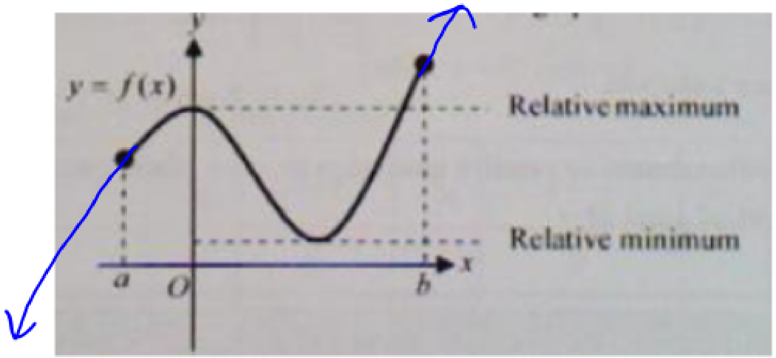
B. Relative Extrema

1st Deriv Test

I. Relative Maximum: The y-value of a function where the graph of the function changes from increasing to decreasing. Another way to define a relative maximum is the y-value where derivative of a function changes from positive to negative.

II. Relative Minimum: The y-value of a function where the graph of the function changes from decreasing to increasing. Another way to define a relative maximum is the y-value where derivative of a function changes from negative to positive.



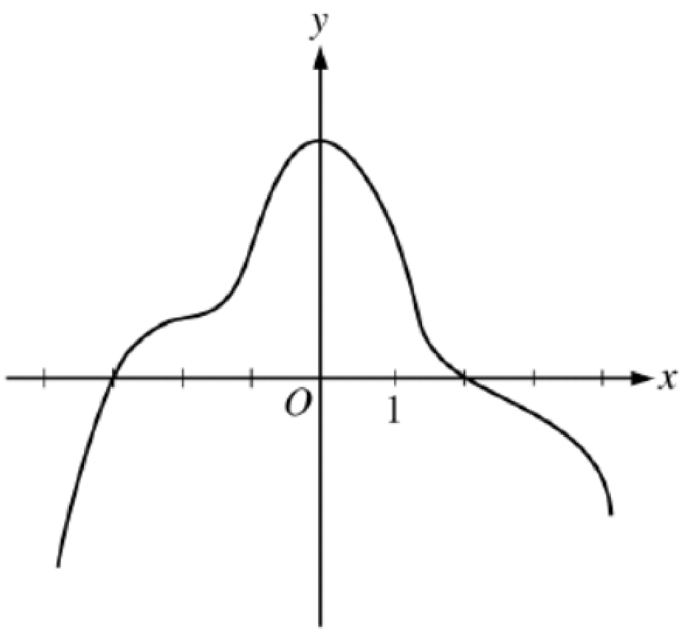
2nd Deriv Test:

① Rel Min: $f'(x)=0$ & $f''(x)>0$

② Rel Max: $f'(x)=0$ & $f''(x)<0$

Point of Inflection

Let *f* be a functions whose second derivative exists on any interval. If *f* is continuous at *x = c*, *f ''(c) = 0* or *f ''(c)* is undefined, and *f ''(x)* changes sign at *x = c*, then the point $(c, f(c))$ is a point of inflection.



Graph of f'

The graph of f', the derivative of the function f, is shown above. Which of the following statements must be true?

I. f has a relative minimum at $x = -3$.

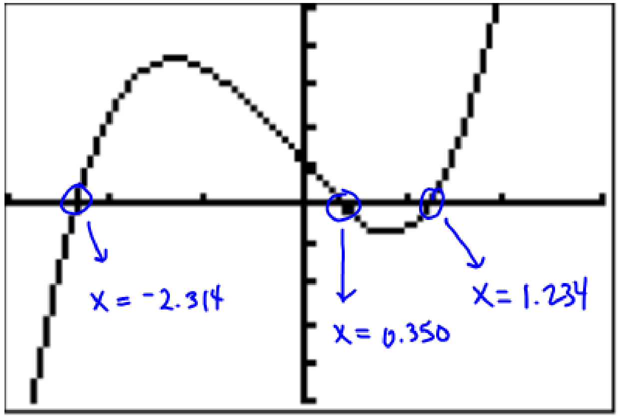
II. The graph of f has a point of inflection at $x = -2$.

