Volumes of Known Cross Sections

If a defined region, bounded by a differentiable function *f*, is used at the base of a solid, then the volume of the solid can be found by integrated using known area formulas.

I. Cross sections are squares

$$\text{Volume} = \int_a^b [f(x)]^2 dx$$

II. Cross sections are equilateral triangles

$$\text{Volume} = \frac{\sqrt{3}}{4}\int_a^b [f(x)]^2 dx$$

III. Cross sections are isosceles right triangles with a leg in the base

$$\text{Volume} = \frac{1}{2}\int_a^b [f(x)]^2 dx$$

IV. Cross sections are isosceles right triangles with the hypotenuse in the base

$$\text{Volume} = \frac{1}{4}\int_a^b [f(x)]^2 dx$$

V. Cross sections are semicircles (with diameter in base)

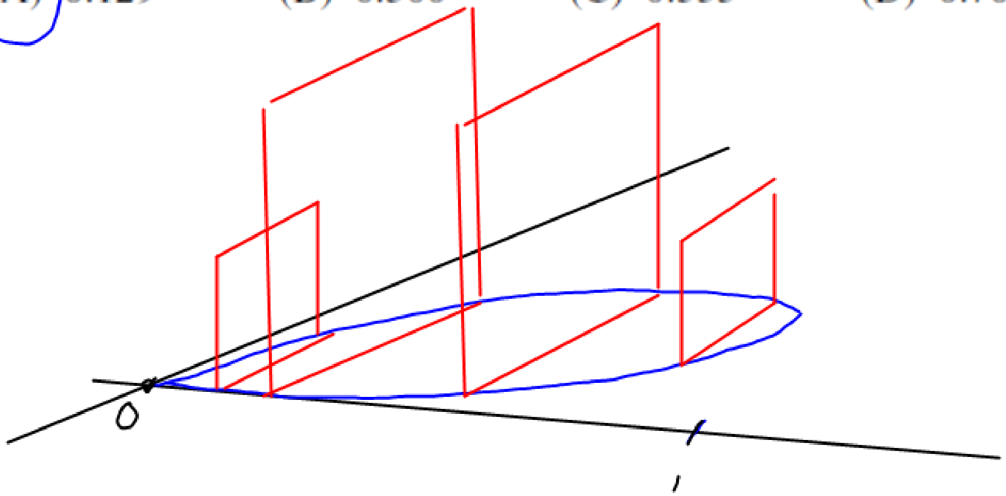
$$\text{Volume} = \frac{\pi}{8}\int_a^b [f(x)]^2 dx$$

VI. Cross sections are semicircles (with radius in base)

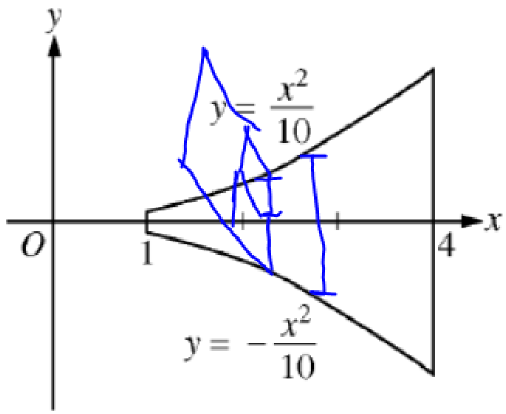
$$\text{Volume} = \frac{\pi}{2}\int_a^b [f(x)]^2 dx$$

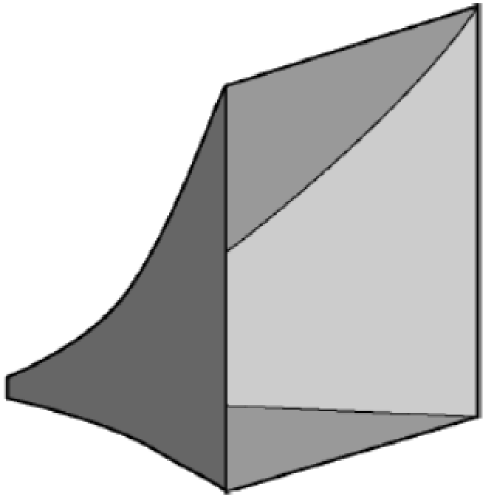
Let R be the region in the first quadrant bounded below by the graph of $y = x^2$ and above by the graph of $y = \sqrt{x}$. R is the base of a solid whose cross sections perpendicular to the x-axis are squares. What is the volume of the solid?

