Differential Equations

A differential equation is an equation involving an unknown function and one or more of its derivatives

$\frac{dy}{dx} = f(x, y)$ ⟶ Usually expressed as a derivative equal to an expression in terms of ***x*** and/or ***y***.

To solve differential equations, use the technique of separation of variables.

Given the differential equation $\frac{dy}{dx} = \frac{xy}{(x^2+1)}$

Step 1: Separate the variables, putting all *y's* on one side, with ***dy*** in the numerator, and all *x's* on the other side, with ***dx*** in the numerator.

$$\frac{1}{y}dy = \frac{x}{(x^2+1)}dx$$

Step 2: Integrate both sides of the equation.

$$\ln|y| = \frac{1}{2}\ln\sqrt{x^2+1} + C$$

Step 3: Solve the equation for y.

$$y = C\sqrt{x^2+1}$$

A student attempted to solve the differential equation $\frac{dy}{dx} = xy$ with initial condition $y = 2$ when $x = 0$. In which step, if any, does an error first appear?

Step 1: $\int \frac{1}{y}\, dy = \int x\, dx$

Step 2: $\ln|y| = \frac{x^2}{2} + C$

Step 3: $|y| = e^{x^2/2} + C$

Step 4: Since $y = 2$ when $x = 0$, $2 = e^0 + C$.

Step 5: $y = e^{x^2/2} + 1$

(A) Step 2

[✓ (B) Step 3]

(C) Step 4

(D) Step 5

(E) There is no error in the solution.

$$e^{\ln|y|} = e^{\frac{x^2}{2}} + c$$

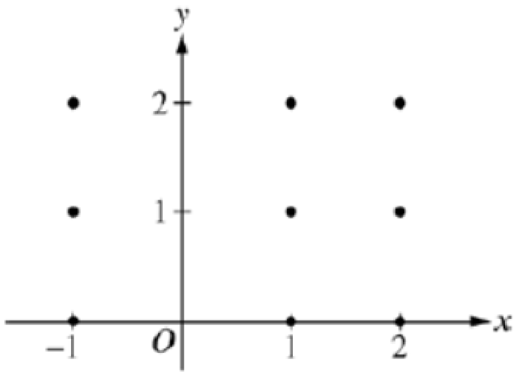
$$|y| = e^{x^2/2} \cdot e^{c} = Ce^{x^2/2}$$

Consider the differential equation $\frac{dy}{dx} = \left(1 - \frac{2}{x^2}\right)(y-1)$, where $x \neq 0$.

Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(1) = 2$.

(a) Find the slope of the line tangent to the graph of f at the point $(1, 2)$.

(b) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

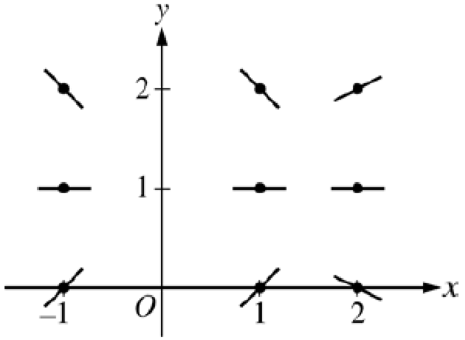


(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \left(1 - \frac{2}{x^2}\right)(y-1)$ with initial condition $f(1) = 2$.

(a) $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(1,\, 2)} = \left(1-\dfrac{2}{1}\right)(2-1) = -1$

1 : answer

(b)



$2 : \begin{cases} 1 : \text{zero slope at each point } (x,\, y) \text{ where } y = 1 \\ 1 : \text{remaining slopes} \end{cases}$

(c) $\dfrac{dy}{dx} = \left(1-\dfrac{2}{x^2}\right)(y-1)$

$$\int \frac{dy}{y-1} = \int\left(1-\frac{2}{x^2}\right)dx$$

$$\ln|y-1| = x + \frac{2}{x} + C$$

$$\ln|2-1| = 1 + \frac{2}{1} + C \Rightarrow C = -3$$

$$\ln|y-1| = x + \frac{2}{x} - 3$$

Note that $y-1>0$ since the solution curve includes the point $(1,\, 2)$.

$$\ln(y-1) = x + \frac{2}{x} - 3$$

$$y = f(x) = e^{\left(x+\frac{2}{x}-3\right)} + 1$$

Note: This solution is valid for $x > 0$.

