BC Only: Improper Integrals

An improper integral is characterized by having a limits of integration that is infinite or the function *f* having an infinite discontinuity (asymptote) on the interval [a, b].

A. Infinite Upper Limit (continuous function)

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

B. Infinite Lower Limit (continuous function)

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

C. Both Infinite Limits (continuous function)

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx \text{, where } c \text{ is an } x \text{ value anywhere on } f.$$

D. Infinite Discontinuity (Let *x* = *k* represent an infinite discontinuity on [a, b])

$$\int_a^{b} f(x)dx = \lim_{x\to k^-}\int_a^k f(x)dx + \lim_{x\to k^+}\int_k^b f(x)dx$$

$$\int_1^\infty \frac{x^2}{(x^3+2)^2}\,dx \text{ is}$$

$u = x^3 + 2$

$du = 3x^2\,dx$

$\frac{1}{3}\int \frac{1}{u^2}\,du = \frac{1}{3}\left[-\frac{1}{u}\right]$

(A) $-\frac{1}{9}$ (B) $\frac{1}{9}$ (C) $\frac{1}{3}$ (D) 1 (E) divergent

$$\lim_{b\to\infty} -\frac{1}{3}\left[\frac{1}{x^3+2}\right]_1^b$$

$$\lim_{b\to\infty} -\frac{1}{3}\left[\frac{1}{b^3+2} - \frac{1}{3}\right]$$

$$-\frac{1}{3}\left[0 - \frac{1}{3}\right]$$

Extra: $\int_{-\sqrt[3]{2}}^{0} \frac{x^2}{(x^3+2)^2}\,dx$ = Divergent

$$\lim_{a\to -\sqrt[3]{2}^+} -\frac{1}{3}\left[\frac{1}{x^3+2}\right]_a^0$$

$$\lim_{a\to -\sqrt[3]{2}^+} -\frac{1}{3}\left[\frac{1}{2} - \frac{1}{a^3+2}\right] = -\frac{1}{3}\left[\frac{1}{2} - \infty\right] = \infty$$

$\lim_{a\to -\sqrt[3]{2}^+} (a^3+2) = 0$

Really Small positive

If $\int_1^x f(t)\,dt = \dfrac{20x}{\sqrt{4x^2+21}} - 4$, then $\boxed{\int_1^\infty f(t)\,dt}$ is

(A) 6 (B) 1 (C) -3 (D) -4 (E) divergent

$$\lim_{x\to\infty} \sqrt{\frac{400x^2}{4x^2+21}}$$

$$\lim_{x\to\infty} \int_1^x f(t)\,dt$$

$$\lim_{x\to\infty} \frac{20x}{\sqrt{4x^2+21}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} - 4$$

$$\lim_{x\to\infty} \frac{20}{\sqrt{4+\frac{21}{x^2}}} - 4$$

$$10 - 4$$

$\int_1^\infty xe^{-x^2}\,dx$ is

(A) $-\frac{1}{e}$ (B) $\frac{1}{2e}$ (C) $\frac{1}{e}$ (D) $\frac{2}{e}$ (E) divergent

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u$

$\lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b$

$\lim_{b\to\infty}\left[-\frac{1}{2e^{b^2}} + \frac{1}{2e^1}\right] = 0 + \frac{1}{2e}$

Let g be the function given by $g(x) = \dfrac{1}{\sqrt{x}}$.

(a) Find the average value of g on the closed interval $[1, 4]$.

(b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

(c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\displaystyle\lim_{b\to\infty}\left[\frac{\int_a^b f(x)\,dx}{b - a}\right]$. Show that the improper integral $\displaystyle\int_4^{\infty} g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

(a) $\frac{1}{3}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\cdot 2\sqrt{x}\Big\|_1^4 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$	$2:\begin{cases} 1: \text{integral} \\ 1: \text{antidifferentiation} \\ \quad \text{and evaluation} \end{cases}$
(b) Volume $= \pi \int_1^4 \frac{1}{x}\,dx = \pi \ln x\Big\|_1^4 = \pi \ln 4$	$2:\begin{cases} 1: \text{integral} \\ 1: \text{antidifferentiation} \\ \quad \text{and evaluation} \end{cases}$
(c) The cross section at x has area $\pi\left(\frac{1}{\sqrt{x}}\right)^2 = \frac{\pi}{x}$ Average value $= \frac{1}{3}\int_1^4 \frac{\pi}{x}\,dx = \frac{1}{3}\pi \ln 4$	1 : answer
(d) $\int_4^\infty g(x)\,dx = \lim_{b\to\infty}\int_4^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b} - 4\right) = \infty$ This limit is not finite, so the integral is divergent. $\frac{\int_4^b g(x)\,dx}{b-4} = \frac{1}{b-4}\int_4^b \frac{1}{\sqrt{x}}\,dx = \frac{2\sqrt{b}-4}{b-4}$ $\lim_{b\to\infty}\frac{2\sqrt{b}-4}{b-4} = 0$	$4:\begin{cases} 1: \int_4^b g(x)\,dx = 2\sqrt{b} - 4 \\ 1: \text{indicates integral diverges} \\ 1: \frac{1}{b-4}\int_4^b g(x)\,dx = \frac{2\sqrt{b}-4}{b-4} \\ 1: \text{finite limit as } b \to \infty \end{cases}$