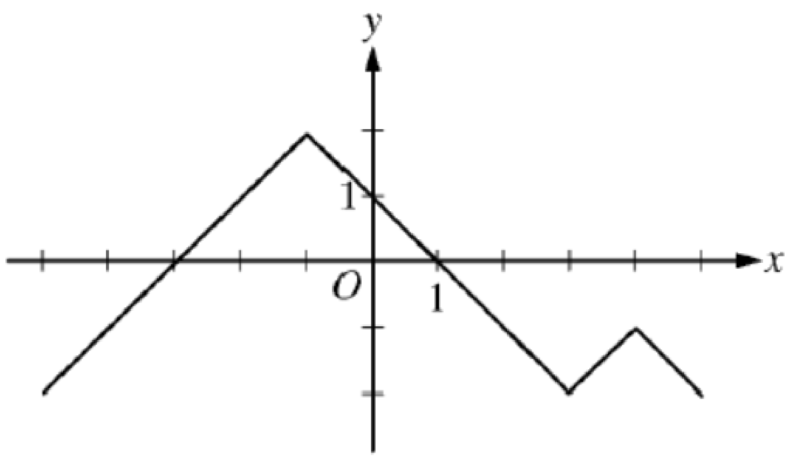
III. The graph of f is concave down for $0 < x < 4$.

(A) I only (B) II only (C) III only (D) I and II only (E) I and III only

If $f'(x) = \sqrt{x^4 + 1} + x^3 - 3x$, then f has a local maximum at $x =$

(A) -2.314 (B) -1.332 (C) 0.350 (D) 0.829 (E) 1.234





Graph of f'

The graph of f', the derivative of f, is shown in the figure above. The function f has a local maximum at $x =$

(A) -3 (B) -1 (C) 1 (D) 3 (E) 4

The function f is defined by $f(x) = \sin x + \cos x$ for $0 \le x \le 2\pi$. What is the x-coordinate of the point of inflection where the graph of f changes from concave down to concave up?

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $\frac{7\pi}{4}$ (E) $\frac{9\pi}{4}$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

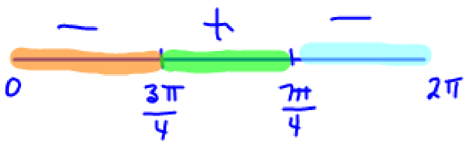
$= -\sin x - \cos x$

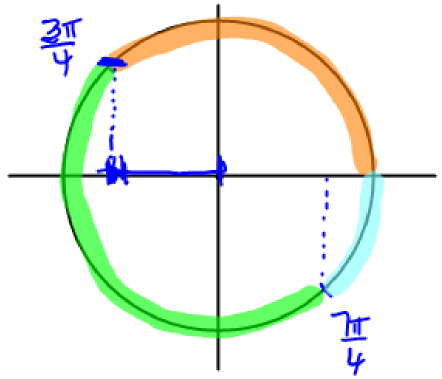
$-\sin x - \cos x = 0$

$-\sin x = \cos x$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f'' = -(\sin x + \cos x)$





Second Fundamental Theorem of Calculus

If a function *f* is continuous on the interval [a, b], let u represent a function of *x*, then