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$$

$$3(G - 27)^{1/3} = -t + C$$

$$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$$

$$3(G - 27)^{1/3} = 12 - t$$

$$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3 \text{ for } 0 \le t < 10$$

The internal temperature of the potato at time $t = 3$ minutes is

$27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad\ \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Exponential Growth and Decay

When the rate of change of a variable *y* is directly proportional to the value of *y*, the function *y = f(x)* is said to grow/decay exponentially.

A. Differential Equation for rate of change: $\boxed{\frac{dy}{dt} = ky}$

B. General Solution: $\boxed{y = Ce^{kt}}$

I. If $k > 0$, then exponential growth occurs.

II. If $k < 0$, then exponential decay occurs.

t	0	2
$f(t)$	4	12

Let $y = f(t)$ be a solution to the differential equation $\boxed{\frac{dy}{dt} = ky,}$ where k is a constant. Values of f for selected values of t are given in the table above. Which of the following is an expression for $f(t)$?

[✓ (A)] $4e^{\frac{t}{2}\ln 3}$

(B) $e^{\frac{t}{2}\ln 9} + 3$

(C) $2t^2 + 4$

(D) $4t + 4$

$y = Ce^{kt}$

$4 = Ce^{0}$

$4 = C$

$y = 4e^{kt}$

$12 = 4e^{2k}$

$3 = e^{2k}$

$\ln 3 = 2k$

$\frac{1}{2}\ln 3 = k$

Bacteria in a certain culture increase at rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

[✓ A)] $\frac{3\ln 3}{\ln 2}$

B) $\frac{2\ln 3}{\ln 2}$

C) $\frac{\ln 3}{\ln 2}$

D) $\ln\left(\frac{27}{2}\right)$

E) $\ln\left(\frac{9}{2}\right)$

$$y = Ce^{kt}$$

$$(0, \text{☺}) \longrightarrow \text{☺} = C$$

$$y = \text{☺}e^{kt}$$

$$(3, 2\text{☺}) \longrightarrow 2\text{☺} = \text{☺}e^{3k}$$

$$2 = e^{3k}$$

$$\frac{1}{3}\ln(2) = k$$

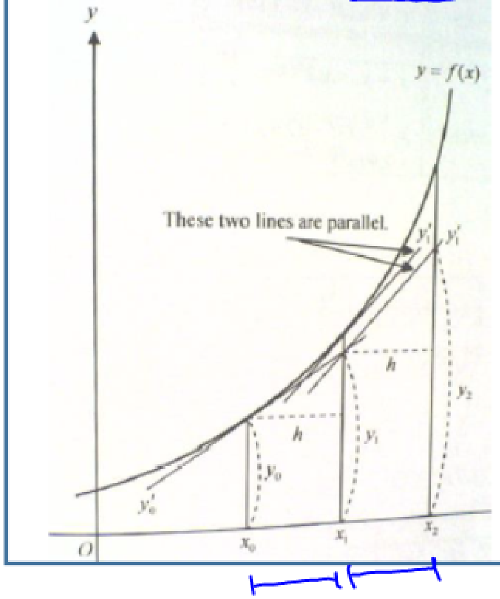
$$(t, 3\text{☺}) \longrightarrow 3\text{☺} = \text{☺}e^{\frac{1}{3}\ln(2)\cdot t}$$

$$3 = e^{\frac{1}{3}\ln(2)t}$$

$$\ln 3 = \frac{1}{3}\ln(2)\cdot t$$

BC Only: Euler's Method for Approximating the Solution of a Differential Equation

Euler's method uses a linear approximation with increments (steps) $\boxed{h,}$ for approximating the solution to a given differential equation $\boxed{\frac{dy}{dx} = F(x,y),}$ with a given initial value.



