{n^2+1}}}}_{a_n}$$

$$b_n = \frac{1}{n\sqrt{n^2}} = \frac{1}{n^2}$$

Convergent p-series

$$\lim_{n\to\infty} \frac{1}{\cancel{n}\sqrt{n^2+1}} \cdot \frac{\cancel{n}\sqrt{n^2}}{1}$$

$$\sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = 1$$

finite & positive

$\therefore \sum$ converges

• Ratio	Given a series $\sum a_n$ **I.** If $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| < 1$, then $\sum a_n$ converges. **II.** If $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| > 1$, then $\sum a_n$ diverges. **III.** If $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = 1$, then the ratio test is inconclusive. *This is a test for absolute convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$\lim_{n\to\infty} \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}}$$

$$\lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

$\therefore \sum$ converges (absolutely)

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \frac{\overset{(n+1)^n \cancel{(n+1)}}{(n+1)^{n+1}}}{\underset{\cancel{(n+1)}\,\cancel{n!}}{(n+1)!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$\therefore \sum$ diverges

The infinite series $\sum_{k=1}^{\infty} a_k$ has nth partial sum $S_n = (-1)^{n+1}$ for $n \geq 1$. What is the sum of the series $\sum_{k=1}^{\infty} a_k$?

$\lim_{n \to \infty} S_n = \text{undefined}$

(A) -1

(B) 0

(C) $\frac{1}{2}$

(D) 1

(E) The series diverges.

What is the sum of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{n+1}}$?

(A) $\frac{-2}{e^2 - 2e}$ (B) $\frac{-2}{e^2 + 2e}$ (C) $\frac{-2}{e + 2}$ (D) $\frac{e}{e + 2}$ (E) The series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{e}\left(\frac{-2}{e}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{1}{e}\left(\frac{-2}{e}\right)^{n+1}$$

$$\sum_{n=0}^{\infty} \frac{-2}{e^2}\left(\frac{-2}{e}\right)^n$$

$$S = \frac{\frac{-2}{e^2}}{1 - \frac{-2}{e}} = \frac{\frac{-2}{e^2}}{\frac{e+2}{e}} = \frac{-2}{e(e+2)}$$

Which of the following series converge?

I. $1 + (-1) + 1 + \cdots + (-1)^{n-1} + \cdots$ $\rightarrow \sum (-1)^{n-1}$ Diverges

II. $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots$ $\rightarrow \sum \frac{1}{2n-1}$ Divergent General Harmonic

III. $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}} + \cdots$ $\rightarrow \sum 3\left(\frac{1}{3}\right)^n$ Convergent geometric

(A) I only

(B) II only

(C) III only

(D) II and III only

(E) I, II, and III

If $0 < b_n < a_n$ for $n \geq 1$, which of the following must be true?

(A) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} b_n$ converges.

(B) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} b_n = 0$.

If the larges series converges, the smaller series must converges as well. $\lim_{n\to\infty} b_n$ always $= 0$ for convergent series.

(C) If $\sum_{n=1}^{\infty} b_n$ diverges, then $\lim_{n\to\infty} a_n = 0$.

(D) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(E) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

What is the value of $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n}$?

(A) $-\frac{15}{8}$ (B) $-\frac{9}{8}$ (C) $-\frac{3}{8}$ (D) $\frac{9}{8}$ (E) $\frac{15}{8}$

$$\sum_{n=1}^{\infty} -3\left(\frac{-3}{5}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{9}{5}\left(-\frac{3}{5}\right)^n$$

$$S = \frac{\frac{9}{5}}{1 - \frac{-3}{5}} = \frac{\frac{9}{5}}{\frac{8}{5}} = \frac{9}{8}$$

Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ II. $\sum_{n=1}^{\infty} e^{-n}$ III. $\sum_{n=1}^{\infty} \frac{n+2}{n^2+n}$

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I and III only

I. $|\sin n| \le 1$

$\frac{|\sin n|}{n^2} \le \frac{1}{n^2} \to$ convergent p-series

II. $\sum \left(\frac{1}{e}\right)^n \to$ convergent geometric

III. $a_n = \frac{n+2}{n^2+n}$ $b_n = \frac{n}{n^2} = \frac{1}{n} \to$ Divergent Harmonic

$\lim_{n\to\infty} \frac{n+2}{n^2+n} \cdot \frac{n^2}{n} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+n^2} = 1$

Which of the following statements are true about the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{(-1)^n}{\sqrt{n} + (-1)^n}$?

