Explicit and Implicit Differentiation

A. Explicit Functions: Function y is written only in terms of the variable x $\left(y = f(x)\right)$. Apply derivatives rules normally.

B. Implicit Differentiation: An expression representing the graph of a curve in terms of both variables x and y.

I. Differentiate both sides of the equation with respect to x. (terms with x differentiate normally, terms with y are multiplied by $\frac{dy}{dx}$ per the chain rule)

II. Group all terms with $\frac{dy}{dx}$ on one side of the equation and all other terms on the other side of the equation.

III. Factor $\frac{dy}{dx}$ and express $\frac{dy}{dx}$ in terms of x and y.

If $x + \boxed{2x}y - y^2 = 2$, then at the point $\boxed{(1,1)}$, $\frac{dy}{dx}$ is

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{3}{2}$ $\boxed{\text{(E) nonexistent}}$

$$1 + 2\left[x\frac{dy}{dx} + y\right] - 2y\frac{dy}{dx} = 0$$

$$1 + 2\left[\frac{dy}{dx} + 1\right] - 2\frac{dy}{dx} = 0 \longrightarrow 3 \neq 0$$

$$1 + 2\frac{dy}{dx} + 2 - 2\frac{dy}{dx} = 0$$

If $y^2 - 2xy = 16$, then $\frac{dy}{dx} =$

(A) $\frac{x}{y-x}$ (B) $\frac{y}{x-y}$ (C) $\frac{y}{y-x}$ (D) $\frac{y}{2y-x}$ (E) $\frac{2y}{x-y}$

$$2y \cdot \frac{dy}{dx} - 2\left[x\frac{dy}{dx} + y\right] = 0$$

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y = 0$$

$$\frac{dy}{dx} = \frac{2y}{2y - 2x} = \frac{y}{y-x}$$

$$\frac{d^2y}{dx^2} = \frac{(y-x)\frac{dy}{dx} - y\left(\frac{dy}{dx} - 1\right)}{(y-x)^2}$$

$$= \frac{\cancel{y\frac{dy}{dx}} - x\frac{dy}{dx} - \cancel{y\frac{dy}{dx}} + y}{(y-x)^2}$$

$$= \frac{-x\left(\frac{y}{y-x}\right) + y}{\frac{(y-x)^2}{1}}$$

$$= \frac{-xy + y(y-x)}{(y-x)^3}$$

$$= \frac{y(y-2x)}{(y-x)^3} = \frac{y^2 - 2xy}{(y-x)^3}$$

Related Rates

A. Identify the known variables, including their rates of change and the rate of change that is to be found. Construct an equation relating the quantities whose rates of change are known and the rate of change to be found.

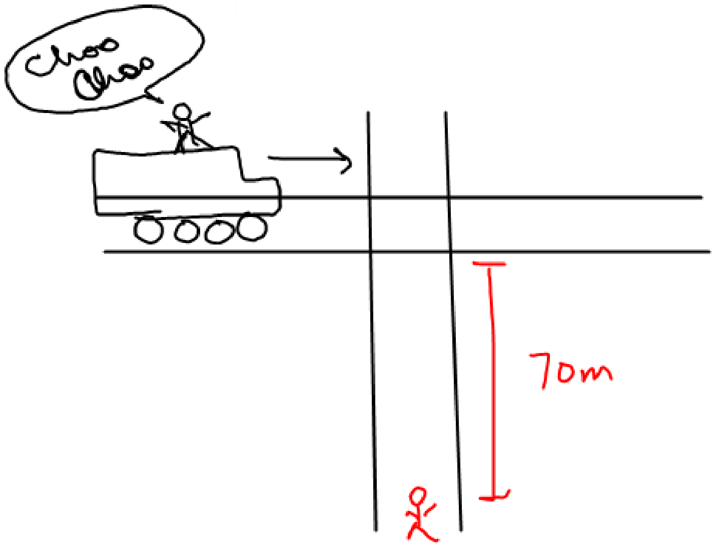
B. Implicitly differentiate both sides of the equation with respect to time (Remember: DO NOT substitute the value of a variable that changes throughout the situation before you differentiate. If the value is constant, you can substitute it into the equation to simplify the derivative calculation).