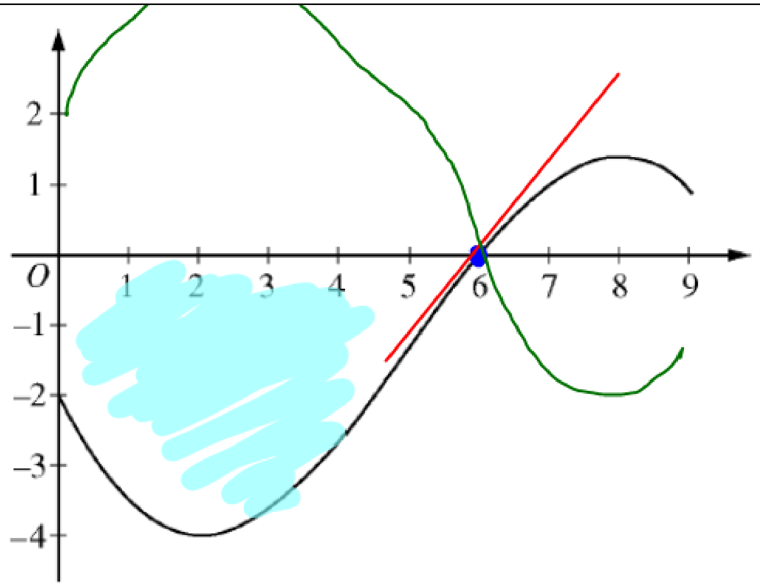
A. $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$

B. $\frac{d}{dx}\left[\int_x^b f(t)dt\right] = -f(x)$

C. $\frac{d}{dx}\left[\int_a^{u(x)} f(t)dt\right] = f(u(x)) \cdot u'(x)$

D. $\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)dt\right] = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$

Ex: $\frac{d}{dt}\left[\int_{2t}^{t^3} \ln\left(\sin\left(\sqrt{x^2+x}\right)\right) dx\right] = \ln\left(\sin\left(\sqrt{t^6+t^3}\right)\right) \cdot 3t^2 - \ln\left(\sin\left(\sqrt{4t^2+2t}\right)\right) \cdot 2$



Graph of f

The graph of a differentiable function f is shown above. If $h(x) = \int_0^x f(t)\,dt$, which of the following is true?

(A) $h(6) < h'(6) < h''(6)$

(B) $h(6) < h''(6) < h'(6)$

(C) $h'(6) < h(6) < h''(6)$

(D) $h''(6) < h(6) < h'(6)$

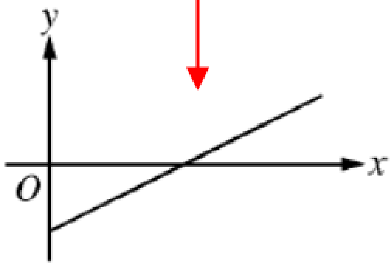
(E) $h''(6) < h'(6) < h(6)$

$h(6) = \int_0^6 f(t)\,dt < 0$

$h'(6) = f(6) = 0$

$h''(6) = f'(6) > 0$

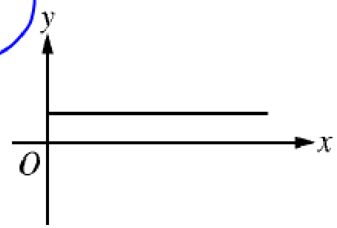
The figure ~~above~~ shows the graph of f. If $f(x) = \int_2^x g(t)\,dt$, which of the following could be the graph of $y = g(x)$?



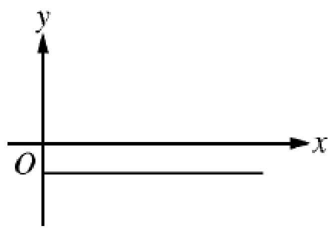
Graph of f

$g(x) = f'(x)$

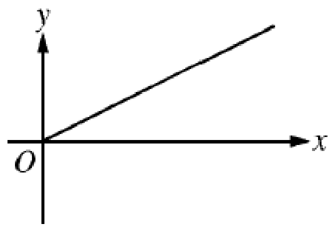
(A)



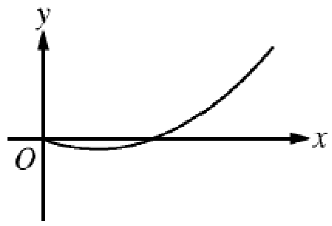
(B)



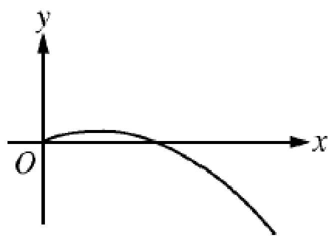
(C)



(D)



(E)



Let g be a function with first derivative given by $g'(x) = \int_0^x e^{-t^2}\,dt$. Which of the following must be true on the interval $0 < x < 2$?