Process: Initial value (x_0, y_0)

$x_1 = x_0 + h \qquad y_1 = y_0 + h \cdot F(x_0, y_0) = y_0 + h\left[\left.\frac{dy}{dx}\right|_{(x_0,y_0)}\right]$

$x_2 = x_1 + h \qquad y_2 = y_1 + h \cdot F(x_1, y_1)$

$x_3 = x_2 + h \qquad y_3 = y_2 + h \cdot F(x_2, y_2)$

$\vdots \qquad \vdots$

* This process repeats until the desired y – value is given.

x	$f'(x)$
1	0.2
1.5	0.5
2	0.9

The table above gives values of f', the derivative of a function f. If $f(1) = 4$, what is the approximation to $f(2)$ obtained by using Euler's method with a step size of 0.5 ?

(A) 2.35

(B) 3.65

[✓ (C) 4.35]

(D) 4.70

(E) 4.80

$f(1) = 4$

$$f(1.5) = f(1) + 0.5 \cdot f'(1)$$

$$= 4 + \frac{1}{2} \cdot \frac{1}{5} = 4 + \frac{1}{10} = \frac{41}{10}$$

$$f(2) = f(1.5) + 0.5 \cdot f'(1.5)$$

$$= \frac{41}{10} + \frac{1}{2} \cdot \frac{1}{2} = \frac{41}{10} + \frac{1}{4} = \frac{87}{20} = \frac{435}{100}$$

Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 2x + y$ with initial condition $\boxed{f(1) = 0.}$ What is the approximation for $f(2)$ obtained by using Euler's method with two steps of equal length, starting at $x = 1$?

(A) 0 (B) 1 (C) 2.75 [✓ (D) 3] (E) 6

$$f(1) = 0$$

$$f(1.5) = f(1) + \frac{1}{2} \cdot \left[\frac{dy}{dx} \bigg|_{(1,0)} \right]$$

$$= 0 + \frac{1}{2} \left[2(1) + 0 \right] = 1$$

$$f(2) = 1 + \frac{1}{2} \left[\frac{dy}{dx} \bigg|_{(1.5,1)} \right]$$

$$= 1 + \frac{1}{2} \left[2(1.5) + 1 \right] = 3$$

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15, \;\; f'(1) = 8$

An equation for the tangent line is
$y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$$2 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$$3 : \begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$$

(d) $T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2$

$= 15 + 8(x - 1) + 10(x - 1)^2$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

$$2 : \begin{cases} 1 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$$

BC Only: Logistic Growth

A population, *P*, that experiences a limit factor in the growth of the population based upon the available resources to support the population is said to experience logistic growth.

A. Differential Equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$

B. General Solution: $P(t) = \frac{L}{1+be^{-kt}}$

P = population

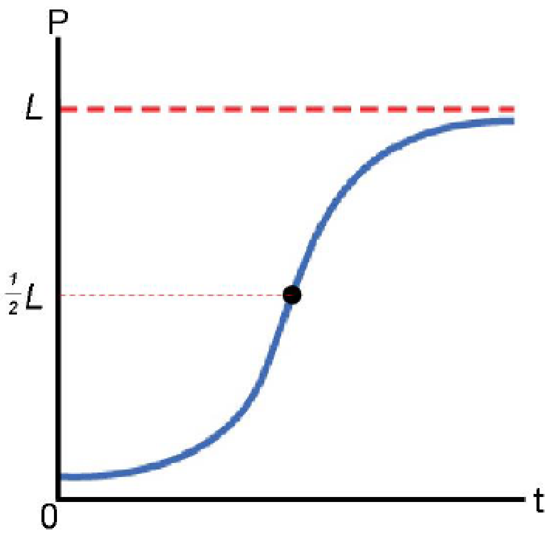
k = constant growth factor

L = carrying capacity

t = time,

b = constant (found with intital condition)

Graph



Characteristics of Logistics

I. The population is growing the fastest where $P = \frac{L}{2}$

II. The point where $P = \frac{L}{2}$ represents a point of inflection

III. $\lim_{t\to\infty} P(t) = L$

The number of antibodies y in a patient's bloodstream at time t is increasing according to a logistic differential equation. Which of the following could be the differential equation?

