The Definition of Continuity

A function $f(x)$ is continuous at c if

I. $\lim_{x \to c} f(x)$ exists

II. $f(c)$ exists

III. $\lim_{x \to c} f(x) = f(c)$

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

I. $\lim_{x \to -3} g(x) \to \lim_{x \to -3^-} f(x) = \sqrt{25 - (-3)^2} = 4$

$\lim_{x \to -3^+} (-3) + 7 = 4$

$\therefore \lim_{x \to -3} g(x) = 4$

II. $g(-3) = f(-3) = 4$

$g(x)$ is continuous @ $x = -3$ since $g(-3) = \lim_{x \to -3} g(x) = 4$.

Discontinuities:

① Removable

↓

Holes

↓

Occurs when:

$$\lim_{x \to c} f(x) = L \rightarrow L \text{ is constant}$$

but

$$\lim_{x \to c} f(x) \neq f(c)$$

Ex. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$ $\quad \frac{0}{0}$ = INDETERMINATE

L'Hops $\lim_{x \to -1} \frac{2x}{1} = -2$

↓

Hole @ $(-1, -2)$

② Non Removable

- Jump (piecewise or $\frac{|x-c|}{x-c}$)

↓

$$\lim_{x \to c^-} f(x) = a$$

$$\lim_{x \to c^+} f(x) = b$$

$\ \& \ a \neq b$

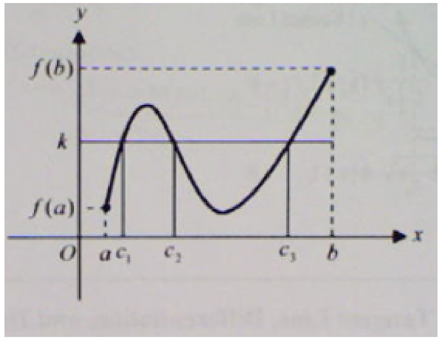
- (Vertical) Asymptote (Infinite Discontinuity)

↓

$$\lim_{x \to c} f(x) = \pm \infty$$

Intermediate Value Theorem

If *f* is a continuous function on the closed interval [a, b] and *k* is any number between *f(a)* and *f(b)*, then there exists at least one value of *c* on [a, b] such that *f(c) = k*. In other words, on a continuous function, if *f(a)< f(b)*, any y – value greater than *f(a)* and less than *f(b)* is guaranteed to exists on the function *f*.



Let f be a continuous function on the closed interval $[-2, 7]$. If $f(-2) = 5$ and $f(7) = -3$, then the Intermediate Value Theorem guarantees that

IVT can guarantee any/all y-values such $-3 \le y \le 5$.

~~(A)~~ $f'(c) = 0$ for at least one c between -2 and 7

~~(B)~~ $f'(c) = 0$ for at least one c between -3 and 5

(C) $f(c) = 0$ for at least one c between -3 and 5

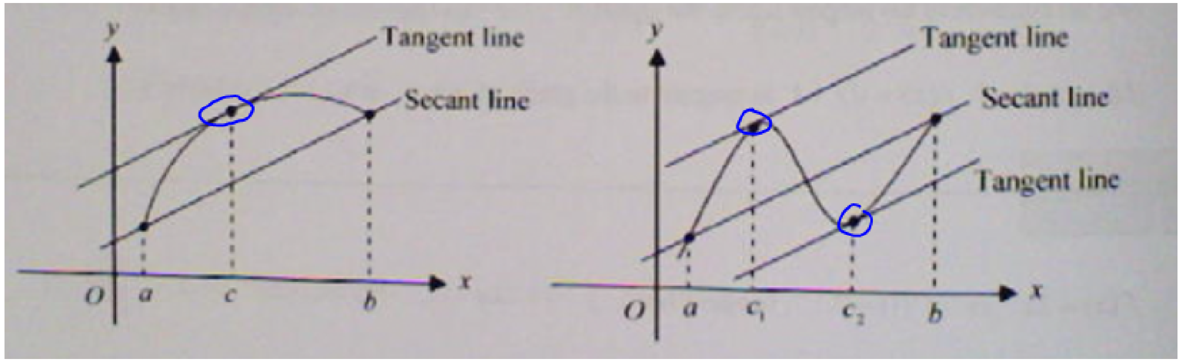
(D) $f(c) = 0$ for at least one c between -2 and 7

Mean Value Theorem for Derivatives

If the function f is continuous on the close interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number c between a and b such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The slope of the tangent line is equal to the slope of the secant line.



Let f be the function given by $f(x) = \frac{x}{x+2}$. What are the values of c that satisfy the Mean Value Theorem on the closed interval $[-1, 2]$?