(A) g is increasing, and the graph of g is concave up.

(B) g is increasing, and the graph of g is concave down.

(C) g is decreasing, and the graph of g is concave up.

(D) g is decreasing, and the graph of g is concave down.

(E) g is decreasing, and the graph of g has a point of inflection on $0 < x < 2$.

$g'(x) = \int_0^x \frac{1}{e^{t^2}}\,dt$

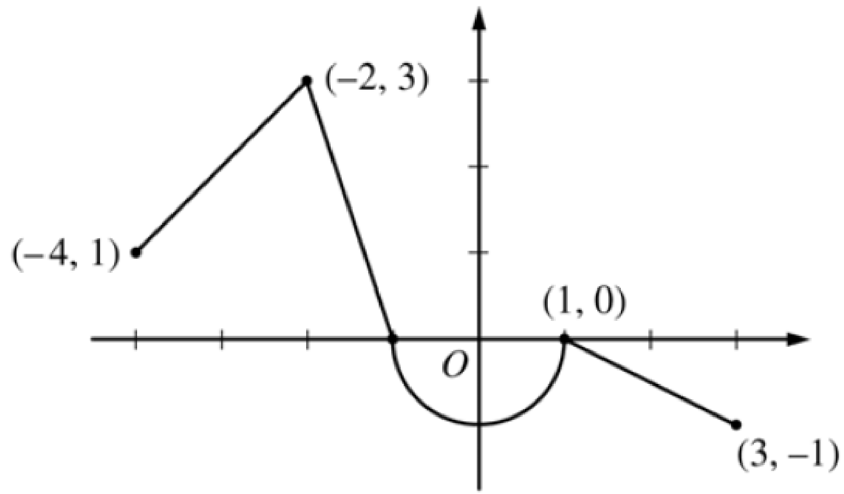
ABOVE X-AXIS

$g''(x) = \frac{1}{e^{x^2}}$

Always positive

e^a such $a \in \mathbb{R} \rightarrow$ always positive

2012 Question 3 (non calculator)



Graph of f

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

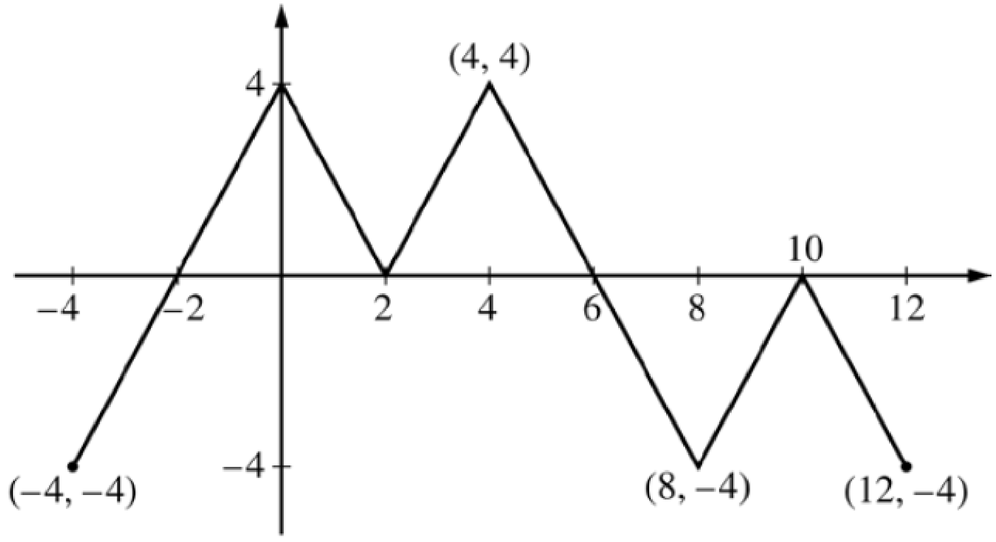
(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$ $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$ $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$	$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$
(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$ $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$	$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$
(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$. $g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$. $g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.	$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$
(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