I. The series is alternating.

II. $|a_{n+1}| \le |a_n|$ for all $n \ge 2$

III. $\lim_{n\to\infty} a_n = 0$

(A) None

(B) I only

(C) I and II only

(D) I and III only

(E) I, II, and III

I. $\frac{1}{\sqrt{2}+1} + \frac{-1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}+1} + \cdots$ ✓

$|a_n| \ge |a_{n+1}|$

II. $\frac{1}{\sqrt{n}+1} \ge \frac{1}{\sqrt{n+1}-1}$ X

$\sqrt{n}+1 \le \sqrt{n+1} - 1$

$(\sqrt{n}+2)^2 \le (\sqrt{n+1})^2$

$\cancel{n} + 4\sqrt{n} + 4 \le \cancel{n} + 1$

$4\sqrt{n} \le -3$

$\sqrt{n} \le -3/4$ → not possible since $\sqrt{n} > 0$

III. $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$ → n even $\frac{1}{\sqrt{n}+1} = 0$ ✓

→ n odd $\frac{-1}{\sqrt{n}-1} = 0$

Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$ II. $\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$ III. $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

I. $\lim_{n\to\infty} \left| \frac{8^{n+1}}{(n+1)!} \cdot \frac{n!}{8^n} \right| = \lim_{n\to\infty} \frac{8}{n+1} = 0 < 1$ ✓

II. $\lim_{n\to\infty} \frac{n!}{n^{100}} = \infty \neq 0$ X

III. $a_n = \frac{n+1}{n(n+2)(n+3)}$ $b_n = \frac{n}{n\cdot n\cdot n} = \frac{1}{n^2}$ convergent p-series

$\lim_{n\to\infty} \frac{n+1}{\cancel{n}(n+2)(n+3)} \cdot \frac{\cancel{n}\cdot n\cdot n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{n}{n+2} \cdot \frac{n}{n+3} = 1\cdot 1\cdot 1 = 1$ ✓

For what values of p will both series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ and $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converge?

(A) $-2 < p < 2$ only

(B) $-\frac{1}{2} < p < \frac{1}{2}$ only

(C) $\frac{1}{2} < p < 2$ only

(D) $p < \frac{1}{2}$ and $p > 2$

(E) There are no such values of p.

p-series converges when $2p > 1$

$p > 1/2$

Geometric converges when $\left|\frac{p}{2}\right| < 1$

$|p| < 2$

$-2 < p < 2$

If the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, which of the following must be true?

(A) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

(B) $|a_n| < 1$ for all n

(C) $\sum_{n=1}^{\infty} a_n = 0$

(D) $\sum_{n=1}^{\infty} n a_n$ diverges.

(E) $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

BC Only: Power Series

A. Power Series Structure and Characteristics

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \longrightarrow \text{power series centered at } x = 0$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots \longrightarrow \text{power series centered at } x = c$$

A function *f* can be represented by a power series , where the power series converges to the function in one of three ways:

I. The power series only converges at the center *x* = *c*.

II. The power series converges for all real values of *x*.

III. The power series converges for some interval of values such that $|x - c| < R$, where *R* is the radius of convergence of the power series.

B. Interval of Convergence: Find this by applying the Ratio to the given series.

I. If $R = 0$, then the series converges only at *x* = *c*.

II. If $R = \infty$, then the series converges for all real values of *x*.

III. If the Ratio Test results in an expression of the form $|x - c| < R$, then the interval of convergence is of the form $c - R < x < c + R$.