BC Only: Integration by Parts

If *u* and *v* are differentiable functions of ***x***, then

$$\int u\,dv = uv - \int v\,du$$

Tips: For your choice of the function ***u***, make the selection following:

A. LIPET: Logarithmic, Inverse Trig, Polynomial, Exponential, Trig

B. LIATE: Logarithmic, Inverse Trig, Algebraic, Trig, Exponential

$*$ Comes from Integration by Parts. MEMORIZE $\int \ln x\,dx = x\ln x - x + C$

$$\int x\sin(6x)\,dx =$$

(A) $-x\cos(6x)+\sin(6x)+C$

(B) $-\frac{x}{6}\cos(6x)+\frac{1}{36}\sin(6x)+C$

(C) $-\frac{x}{6}\cos(6x)+\frac{1}{6}\sin(6x)+C$

(D) $\frac{x}{6}\cos(6x)+\frac{1}{36}\sin(6x)+C$

(E) $6x\cos(6x)-\sin(6x)+C$

$u = x \qquad dv = \sin(6x)\,dx$

$du = dx \qquad v = -\frac{1}{6}\cos(6x)$

$$-\frac{1}{6}x\cos(6x)+\frac{1}{6}\int\cos(6x)\,dx$$

$$-\frac{1}{6}x\cos(6x)+\frac{1}{36}\sin(6x)+C$$

Let f be a differentiable function such that $\int f(x)\sin x\,dx = -f(x)\cos x + \int 4x^3 \cos x\,dx$. Which of the following could be $f(x)$?

(A) $\cos x$ (B) $\sin x$ (C) $4x^3$ (D) $-x^4$ (E) x^4

$u = f(x)$ $dv = \sin x\,dx$

$du = f'(x)\,dx$ $v = -\cos x$

$-f(x)\cos x + \int f'(x)\cos x\,dx$

$f'(x) = 4x^3$

$f(x) = \int 4x^3\,dx$

$= x^4$

If f is a function such that $\boxed{f'(x) = -f(x)}$, then $\int x f(x)\,dx =$

(A) $f(x)(x+1) + C$

(B) $-f(x)(x+1) + C$

(C) $\dfrac{x^2}{2} f(x) + C$

(D) $-\dfrac{x^2}{2} f(x) + C$

(E) $-\dfrac{x^2}{2} f(x)\left(1 + \dfrac{x}{3}\right) + C$

$u = f(x) \qquad dv = x\,dx$

$du = f'(x)\,dx \qquad v = \frac{1}{2}x^2$

$\frac{1}{2}x^2 f(x) + \boxed{\frac{1}{2}\int x^2 f'(x)\,dx}$

$u = x \qquad dv = f(x)\,dx$

$du = dx \qquad dv = -f'(x)\,dx$

$v = -f(x)$

$-xf(x) + \int f(x)\,dx$

$-xf(x) - \int f'(x)\,dx$

$-x f(x) - f(x) + C$

x	2	4
$f(x)$	7	13
$g(x)$	2	9
$g'(x)$	1	7
$g''(x)$	5	8

The table above gives selected values of twice-differentiable functions f and g, as well as the first two derivatives of g. If $f'(x) = 3$ for all values of x, what is the value of $\int_2^4 f(x)g''(x)\,dx$?

(A) 63 (B) 69 (C) 78 (D) 84 (E) 103

$u = f(x) \qquad dv = g''(x)dx$

$du = f'(x)dx \qquad v = g'(x)$

$\left[f(x)g'(x) - \int f'(x)g'(x)dx\right]_2^4$

$\left[f(x)g'(x) - 3\int g'(x)dx\right]_2^4$

$\left[f(x)g'(x) - 3g(x)\right]_2^4$

$\left(f(4)g'(4) - 3g(4)\right) - \left(f(2)g'(2) - 3g(2)\right)$

The derivative of a function f is given by $f'(x) = (x - 3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$ Therefore, f has a relative minimum at $x = 3$.	$2 : \begin{cases} 1\text{: minimum at } x = 3 \\ 1\text{: justification} \end{cases}$
(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$ $f''(x) > 0$ for $x > 2$ $f'(x) < 0$ for $0 < x < 3$ Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.	$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$
(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$ $u = x - 3 \quad dv = e^x\,dx$ $du = dx \quad v = e^x$ $f(3) = 7 + (x-3)e^x\Big\vert_1^3 - \int_1^3 e^x\,dx$ $= 7 + \left((x-3)e^x - e^x\right)\Big\vert_1^3$ $= 7 + 3e - e^3$	$4 : \begin{cases} 1 : \text{uses initial condition} \\ 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

BC Only: Partial Fractions

Let *R(x)* represent a rational function of the form $R(x) = \frac{N(x)}{D(x)}$. If *D(x)* is a factorable polynomial, Partial Fractions can be used to rewrite *R(x)* as the sum or difference of simpler rational functions. Then, integration using natural log.