2016 Question 3 (non calculator)



Graph of f

The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

4 : { 1 : considers $x = -2$ and $x = 6$ as candidates
1 : considers $x = -4$ and $x = 12$
2 : answers with justification

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals

Riemann Sum (Approximations)

A Riemann Sum is the use of geometric shapes (rectangles and trapezoids) to approximate the area under a curve, therefore approximating the value of a definite integral.

If the interval [a, b] is partitioned into n subintervals, then each subinterval, Δx, has a width: $\Delta x = \frac{b-a}{n}$.

Therefore, you find the sum of the geometric shapes, which approximates the area by the following formulas:

A. Right Riemann Sum

$$Area \approx \Delta x\,[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})]$$

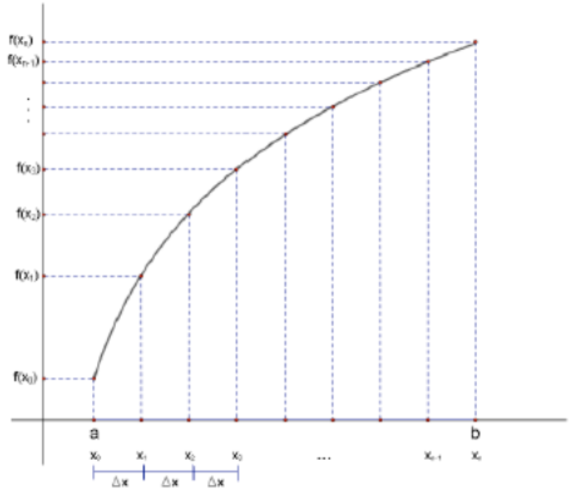
B. Left Riemann Sum

$$Area \approx \Delta x\,[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)]$$

C. Midpoint Riemann Sum

$$Area \approx \Delta x\,[f(x_{1/2}) + f(x_{3/2}) + f(x_{5/2}) + \cdots + f(x_{(2n-1)/2})]$$

D. Trapezoidal Sum



$$Area \approx \frac{1}{2}\Delta x\,[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Properties of Riemann Sums

A. The area under the curve is under approximated when

I. A Left Riemann sum is used on an increasing function.

II. A Right Riemann sum is used on a decreasing function.

III. A Trapezoidal sum is used on a concave down function.

B. The area under the curve is over approximated when

I. A Left Riemann sum is used on a decreasing function.

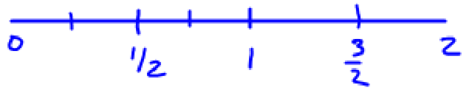
II. A Right Riemann sum is used on an increasing function.

III. A Trapezoidal sum is used on a concave up function.

Let f be the function given by $f(x) = 9^x$. If four subintervals of equal length are used, what is the value of the right Riemann sum approximation for $\int_0^2 f(x)\,dx$?

(A) 20 (B) 40 (C) 60 (D) 80 (E) 120

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$



Right: $\frac{1}{2}\left[9^{1/2} + 9^{1} + 9^{3/2} + 9^{2}\right] = 60$

Left: $\frac{1}{2}\left[9^{0} + 9^{1/2} + 9^{1} + 9^{3/2}\right] = 20$

Mid: $\frac{1}{2}\left[9^{1/4} + 9^{3/4} + 9^{5/4} + 9^{7/4}\right]$

Trap: $\frac{1}{2}\cdot\frac{1}{2}\left[9^{0} + 2\cdot 9^{1/2} + 2\cdot 9^{1} + 2\cdot 9^{3/2} + 9^{2}\right]$ ($\frac{1}{2} \to \Delta x$)