*The convergence at the endpoints of the interval of convergence should be tested separately.

What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n}$?

(A) $1 < x < 5$

(B) $1 \le x < 5$

(C) $1 \le x \le 5$

(D) $2 < x < 4$

(E) $2 \le x \le 4$

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)\, 2^{n+1}} \cdot \frac{n \cdot 2^n}{(x-3)^n} \right|$$

$$\left| \frac{x-3}{2} \right| \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right|$$

$$\left| \frac{x-3}{2} \right| < 1$$

$$|x-3| < 2$$

Center = 3 R = 2

$|2x-1| < 4$

$|x - 1/2| < 2$

$\boxed{x=1}$ $\sum \frac{(-2)^n}{n \cdot 2^n} = \sum \frac{(-1)^n}{n}$

Converges

$\boxed{x=5}$ $\sum \frac{2^n}{n \cdot 2^n} = \sum \frac{1}{n}$

Diverges

What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x + \frac{3}{2}\right)^n$ converges?

(A) $-\frac{5}{2} < x < -\frac{1}{2}$

(B) $-\frac{5}{2} < x \le -\frac{1}{2}$

(C) $-\frac{5}{2} \le x < -\frac{1}{2}$

(D) $-\frac{1}{2} < x < \frac{1}{2}$

(E) $x \le -\frac{1}{2}$

$$\lim_{n\to\infty} \left| \frac{(x+\frac{3}{2})^{n+1}}{n+1} \cdot \frac{n}{(x+\frac{3}{2})^n} \right|$$

$$\left|x+\frac{3}{2}\right| \cdot \lim_{n\to\infty} \left|\frac{n}{n+1}\right|$$

$$\left|x+\frac{3}{2}\right| < 1$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\left(x+\frac{3}{2}\right)^n\right|}$$

$$\left|x+\frac{3}{2}\right| \cdot \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}}$$

$$\left|x+\frac{3}{2}\right| < 1$$

$\boxed{x = -\frac{5}{2}}$ $\sum \frac{(-1)^n}{n}(-1)^n = \sum \frac{1}{n}$ Diverges

$\boxed{x = -\frac{1}{2}}$ $\sum \frac{(-1)^n}{n}(1)^n = \sum \frac{(-1)^n}{n}$ converges

What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-4)^{2n}}{3^n}$?

(A) $2\sqrt{3}$ (B) 3 (C) $\sqrt{3}$ (D) $\frac{\sqrt{3}}{2}$ (E) 0

$$\lim_{n\to\infty} \left| \frac{(x-4)^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{(x-4)^{2n}} \right|$$

$$\left| \frac{(x-4)^2}{3} \right| \cdot \lim_{n\to\infty} (1)$$

$$\left| \frac{(x-4)^2}{3} \right| < 1$$

$$\left| (x-4)^2 \right| < 3$$

$$|x-4| < \sqrt{3}$$

The power series $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 5$. Which of the following must be true?

(A) The series diverges at $x = 0$.

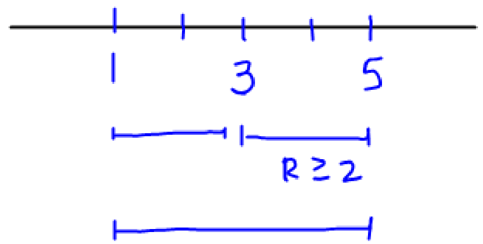
(B) The series diverges at $x = 1$.

(C) The series converges at $x = 1$.

(D) The series converges at $x = 2$.

(E) The series converges at $x = 6$.