(A) 0.129 (B) 0.300 (C) 0.333 (D) 0.700 (E) 1.271



$$\int_0^1 \left[\sqrt{x} - x^2\right]^2 dx$$

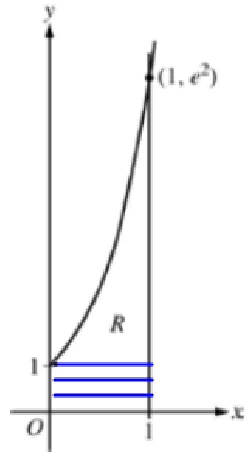




The base of a loudspeaker is determined by the two curves $y = \frac{x^2}{10}$ and $y = -\frac{x^2}{10}$ for $1 \le x \le 4$, as shown in the figure above. For this loudspeaker, the cross sections perpendicular to the x-axis are squares. What is the volume of the loudspeaker, in cubic units?

(A) 2.046 (B) 4.092 (C) 4.200 (D) 8.184 (E) 25.711

$$\int_1^4 \left(\frac{x^2}{10} - \frac{-x^2}{10} \right)^2 dx = \int_1^4 \left(\frac{1}{5}x^2 \right)^2 dx = \frac{1}{25} \int_1^4 x^4 \, dx$$



Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of $y = f(x)$ and the vertical line $x = 1$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$.

(b) Find the area of R.

(c) Region R forms the base of a solid whose cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(a) $f(1) = e^2$ $f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$ An equation for the tangent line is $y = e^2 + 2e^2(x-1)$.	$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$
(b) Area $= \int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_{x=0}^{x=1} = \frac{1}{2}\left(e^2 - 1\right)$	$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$
(c) Volume $= 1 + \int_1^{e^2} \left(1 - \frac{1}{2}\ln y\right)^2 dy$ $\int_0^1 dy$	$4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

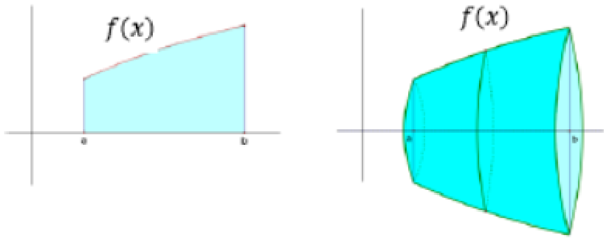
	1 : units in parts (a), (c), and (d)
(a) Volume $= \int_0^{10} A(h)\,dh$ $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ $= 176.3$ cubic feet	$2 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h)\,dh = 101.325338$ The volume is 101.325 cubic feet.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x)\,dx$. $\left.\frac{dV}{dt}\right\|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ $= f(5)\cdot 0.26 = 1.694419$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

Volumes of a Solid of Revolution: Disk Method

If a defined region, bounded by a differentiable function *f*, on a graph is rotated about a line, the resulting solid is called a solid of revolution and the line is called the axis of revolution. The disk method is used when the defined region boarders the axis of revolution over the entire interval [a, b]