$\int_0^2 9^x\,dx = \left[\frac{1}{\ln 9}\cdot 9^x\right]_0^2$

$\int e^x\,dx = e^x$

t (hours)	4	7	12	15
$R(t)$ (liters/hour)	6.5	6.2	5.9	5.6

A tank contains 50 liters of oil at time $t = 4$ hours. Oil is being pumped into the tank at a rate $R(t)$, where $R(t)$ is measured in liters per hour, and t is measured in hours. Selected values of $R(t)$ are given in the table above. Using a right Riemann sum with three subintervals and data from the table, what is the approximation of the number of liters of oil that are in the tank at time $t = 15$ hours?

(A) 64.9 (B) 68.2 (C) 114.9 (D) 116.6 (E) 118.2

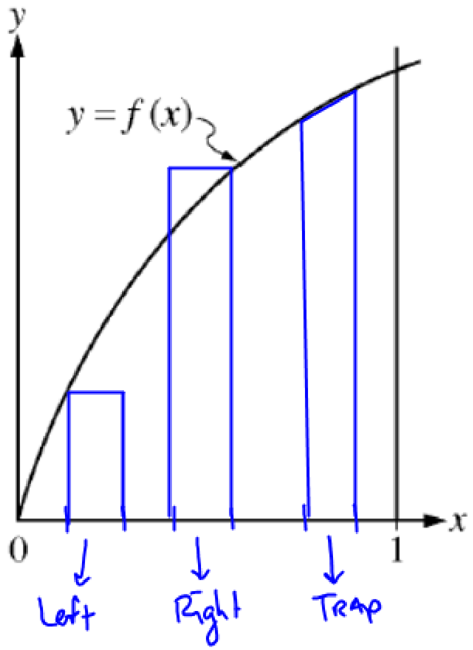
$$50 + \int_4^{15} R(t)\,dt = 50 + \left[3 \cdot R(7) + 5 \cdot R(12) + 3 \cdot R(15)\right]$$

A left Riemann sum, a right Riemann sum, and a trapezoidal sum are used to approximate the value of $\int_0^1 f(x)\,dx$, each using the same number of subintervals. The graph of the function f is shown in the figure above. Which of the sums give an underestimate of the value of $\int_0^1 f(x)\,dx$?

I. Left sum

II. Right sum

III. Trapezoidal sum



(A) I only

(B) II only

(C) III only

(D) I and III only

(E) II and III only

x	0	2	4	6
$f(x)$	4	k	8	12

The function f is continuous on the closed interval $[0, 6]$ and has the values given in the table above. The trapezoidal approximation for $\int_0^6 f(x)\,dx$ found with 3 subintervals of equal length is 52. What is the value of k ?

(A) 2 (B) 6 (C) 7 (D) 10 (E) 14

$$\int_0^6 f(x)\,dx = \frac{1}{2}\cdot 2\left[f(0) + 2\cdot f(2) + 2\cdot f(4) + f(6)\right] = 52$$

$$4 + 2k + 16 + 12 = 52$$

$$2k = 20$$

Calculator Active

t (minutes)	0	4	8	12	16
$H(t)$ (°C)	65	68	73	80	90

The temperature, in degrees Celsius (°C), of an oven being heated is modeled by an increasing differentiable function H of time t, where t is measured in minutes. The table above gives the temperature as recorded every 4 minutes over a 16-minute period.

(a) Use the data in the table to estimate the instantaneous rate at which the temperature of the oven is changing at time $t = 10$. Show the computations that lead to your answer. Indicate units of measure.

(b) Write an integral expression in terms of H for the average temperature of the oven between time $t = 0$ and time $t = 16$. Estimate the average temperature of the oven using a left Riemann sum with four subintervals of equal length. Show the computations that lead to your answer.

(c) Is your approximation in part (b) an underestimate or an overestimate of the average temperature? Give a reason for your answer.

(d) Are the data in the table consistent with or do they contradict the claim that the temperature of the oven is increasing at an increasing rate? Give a reason for your answer.

(a) $H'(10) \approx \frac{H(12) - H(8)}{12 - 8} = \frac{80 - 73}{4} = \frac{7}{4}\,°\text{C/min}$	$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer with units} \end{cases}$
(b) Average temperature is $\frac{1}{16}\int_0^{16} H(t)\,dt$ $\int_0^{16} H(t)\,dt \approx 4 \cdot (65 + 68 + 73 + 80)$ Average temperature $\approx \frac{4 \cdot 286}{16} = 71.5°\text{C}$	$3 : \begin{cases} 1 : \frac{1}{16}\int_0^{16} H(t)\,dt \\ 1 : \text{left Riemann sum} \\ 1 : \text{answer} \end{cases}$
(c) The left Riemann sum approximation is an underestimate of the integral because the graph of H is increasing. Dividing by 16 will not change the inequality, so $71.5°\text{C}$ is an underestimate of the average temperature.	1 : answer with reason
(d) If a continuous function is increasing at an increasing rate, then the slopes of the secant lines of the graph of the function are increasing. The slopes of the secant lines for the four intervals in the table are $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, and $\frac{10}{4}$, respectively. Since the slopes are increasing, the data are consistent with the claim. OR By the Mean Value Theorem, the slopes are also the values of $H'(c_k)$ for some times $c_1 < c_2 < c_3 < c_4$, respectively. Since these derivative values are positive and increasing, the data are consistent with the claim.	$3 : \begin{cases} 1 : \text{considers slopes of four secant lines} \\ 1 : \text{explanation} \\ 1 : \text{conclusion consistent with explanation} \end{cases}$

Calculator Active

t (hours)	0	0.4	0.8	1.2	1.6	2.0	2.4
$v(t)$ (miles per hour)	0	11.8	9.5	17.2	16.3	16.8	20.1

Ruth rode her bicycle on a straight trail. She recorded her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 2.4$ hours, as shown in the table above. For $0 < t \le 2.4$, $v(t) > 0$.

(a) Use the data in the table to approximate Ruth's acceleration at time $t = 1.4$ hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{2.4} v(t)\,dt$ in the context of the problem. Approximate $\int_0^{2.4} v(t)\,dt$ using a midpoint Riemann sum with three subintervals of equal length and values from the table.

(c) For $0 \le t \le 2.4$ hours, Ruth's velocity can be modeled by the function g given by $g(t) = \dfrac{24t + 5\sin(6t)}{t + 0.7}$. According to the model, what was Ruth's average velocity during the time interval $0 \le t \le 2.4$?

(d) According to the model given in part (c), is Ruth's speed increasing or decreasing at time $t = 1.3$? Give a reason for your answer.

(a) $a(1.4) \approx \frac{v(1.6)-v(1.2)}{1.6-1.2} = \frac{16.3-17.2}{1.6-1.2} = -2.25 \text{ miles/hr}^2$	$2: \begin{cases} 1: \text{approximation} \\ 1: \text{units} \end{cases}$
(b) $\int_0^{2.4} v(t)\,dt$ is the total distance Ruth traveled, in miles, from time $t = 0$ to time $t = 2.4$ hours. $\int_0^{2.4} v(t)\,dt \approx (0.8)(11.8) + (0.8)(17.2) + (0.8)(16.8)$ $= 36.64$ miles	$3: \begin{cases} 1: \text{interpretation} \\ 1: \text{midpoint Riemann sum} \\ 1: \text{approximation} \end{cases}$
(c) Average velocity $= \frac{1}{2.4}\int_0^{2.4} g(t)\,dt = 14.064$ miles/hr	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(d) Velocity $= g(1.3) = 18.096358 > 0$ Acceleration $= g'(1.3) = 3.761152 > 0$ Ruth's speed is increasing at time $t = 1.3$ since velocity and acceleration have the same sign at this time.	2 : conclusion with reason