$\sum$ centered @ $x = 3$



Convergent anywhere in between

$1 < x < 5$

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$.
The series converges if and only if $|x+1| < 1$.
Therefore, the interval of convergence is $-2 < x < 0$.

$$3 : \begin{cases} 1 : \text{identifies as geometric} \\ 1 : |x+1| < 1 \\ 1 : \text{interval of convergence} \end{cases}$$

OR

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1| < 1 \text{ when } -2 < x < 0$$

At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

OR

$$3 : \begin{cases} 1 : \text{sets up limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{interval of convergence} \end{cases}$$

(b) Since the series is geometric,

$$f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x} \text{ for } -2 < x < 0.$$

1 : answer

(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln|x| \Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{value} \end{cases}$$

(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$

$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$

$$3 : \begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \\ 1 : \text{value of } h\left(\frac{1}{2}\right) \end{cases}$$

BC Only: Taylor and Maclaurin Series (specific power series)

If a function of *f* has derivatives of all orders at $x = c$, then the series is called a Taylor Series for *f* centered at *c*. A Taylor series centered at 0 is also known as a Maclaurin Series.

A. Maclaurin Series

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

B. Taylor Series

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + f'''(c)\frac{(x-c)^3}{3!} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(c)\frac{(x-c)^n}{n!}$$

BC Only: Common Series to MEMORIZE

Series	Interval of Convergence
A. $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$ $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum (-x)^n$	$-1 < x < 1$
B. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
C. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
D. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
E. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^n x^n}{n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$	$-1 < x \le 1$
F. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-\infty < x < \infty$

. Let $P(x) = 3 - 3x^2 + \boxed{6x^4}$ be the fourth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f^{(4)}(0)$?

(A) 0 (B) $\frac{1}{4}$ (C) 6 (D) 24 (E) 144

$$6x^4 = f^{(4)}(0) \cdot \frac{x^4}{4!}$$

$$6 = \frac{1}{4!} \cdot f^{(4)}(0)$$

$$6 \cdot 4! = f^{(4)}(0)$$

Which of the following is the Maclaurin series for e^{3x} ?

(A) $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(B) $3 + 9x + \frac{27x^2}{2} + \frac{81x^3}{3!} + \frac{243x^4}{4!} + \cdots$

(C) $1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \cdots$

(D) $1 + 3x + \frac{3x^2}{2} + \frac{3x^3}{3!} + \frac{3x^4}{4!} + \cdots$

(E) $1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots$$

What is the coefficient of x^2 in the Taylor series for $\sin^2 x$ about $x = 0$?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= x^2 - \left(\frac{x^4}{3!} + \frac{x^4}{3!}\right) + \left(\frac{x^6}{5!} + \frac{x^6}{5!} + \frac{x^6}{3!\cdot 3!}\right) + \ldots$$

What is the coefficient of x^6 in the Taylor series for $\dfrac{e^{3x^2}}{2}$ about $x = 0$?

(A) $\dfrac{1}{1440}$ (B) $\dfrac{81}{160}$ (C) $\dfrac{9}{4}$ (D) $\dfrac{9}{2}$ (E) $\dfrac{27}{2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{3x^2} = 1 + (3x^2) + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{3!}$$

$$\frac{e^{3x^2}}{2} = \frac{1}{2} + \frac{3x^2}{2} + \frac{(3x^2)^2}{2 \cdot 2!} + \frac{(3x^2)^3}{2 \cdot 3!}$$

$$\downarrow$$

$$\frac{27}{12}x^6$$

Which of the following is the Maclaurin series for $\dfrac{1}{(1-x)^2}$?

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$$

(A) $1 - x + x^2 - x^3 + \cdots$

(B) $1 - 2x + 3x^2 - 4x^3 + \cdots$

(C) $1 + 2x + 3x^2 + 4x^3 + \cdots$

(D) $1 + x^2 + x^4 + x^6 + \cdots$

(E) $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[1 + x + x^2 + x^3 + \ldots\right]$$

$$= 1 + 2x + 3x^2 + \ldots$$

The third-degree Taylor polynomial for a function f about $x = 4$ is $\dfrac{(x-4)^3}{512} - \dfrac{(x-4)^2}{64} + \dfrac{(x-4)}{4} + 2$. What is the value of $f'''(4)$?