A. Constant Numerator

$$\int \frac{1}{x^2-5x+6}\,dx \quad \textbf{(Rule 1)}$$

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)}$$

$$\frac{A}{(x-3)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)} = \frac{(A+B)x-(2A+3B)}{(x-3)(x-2)}$$

Since

$$\frac{1}{(x-3)(x-2)} = \frac{(A+B)x-(2A+3B)}{(x-3)(x-2)},$$

$$A+B=0 \text{ and } 2A+3B=-1 \Rightarrow A=1 \text{ and } B=-1.$$

$$\int \frac{1}{x^2-5x+6}\,dx = \int\left[\frac{1}{(x-3)} + \frac{-1}{(x-2)}\right]dx = \ln|x-3| - \ln|x-2| + C$$

B. Polynomial Numerator

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx \quad \textbf{(Rule 1)}$$

$$\frac{x^2+12x+12}{x^3-4x} = \frac{x^2+12x+12}{x(x+2)(x-2)} \Rightarrow \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} = \frac{A(x^2-4)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$\begin{cases} A+B+C=1 \\ 2C-2B=12 \end{cases} \Rightarrow \begin{cases} A=-3 \\ B=-1 \end{cases}$$

$$\int \frac{1}{x^2 - 7x + 10}\,dx =$$

(A) $\ln|(x-2)(x-5)| + C$

(B) $\frac{1}{3}\ln|(x-2)(x-5)| + C$

(C) $\frac{1}{3}\ln\left|\frac{2x-7}{(x-2)(x-5)}\right| + C$

(D) $\frac{1}{3}\ln\left|\frac{x-2}{x-5}\right| + C$

(E) $\frac{1}{3}\ln\left|\frac{x-5}{x-2}\right| + C$

$$\int \frac{1}{(x-5)(x-2)}\,dx = \int \frac{A}{x-5} + \frac{B}{x-2}\,dx$$

$$1 = A(x-2) + B(x-5)$$

$$2(A+B=0)$$

$$-2A-5B=1$$

$$-3B=1$$

$$B=-1/3 \quad A=1/3$$

$$= \int \frac{1/3}{x-5} - \frac{1/3}{x-2}\,dx$$

$$= \frac{1}{3}\ln|x-5| - \frac{1}{3}\ln|x-2| + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int_0^1 \frac{5x+8}{x^2+3x+2}\,dx \text{ is}$$

(A) $\ln(8)$ (B) $\ln\left(\frac{27}{2}\right)$ (C) $\ln(18)$ (D) $\ln(288)$ (E) divergent

$$\int_0^1 \frac{5x+8}{(x+2)(x+1)}\,dx = \int_0^1 \frac{A}{x+2} + \frac{B}{x+1}\,dx$$

$5x+8 = A(x+1)+B(x+2)$

$$= \int_0^1 \frac{2}{x+2} + \frac{3}{x+1}\,dx$$

$A+B=5$

$-(A+2B=8)$

$-B=-3$

$B=3$ $A=2$

$$= \Big[2\ln|x+2| + 3\ln|x+1|\Big]_0^1$$

$$= \big(2\ln(3)+3\ln(2)\big) - \big(2\ln(2)+3\ln(1)\big)$$

$$= 2\ln(3)+\ln(2)$$

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\, dx$.

(a) $f(4) = \frac{1}{4^2 - 3\cdot 4} = \frac{1}{4}$ $\quad f'(4) = \frac{3 - 2\cdot 4}{\left(4^2 - 3\cdot 4\right)^2} = -\frac{5}{16}$ An equation for the line tangent to the graph of f at the point whose x-coordinate is 4 is $y = -\frac{5}{16}(x-4) + \frac{1}{4}$.	$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$
(b) $f'(x) = \frac{4 - 2x}{\left(x^2 - 4x\right)^2}$ $\quad f'(2) = \frac{4 - 2\cdot 2}{\left(2^2 - 4\cdot 2\right)^2} = 0$ $f'(x)$ changes sign from positive to negative at $x = 2$. Therefore, f has a relative maximum at $x = 2$.	$2 : \begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$
(c) $f'(-5) = \frac{k - 2\cdot(-5)}{\left((-5)^2 - k\cdot(-5)\right)^2} = 0 \Rightarrow k = -10$	1 : answer
(d) $\frac{1}{x^2 - 6x} = \frac{1}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$ $x = 0 \Rightarrow 1 = A\cdot(-6) \Rightarrow A = -\frac{1}{6}$ $x = 6 \Rightarrow 1 = B\cdot(6) \Rightarrow B = \frac{1}{6}$ $\frac{1}{x(x-6)} = \frac{-1/6}{x} + \frac{1/6}{x-6}$ $\int f(x)\,dx = \int\left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right)dx$ $= -\frac{1}{6}\ln\lvert x\rvert + \frac{1}{6}\ln\lvert x-6\rvert + C = \frac{1}{6}\ln\left\lvert\frac{x-6}{x}\right\rvert + C$	$4 : \begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$