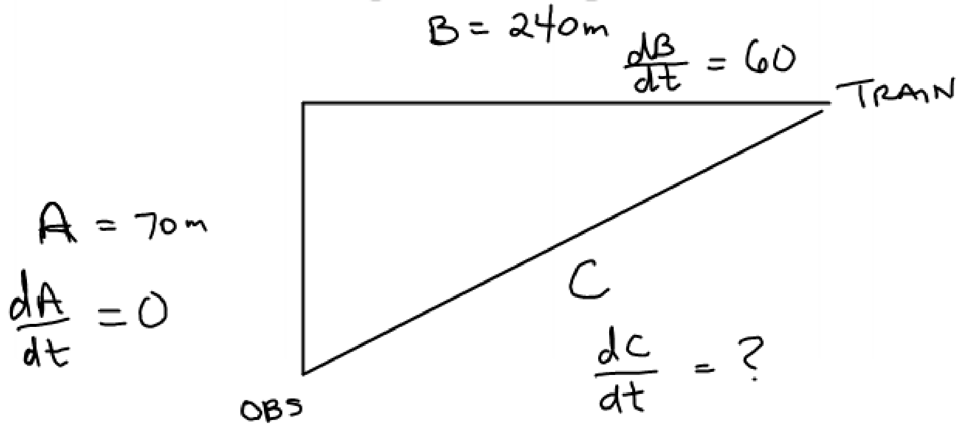
C. Substitute the known rates of change and the known values of the variables into the equation. Then solve for the required rate of change.

*Keep in mind, the variables present can be related in different ways which often involves the use of similar geometric shapes, Pythagorean Theorem, etc.

A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

a. 57.60
b. 57.88
c. 59.20
d. 60.00
e. 67.40





$$A^2 + B^2 = C^2 \longrightarrow 70^2 + 240^2 = C^2$$

$$C = 250$$

$$\frac{d}{dt}\left[70^2 + B^2 = C^2\right]$$

$$\cancel{2}B\frac{dB}{dt} = \cancel{2}C\frac{dC}{dt}$$

$$\frac{dC}{dt} = \frac{B\frac{dB}{dt}}{C} = \frac{240(60)}{250} = \frac{1440}{25}$$

$$= \frac{288}{5}$$

An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic meters per hour. At what rate, in square meters per hour, is the surface area of the sphere decreasing at the moment when the radius is $\boxed{5 \text{ meters}}$? (Note: For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $\frac{4}{3}\pi r^3$.)

(A) $\frac{4\pi}{5}$

(B) 40π

(C) $80\pi^2$

(D) 100π

$$\frac{d}{dt}\left[V = \frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$-2\pi = 4\pi(5)^2 \cdot \frac{dr}{dt}$$

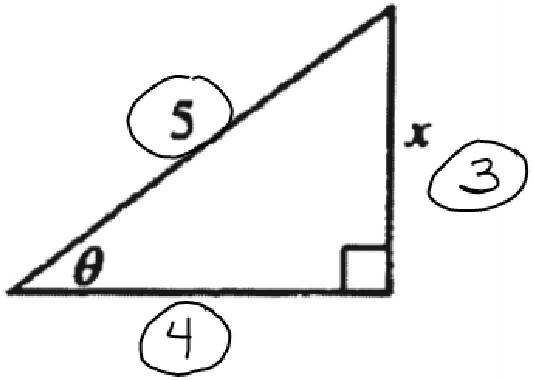
$$-\frac{1}{50} = \frac{dr}{dt}$$

$$\frac{d}{dt}\left[SA = 4\pi r^2\right]$$

$$\frac{d(SA)}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$= 40\pi\left(-\frac{1}{50}\right)$$

$$= -\frac{4}{5}\pi$$



In the triangle shown above, if θ increases at a constant rate of 3 radians per minute, at what rate is x increasing in units per minute when x equals 3 units?