(A) $-\dfrac{1}{64}$ (B) $-\dfrac{1}{32}$ (C) $\dfrac{1}{512}$ (D) $\dfrac{3}{256}$ (E) $\dfrac{81}{256}$

General Term:

$$f'''(4) \cdot \frac{(x-4)^3}{3!} = \frac{(x-4)^3}{512}$$

$$f'''(4) = \frac{3!}{512} = \frac{6}{512} = \frac{3}{256}$$

Let f be a function having derivatives of all orders for $x > 0$ such that $f(3) = 2$, $f'(3) = -1$, $f''(3) = 6$, and $f'''(3) = 12$. Which of the following is the third-degree Taylor polynomial for f about $x = 3$?

(A) $2 - x + 6x^2 + 12x^3$

(B) $2 - x + 3x^2 + 2x^3$

(C) $2 - (x-3) + 6(x-3)^2 + 12(x-3)^3$

(D) $2 - (x-3) + 3(x-3)^2 + 4(x-3)^3$

(E) $2 - (x-3) + 3(x-3)^2 + 2(x-3)^3$

$P_3(x) = f(3) + f'(3)\cdot(x-3) + f''(3)\cdot\frac{(x-3)^2}{2!} + f'''(3)\cdot\frac{(x-3)^3}{3!}$

$P_3(x) = 2 - (x-3) + 3(x-3)^2 + 2(x-3)^3$

For $x > 0$, the power series $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$ converges to which of the following?

(A) $\cos x$ (B) $\sin x$ (C) $\frac{\sin x}{x}$ (D) $e^x - e^{x^2}$ (E) $1 + e^x - e^{x^2}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k\,f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$

$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$

$3 : \begin{cases} 1 : \text{two of } 6, -2x, \frac{x^2}{3}, -\frac{x^3}{27} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so

$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$

$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2ke^{-ax/3}$

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$

$-2ke^{-ax/3} = e^x$

$\frac{-a}{3} = 1$ and $-2k = 1$

$a = -3$ and $k = -\frac{1}{2}$

OR

$f'(x) = -2 + \frac{2}{3}x + \cdots$, so

$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$

$h(x) = 1 + x + \cdots$

$-2k = 1$ and $\frac{2}{3}ak = 1$

$k = -\frac{1}{2}$ and $a = -3$

$3 : \begin{cases} 1 : \text{computes } k f'(ax) \\ 1 : \text{recognizes } h(x) = e^x, \\ \qquad \text{or} \\ \quad \text{equates 2 series for } h(x) \\ 1 : \text{values for } a \text{ and } k \end{cases}$

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \dfrac{1}{100}$. Justify your answer.

(a) $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$

$\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$

$\dfrac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$

$3 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) No, the series does not converge when $x = 1$ because when $x = 1,$ the terms of the series do not converge to 0.

1 : answer with reason

(c) $\ln\left(1 + x^2\right) = \int_0^x \dfrac{2t}{1+t^2}\, dt$

$= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$

$= x^2 - \dfrac{1}{2}x^4 + \dfrac{1}{3}x^6 - \dfrac{1}{4}x^8 + \cdots$

$2 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \end{cases}$

(d) $\ln\left(\dfrac{5}{4}\right) = \ln\left(1 + \dfrac{1}{4}\right) = \left(\dfrac{1}{2}\right)^2 - \dfrac{1}{2}\left(\dfrac{1}{2}\right)^4 + \dfrac{1}{3}\left(\dfrac{1}{2}\right)^6 - \dfrac{1}{4}\left(\dfrac{1}{2}\right)^8 + \cdots$

Let $A = \left(\dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)^4 = \dfrac{7}{32}.$

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \left|\dfrac{1}{3}\left(\dfrac{1}{2}\right)^6\right| = \dfrac{1}{3} \cdot \dfrac{1}{64} < \dfrac{1}{100}.$

$3 : \begin{cases} 1 : \text{uses } x = \frac{1}{2} \\ 1 : \text{value of } A \\ 1 : \text{justification} \end{cases}$