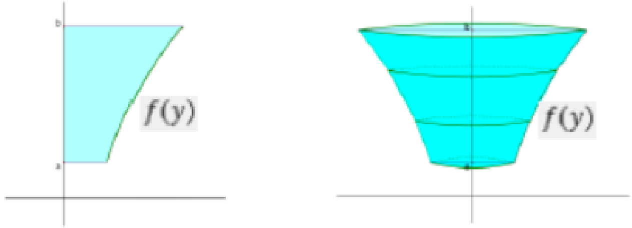
A. Revolving around the x – axis

$$\text{Volume} = \pi \int_a^b (f(x))^2 dx$$



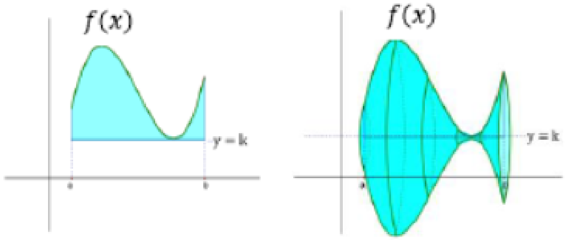
B. Revolving around the y – axis

$$\text{Volume} = \pi \int_a^b (f(y))^2 dy$$



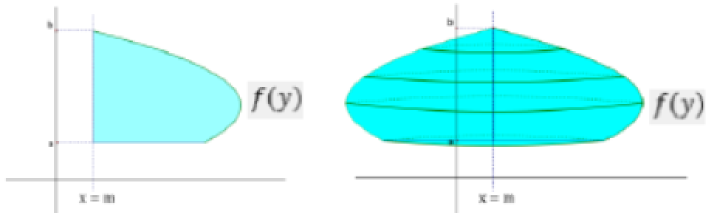
C. Revolving around a horizontal line *y = k*

$$\text{Volume} = \pi \int_a^b (f(x) - k)^2 dx$$



D. Revolving around a vertical line *x = m*

$$\text{Volume} = \pi \int_a^b (f(y) - m)^2 dy$$

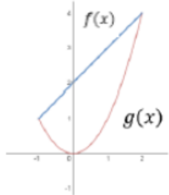


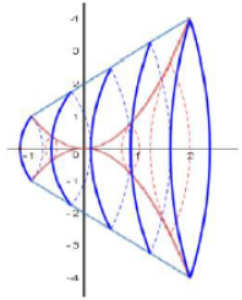
Volumes of a Solid of Revolution: Washer Method

If a defined region, bounded by a differentiable function *f*, on a graph is rotated about a line, the resulting solid is called a solid of revolution and the line is called the axis of revolution. The washer method is used when the defined region has space between the axis of revolution on the interval [a, b]

A. Revolving around the x – axis, where $f(x) \geq g(x)$(meaning the function *f* is always above the function *g* on the graph) for every *x* on the interval [a, b].

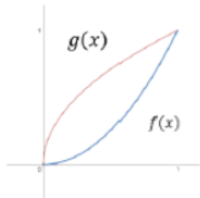
$$\text{Volume} = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

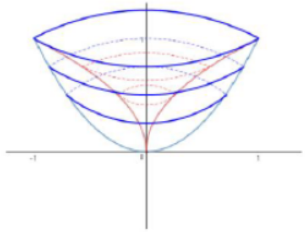




B. Revolving around the y – axis, where $f(y) \geq g(y)$(meaning the function *f* is always to the right of the function *g* on the graph)

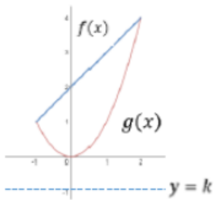
$$\text{Volume} = \pi \int_a^b ([f(y)]^2 - [g(y)]^2)dy$$

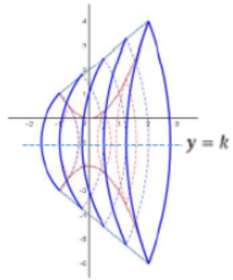




C. Revolving around a horizontal line *y* = *k*, where $f(x) \geq g(x)$(meaning the function *f* is always above the function *g* on the graph) for every *x* on the interval [a, b].

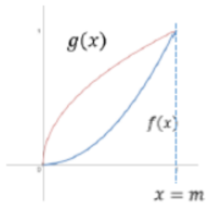
$$\text{Volume} = \pi \int_a^b ([f(x) - k]^2 - [g(x) - k]^2)dx$$

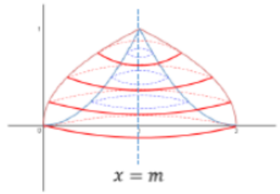




D. Revolving around a vertical line *x* = *m*, where $f(y) \geq g(y)$(meaning the function *f* is always to the right of the function *g* on the graph)

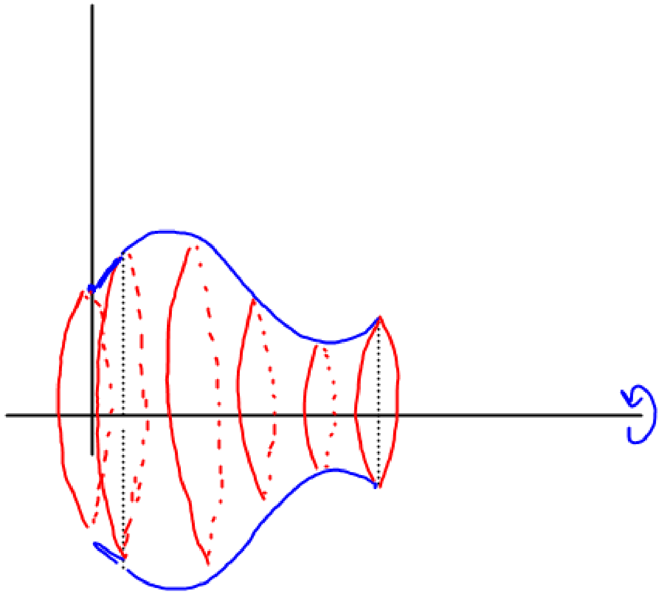
$$\text{Volume} = \pi \int_a^b ([f(y) - m]^2 - [g(y) - m]^2)dy$$





A vase has the shape obtained by revolving the curve $y = 2 + \sin x$ from $x = 0$ to $x = 5$ about the x-axis, where x and y are measured in inches. What is the volume, in cubic inches, of the vase?

(A) 10.716 (B) 25.501 (C) 33.666 (D) 71.113 (E) 80.115

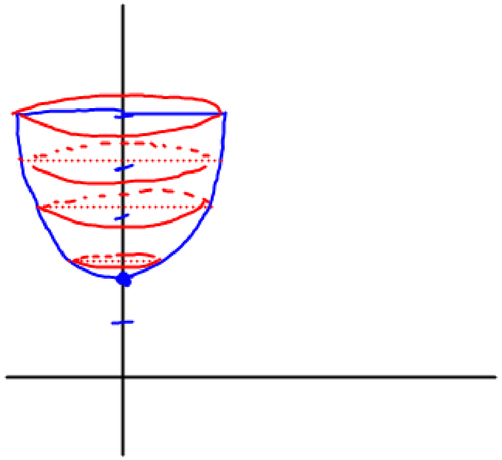


$$\pi \int_0^5 (2+\sin x)^2 \, dx$$

What is the volume of the solid generated when the region bounded by the graph of $x = \sqrt{y-2}$ and the lines $x = 0$ and $y = 5$ is revolved about the y-axis?

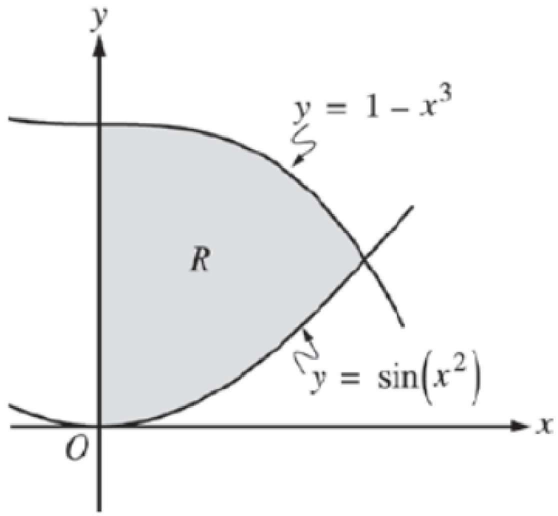
(A) 3.464 (B) 4.500 (C) 7.854 (D) 10.883 (E) 14.137

$y = x^2 + 2$



$$\pi \int_2^5 \left[\sqrt{y-2}\right]^2 dy$$

Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$, as shown in the figure above.