a. 3
b. $\frac{15}{4}$
c. 4
d. 9
e. 12

$$\sin\theta = \frac{x}{5}$$

$$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$$

$$\frac{4}{5}\cdot 3 = \frac{1}{5}\frac{dx}{dt}$$

$$\boxed{12 = \frac{dx}{dt}}$$

$$\theta = \arcsin\left(\frac{x}{5}\right)$$

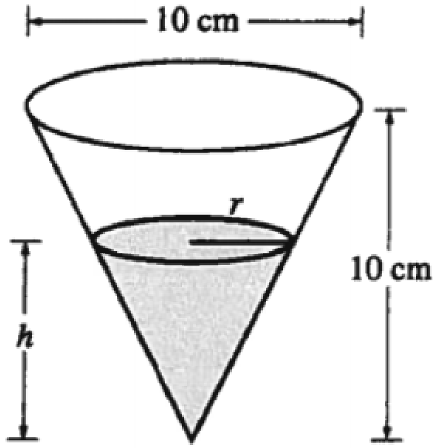
$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\cdot\frac{1}{5}\frac{dx}{dt}$$

$$3 = \frac{1}{\sqrt{1-\frac{9}{25}}}\cdot\frac{1}{5}\frac{dx}{dt}$$

$$3 = \frac{\cancel{5}}{4}\cdot\frac{1}{\cancel{5}}\frac{dx}{dt}$$

$$\boxed{12 = \frac{dx}{dt}}$$

2002 Question 5 (non Calculator)



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi \text{ cm}^3$

(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3;\ \frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi \ \text{cm}^3/\text{hr}$$

OR

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right);\ \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$$

$$= -\frac{15}{8}\pi \ \text{cm}^3/\text{hr}$$

(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} = -\frac{3}{40}\pi h^2$

$$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot \text{area}$$

The constant of proportionality is $-\frac{3}{10}$.

units of cm^3 in (a) and cm^3/hr in (b)

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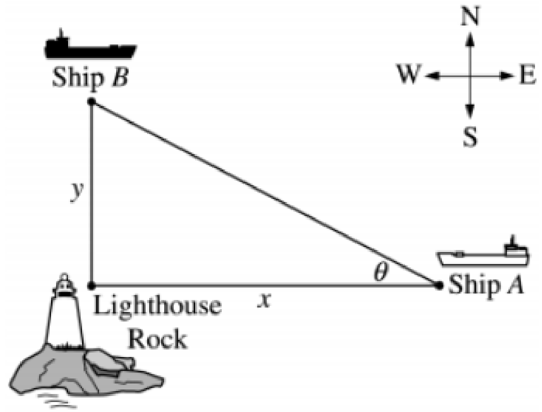
1 : V when $h = 5$

5:
- 1 : $r = \frac{1}{2}h$ in (a) or (b)
- 1 : V as a function of one variable in (a) or (b) OR $\frac{dr}{dt}$
- 2 : $\frac{dV}{dt}$
  < −2 > chain rule or product rule error
- 1 : evaluation at $h = 5$

2:
- 1 : shows $\frac{dV}{dt} = k \cdot \text{area}$
- 1 : identifies constant of proportionality

1 : correct units in (a) and (b)

2002 Question 6 Form B (non Calculator)



Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km

1 : answer

(b) $r^2 = x^2 + y^2$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or explicitly:

$$r = \sqrt{x^2 + y^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

At $x = 4$, $y = 3$,

$$\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6 \text{ km/hr}$$

$$4\begin{cases} 1 : \text{expression for distance} \\ 2 : \text{differentiation with respect to } t \\ \quad < -2 > \text{ chain rule error} \\ 1 : \text{evaluation} \end{cases}$$

(c) $\tan\theta = \dfrac{y}{x}$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

At $x = 4$ and $y = 3$, $\sec\theta = \dfrac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$$

$$= \frac{85}{25} = \frac{17}{5} \text{ radians/hr}$$

$$4\begin{cases} 1 : \text{expression for } \theta \text{ in terms of } x \text{ and } y \\ 2 : \text{differentiation with respect to } t \\ \quad < -2 > \text{ chain rule, quotient rule, or} \\ \qquad\qquad \text{transcendental function error} \\ \quad \text{note: 0/2 if no trig or inverse trig} \\ \qquad \text{function} \\ 1 : \text{evaluation} \end{cases}$$

Optimization

Finding the largest or smallest value of a function subject to some kind of constraints.

A. Define the primary equation for the quantity to be maximized or minimized. Define a feasible domain for the variables present in the equation.