$\to x \neq -2 \to \lim_{x \to -2^-} \frac{x}{x+2} = \frac{-2}{\text{SMALL NEG}} = \infty$

$\lim_{x \to -2^+} \frac{x}{x+2} = \frac{-2}{\text{Small pos}} = -\infty$

(A) -4 only (B) $\boxed{0 \text{ only}}$ (C) 0 and $\frac{3}{2}$ (D) -4 and 0

$f'(x) = \frac{(x+2) - x}{(x+2)^2}$

$= \frac{2}{(x+2)^2}$

MVT: $\frac{2}{(x+2)^2} = \frac{\frac{1}{2} + 1}{3}$

$\frac{2}{(x+2)^2} = \frac{1}{2}$

$(x+2)^2 = 4$

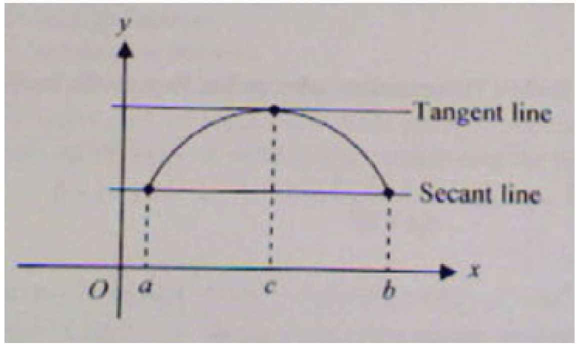
$x+2 = \pm 2$

$x = 0, -4$

Rolle's Theorem (Special Case of Mean Value Theorem)

If the function *f* is continuous on the close interval [a, b] and differentiable on the open interval (a, b), and *f(a) = f(b)*, then there exists at least one number *c* between *a* and *b* such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$$



Let f be the function given by $f(x) = \sin(\pi x)$. What are the values of c that satisfy Rolle's Theorem on the closed interval $[0, 2]$?

$f(0) = \sin(0) = 0$

$f(2) = \sin(2\pi) = 0$

(A) $\frac{1}{4}$ only (B) $\frac{1}{2}$ only (C) $\frac{1}{4}$ and $\frac{1}{2}$ (D) $\frac{1}{2}$ and $\frac{3}{2}$

$f'(x) = \pi\cos(\pi x)$

Rolle's: $\pi\cos(\pi x) = 0$

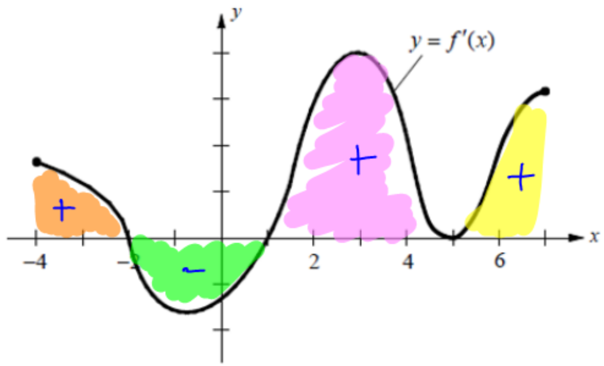
$\pi x = \frac{\pi}{2}$ $\quad \pi x = \frac{3\pi}{2}$

$x = \frac{1}{2}$ $\quad x = \frac{3}{2}$

Extreme Value Theorem

If the function *f* continuous on the closed interval [a, b], then the absolute extrema of the function *f* on the closed interval will occur at the endpoints or critical values of *f*.

> *After identifying critical values, create a table with endpoints and critical values. Calculate the y – value at each of these x values to identify the extrema.



The figure above shows the graph of f', the derivative of the function f, for $-4 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -1$, $x = 3$, and $x = 5$.

(a) Find all values of x, for $-4 \le x \le 7$, at which f attains a relative minimum. Justify your answer. $x = 1$

(b) Find all values of x, for $-4 \le x \le 7$, at which f attains a relative maximum. Justify your answer. $x = -2$

(c) At what value of x, for $-4 \le x \le 7$, does f attain its absolute maximum? Justify your answer. $x = 7$

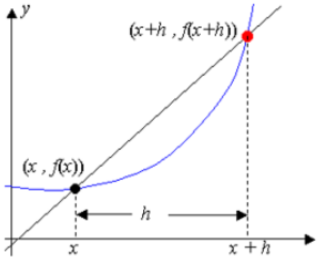
x	$f(x)$
-4	$f(-4)$
-2	$f(-4) + \int_{-4}^{-2} f'(x)\,dx$
1	$f(-4) + \int_{-4}^{1} f'(x)\,dx$
5	$f(-4) + \int_{-4}^{5} f'(x)\,dx$
7	$f(-4) + \int_{-4}^{7} f'(x)\,dx$

$f(7) > f(5) > f(-2) > f(-4) > f(1)$

f attains its abs maximum @ $x=7$ since $f(x) = f(-4) + \int_{-4}^{x} f'(t)\,dt$ and $\int_{-4}^{x} f'(t)\,dt$ acculates the most positive area at $x=7$.