(a) Find the area of R.

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$ intersect. Find k and determine whether this line divides R into two regions of equal area. Show the work that leads to your conclusion.

(c) Find the volume of the solid generated when R is revolved about the line $y = -3$.

The graphs of $y = 1 - x^3$ and $y = \sin\left(x^2\right)$ intersect in the first quadrant at the point $(A, B) = (0.764972, 0.552352)$.

1 : correct limits in an integral in (a), (b), or (c)

(a) $$\text{Area} = \int_0^A \left(1 - x^3 - \sin\left(x^2\right)\right) dx = 0.533 \text{ (or } 0.534)$$

2 : { 1 : integrand; 1 : answer }

(b) $k = B = 0.552352$

$$\int_0^A \left(1 - x^3 - k\right) dx = 0.257 \text{ (or } 0.256)$$

$$\int_0^A \left(k - \sin\left(x^2\right)\right) dx = 0.277 \text{ (or } 0.276)$$

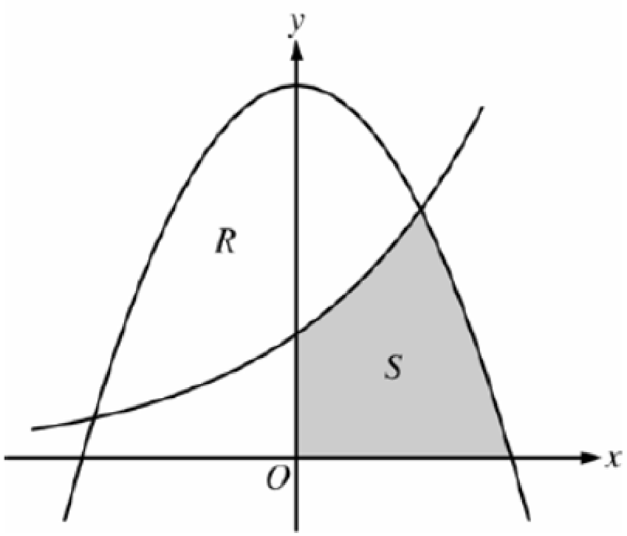
The two regions have unequal areas.

3 : { 1 : integral(s) with k value; 1 : value(s) of integral(s); 1 : conclusion tied to part (a) or comparison of two integrals }

Note: Stating k value only does not earn a point.

(c) $$\text{Volume} = \pi \int_0^A \left(\left(1 - x^3 + 3\right)^2 - \left(\sin\left(x^2\right) + 3\right)^2\right) dx = 11.841 \text{ (or } 11.840)$$

3 : { 2 : integrand; 1 : answer }



Let R and S in the figure above be defined as follows: R is the region in the first and second quadrants bounded by the graphs of $y = 3 - x^2$ and $y = 2^x$. S is the shaded region in the first quadrant bounded by the two graphs, the x-axis, and the y-axis.

(a) Find the area of S.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with one leg across the base of the solid. Write, but do not evaluate, an integral expression that gives the volume of the solid.

$3 - x^2 = 2^x$ when $x = -1.63658$ and $x = 1$
Let $a = -1.63658$

(a) Area of $S = \int_0^1 2^x \, dx + \int_1^{\sqrt{3}} \left(3 - x^2\right) dx$
$= 2.240$

$3 : \begin{cases} 1 : \text{integrands} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_a^1 \left(\left(3 - x^2 + 1\right)^2 - \left(2^x + 1\right)^2\right) dx$
$= 63.106$ or 63.107

$4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \frac{1}{2} \int_a^1 \left(3 - x^2 - 2^x\right)^2 dx$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$