B. If necessary, define a secondary equation that relates the variables present in the primary equation. Solve this equation for one of the variables and substitute into the primary equation.

C. Once the primary equation is represented in a single variable, take the derivative of the primary equation.

D. Find the critical values using the derivative calculated.

E. The optimal solution will more than likely be found at a critical value from **D.** Keep in mind, if the critical values do not represent a minimum or a maximum, the optimal solution may be found at an endpoint of the feasible domain.

The volume of a cylindrical tin can with a top and bottom is to be 16π cubic inches. If a minimum amount of tin is to be used to construct the can, what must be the height, in inches, of the can? (No Calculator)

A. $2\sqrt[3]{2}$ B. $2\sqrt{2}$ C. $2\sqrt[3]{4}$ D. 4 E. 8

$16\pi = \pi r^2 h$

$h = \frac{16}{r^2} \rightarrow r^2 = \frac{16}{h}$

FEASIBLE:

$0 < r < \infty$

$SA = 2\pi r^2 + 2\pi r h$

$SA = 2\pi r^2 + \frac{32\pi}{r}$

$SA = 2\pi\left[r^2 + 16r^{-1}\right]$

$(SA)' = 2\pi\left[2r - 16r^{-2}\right]$

$= 4\pi\left[\frac{r^3 - 8}{r^2}\right]$

CV: $r^3 - 8 = 0$ $\cancel{r^2 = 0}$

$r = 2$

$(SA)'' = 4\pi\left[1 + 16r^{-3}\right]$

1st Deriv:

$-$ $+$

0 2 ∞

OR

2nd Deriv:

$(SA)'(2) = 0$

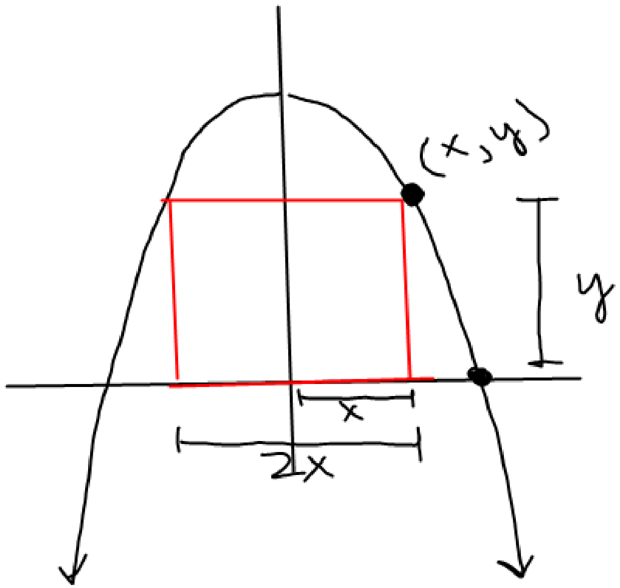
$(SA)''(2) > 0$

What is the area of the largest rectangle that has its base on the x-axis and its other two vertices on the parabola $y = 6 - x^2$?

$A(\sqrt{2}) = 2\sqrt{2}(4)$

(A) $8\sqrt{2}$ (B) $6\sqrt{2}$ (C) $4\sqrt{3}$ (D) $3\sqrt{2}$

FEASIBLE:
$0 \le x \le \sqrt{6}$



$A = 2xy$

$\boxed{A = 2x(6 - x^2)}$

$A = 12x - 2x^3$

$A' = 12 - 6x^2 \longrightarrow A'' = -12x$

$= 6(2 - x^2)$

$x = \sqrt{2}$

2008 Question 3 (Calculator)

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

(a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000\ \text{cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5$ cm/min. $$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$ $$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$ $$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm/min}$$	4 : { 1 : $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$; 2 : expression for $\frac{dV}{dt}$; 1 : answer
(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$. This occurs when $t = 25$ minutes. Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working.	3 : { 1 : $R(t) = 2000$; 1 : answer; 1 : justification
(c) The volume of oil, in cm^3, in the slick at time $t = 25$ minutes is given by $60{,}000 + \int_0^{25} (2000 - R(t))\, dt$.	2 : { 1 : limits and initial condition; 1 : integrand