Definition of the Derivative

The derivative of the function *f*, or instantaneous rate of change, is given by converting the slope of the secant line to the slope of the tangent line by making the change is x, Δx or h, approach zero.

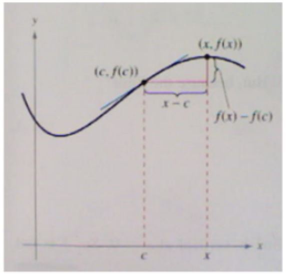


$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

* Use to: find $f'(x)$ or $f'(c)$

Alternate Definition



$$\boxed{f'(c)} = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$\lim_{\Delta x\to 0}\frac{3(x+\Delta x)^2-3x^2}{\Delta x}$$

$f(x) = 3x^2$

$\boxed{f'(x) = 6x}$

$$f(x+h) \quad \lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$$

$f(9) = 3$

@ $x = 9$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$\boxed{f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}}$

$$\lim_{x\to 3}\frac{\frac{2}{x}-\frac{2}{3}}{x-3}$$

$f(x) = \dfrac{2}{x}$

$f'(x) = \dfrac{-2}{x^2}$

$\boxed{f'(3) = \dfrac{-2}{9}}$

Riemann Sum (Limit Definition of Area)

Let f be a continuous function on the interval [a, b]. The area of the region bounded by the graph of the function f and the x – axis (i.e. the value of the definite integral) can be found using

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\cdot\Delta x$$

Where c_i is either the left endpoint ($c_i = a + (i-1)\Delta x$) or right endpoint ($c_i = a + i\Delta x$) and $\Delta x = (b-a)/n$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)^3\cdot\frac{2}{n}$$

$\Delta x = \frac{2}{n} \to b-a=2$

$c_k = \underbrace{1 + k\cdot\frac{2}{n}}_{\downarrow} \to a=1,\ b=3$

$f(x) = x^3$

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big|_1^3 = 20$$

$$\lim_{n\to\infty}\left(\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{\sqrt{n^3}}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n}$$

$\Delta x = \frac{1}{n} \to b-a=1$

$c_i = i\cdot\frac{1}{n} \to a=0,\ b=1$

$f(x)=\sqrt{x}$

$$\int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1$$

$$= \frac{2}{3}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2+\frac{3k}{n}\right)^2\cdot\frac{1}{3n}$$

$$\lim_{n\to\infty}\frac{1}{3}\sum_{k=1}^{n}\left(2+3k\cdot\frac{1}{n}\right)^2\cdot\frac{1}{n}$$

$\Delta x = \frac{1}{n} \to b-a=1$

$c_i = k\cdot\frac{1}{n} \to a=0,\ b=1$

$f(x) = (2+3x)^2$

$$\frac{1}{3}\int_0^1 (2+3x)^2\,dx$$

$u = 2+3x$

$du = 3dx$

$$\frac{1}{9}\int_2^5 u^2\,du$$

$$\left[\frac{1}{27}u^3\right]_2^5 = \frac{13}{3}$$

2011 Question 5 Form B

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

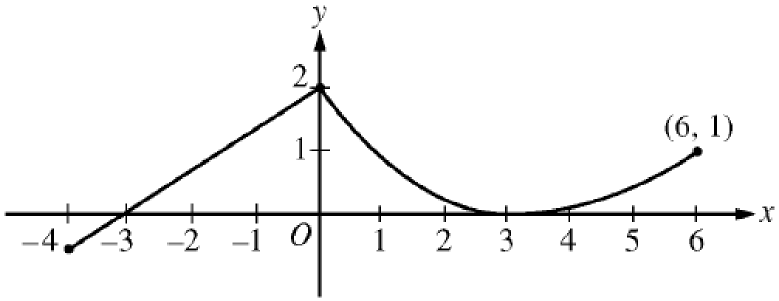
(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\, dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\, dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

(d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

2009 Question 3 Form B



Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0 ? Give a reason for your answer.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

2005 Question 2

The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t) = \frac{15t}{1 + 3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

(b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.