(A) $\frac{dy}{dt} = 0.025t$

(B) $\frac{dy}{dt} = 0.025t(5000 - t)$

(C) $\frac{dy}{dt} = 0.025y$

(D) $\frac{dy}{dt} = 0.025(5000 - y)$

[✓ (E)] $\frac{dy}{dt} = 0.025y(5000 - y)$

The rate of change, $\frac{dP}{dt}$, of the number of people on an ocean beach is modeled by a logistic differential equation. The maximum number of people allowed on the beach is 1200. At 10 A.M., the number of people on the beach is 200 and is increasing at the rate of 400 people per hour. Which of the following differential equations describes the situation?

P

$\left.\frac{dp}{dt}\right|_{p=200}$

L

(A) $\frac{dP}{dt} = \frac{1}{400}(1200 - P) + 200$

(B) $\frac{dP}{dt} = \frac{2}{5}(1200 - P)$

[✓ (C)] $\frac{dP}{dt} = \frac{1}{500}P(1200 - P)$

(D) $\frac{dP}{dt} = \frac{1}{400}P(1200 - P)$

(E) $\frac{dP}{dt} = 400P(1200 - P)$

$400 = k(200)\left(1 - \frac{200}{1200}\right)$

$2 = k\left(\frac{5}{6}\right)$

$\frac{12}{5} = k$

$$\frac{dp}{dt} = \frac{12}{5}p\left(1 - \frac{p}{1200}\right)$$

$$= \frac{12}{5} \cdot \frac{1}{1200} \cdot p(1200 - p)$$

A population y changes at a rate modeled by the differential equation $\frac{dy}{dt} = 0.2y(1000 - y)$, where t is measured in years. What are all values of y for which the population is increasing at a decreasing rate?

L=1000

(A) 500 only

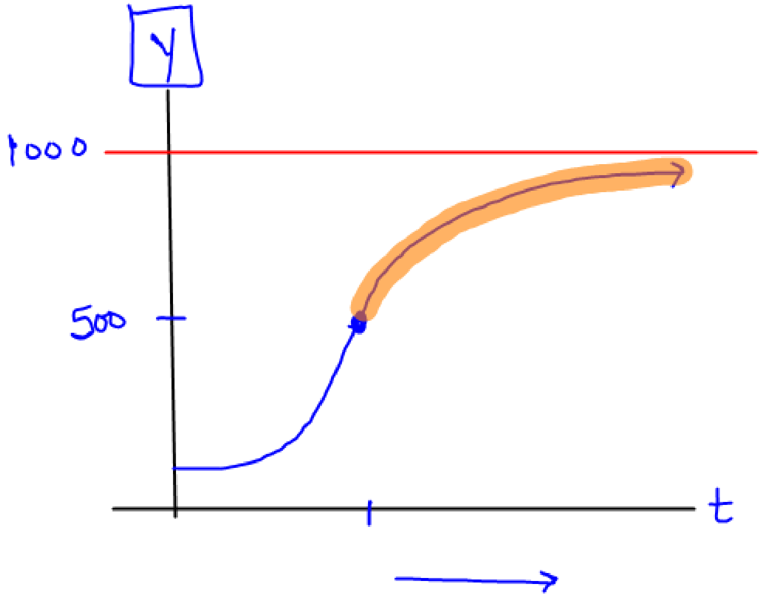
(B) $0 < y < 500$ only

[✓ (C)] $500 < y < 1000$ only

(D) $0 < y < 1000$

(E) $y > 1000$

Graph has a positive slope & is concave down



Let f be a function with $f(4) = 1$ such that all points (x, y) on the graph of f satisfy the differential equation

$$\frac{dy}{dx} = 2y(3 - x).$$

Let g be a function with $g(4) = 1$ such that all points (x, y) on the graph of g satisfy the logistic differential equation

$$\boxed{\frac{dy}{dx} = 2y(3 - y).}$$

(a) Find $y = f(x)$.

(b) Given that $g(4) = 1$, find $\underline{\lim_{x \to \infty} g(x)}$ and $\underline{\lim_{x \to \infty} g'(x).}$ (It is not necessary to solve for $g(x)$ or to show how you arrived at your answers.)

(c) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection. (It is not necessary to solve for $g(x)$.)

(a) $\dfrac{dy}{dx} = 2y(3 - x)$

$\dfrac{1}{y}\,dy = 2(3 - x)\,dx$

$\ln|y| = 6x - x^2 + C$

$0 = 24 - 16 + C$

$C = -8$

$\ln|y| = 6x - x^2 - 8$

$y = e^{6x - x^2 - 8}$ for $-\infty < x < \infty$

(b) $\lim\limits_{x\to\infty} g(x) = 3$

$\lim\limits_{x\to\infty} g'(x) = 0$

(c) $\dfrac{d^2y}{dx^2} = (6 - 4y)\dfrac{dy}{dx}$

Because $\dfrac{dy}{dx} \neq 0$ at any point on the graph of g, the concavity only changes sign at $y = \dfrac{3}{2}$, half the carrying capacity.

$$\left.\frac{dy}{dx}\right|_{y=3/2} = 2\left(\frac{3}{2}\right)\left(3 - \frac{3}{2}\right) = \frac{9}{2}$$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solution} \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

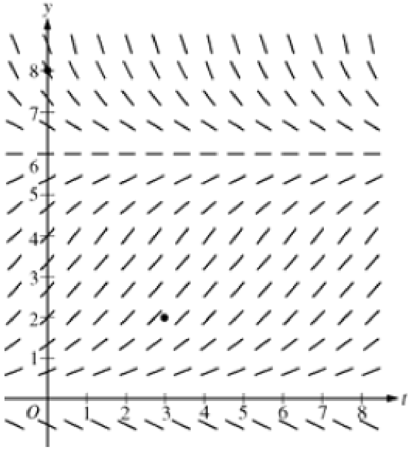
Note: $0/5$ if no separation of variables

$$2 : \begin{cases} 1 : \lim\limits_{x\to\infty} g(x) = 3 \\ 1 : \lim\limits_{x\to\infty} g'(x) = 0 \end{cases}$$

$$2 : \begin{cases} 1 : y = \dfrac{3}{2} \\ 1 : \left.\dfrac{dy}{dx}\right|_{y=3/2} \end{cases}$$

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

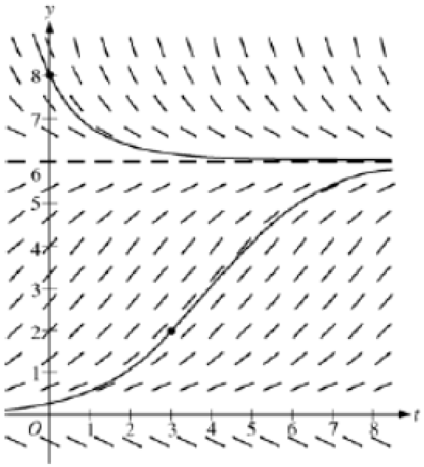


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

(a)



$$2 : \begin{cases} 1\text{: solution curve through } (0, 8) \\ 1\text{: solution curve through } (3, 2) \end{cases}$$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8;\ f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2;$ and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

$$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$$

(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer