18. $s(t) = \int\left[-32t + 12{,}000 \ln \frac{50{,}000}{50{,}000 - 400t}\right] dt$

$$= -16t^2 + 12{,}000\int[\ln 50{,}000 - \ln(50{,}000 - 400t)]\, dt$$

$$= 16t^2 + 12{,}000t \ln 50{,}000 - 12{,}000\left[t\ln(50{,}000 - 400t) - \int \frac{-400t}{50{,}000 - 400t}\, dt\right]$$

$$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t\int\left[1 - \frac{50{,}000}{50{,}000 - 400t}\right] dt$$

$$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t + 1{,}500{,}000 \ln(50{,}000 - 400t) + C$$

$s(0) = 1{,}500{,}000 \ln 50{,}000 + C = 0$

$C = -1{,}500{,}000 \ln 50{,}000$

$$s(t) = -16t^2 + 12{,}000t\left[1 + \ln \frac{50{,}000}{50{,}000 - 400t}\right] + 1{,}500{,}000 \ln \frac{50{,}000 - 400t}{50{,}000}$$

When $t = 100$, $s(100) \approx 557{,}168.626$ feet.

19. By parts,

$$\int_a^b f(x)g''(x)\, dx = \left[f(x)g'(x)\right]_a^b - \int_a^b f'(x)g'(x)\, dx\,[u = f(x),\ dv = g''(x)\, dx]$$

$$= -\int_a^b f'(x)g'(x)\, dx$$

$$= \left[-f'(x)g(x)\right]_a^b + \int_a^b g(x)f''(x)\, dx\,[u = f'(x),\ dv = g'(x)\, dx]$$

$$= \int_a^b f''(x)g(x)\, dx.$$

20. Let $u = (x - a)(x - b)$, $du = [(x - a) + (x - b)]\, dx$, $dv = f''(x)\, dx$, $v = f'(x)$.

$$\int_a^b (x - a)(x - b)\, dx = \left[(x - a)(x - b)f'(x)\right]_a^b - \int_a^b [(x - a) + (x - b)]f'(x)\, dx$$

$$= -\int_a^b (2x - a - b)f'(x)\, dx \qquad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x)\, dx \end{pmatrix}$$

$$= \left[-(2x - a - b)f(x)\right]_a^b + \int_a^b 2f(x)\, dx$$

$$= 2\int_a^b f(x)\, dx$$

21. $$\int_2^\infty\left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}}\right] dx < \int_2^\infty \frac{1}{x^5 - 1}\, dx < \int_2^\infty\left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}\right] dx$$

$$\lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}}\right]_2^b < \int_2^\infty \frac{1}{x^5 - 1}\, dx < \lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}}\right]_2^b$$

$$0.015846 < \int_2^\infty \frac{1}{x^5 - 1}\, dx < 0.015851$$

CHAPTER 9
Infinite Series

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Infinite Series

Section 9.1 Sequences

1. $a_n = 2^n$

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

2. $a_n = \dfrac{3^n}{n!}$

$a_1 = \dfrac{3}{1!} = 3$

$a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2}$

$a_3 = \dfrac{3^3}{3!} = \dfrac{27}{6} = \dfrac{9}{2}$

$a_4 = \dfrac{3^4}{4!} = \dfrac{81}{24} = \dfrac{27}{8}$

$a_5 = \dfrac{3^5}{5!} = \dfrac{243}{120} = \dfrac{81}{40}$

3. $a_n = \left(-\dfrac{1}{2}\right)^n$

$a_1 = \left(-\dfrac{1}{2}\right)^1 = -\dfrac{1}{2}$

$a_2 = \left(-\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$a_3 = \left(-\dfrac{1}{2}\right)^3 = -\dfrac{1}{8}$

$a_4 = \left(-\dfrac{1}{2}\right)^4 = \dfrac{1}{16}$

$a_5 = \left(-\dfrac{1}{2}\right)^5 = -\dfrac{1}{32}$

4. $a_n = \left(-\dfrac{2}{3}\right)^n$

$a_1 = -\dfrac{2}{3}$

$a_2 = \dfrac{4}{9}$

$a_3 = -\dfrac{8}{27}$

$a_4 = \dfrac{16}{81}$

$a_5 = -\dfrac{32}{243}$

5. $a_n = \sin\dfrac{n\pi}{2}$

$a_1 = \sin\dfrac{\pi}{2} = 1$

$a_2 = \sin\pi = 0$

$a_3 = \sin\dfrac{3\pi}{2} = -1$

$a_4 = \sin 2\pi = 0$

$a_5 = \sin\dfrac{5\pi}{2} = 1$

6. $a_n = \dfrac{2n}{n+3}$

$a_1 = \dfrac{2}{4} = \dfrac{1}{2}$

$a_2 = \dfrac{4}{5}$

$a_3 = \dfrac{6}{6} = 1$

$a_4 = \dfrac{8}{7}$

$a_5 = \dfrac{10}{8} = \dfrac{5}{4}$

7. $a_n = \dfrac{(-1)^{n(n+1)/2}}{n^2}$

$a_1 = \dfrac{(-1)^1}{1^2} = -1$

$a_2 = \dfrac{(-1)^3}{2^2} = -\dfrac{1}{4}$

$a_3 = \dfrac{(-1)^6}{3^2} = \dfrac{1}{9}$

$a_4 = \dfrac{(-1)^{10}}{4^2} = \dfrac{1}{16}$

$a_5 = \dfrac{(-1)^{15}}{5^2} = -\dfrac{1}{25}$

8. $a_n = (-1)^{n+1}\left(\dfrac{2}{n}\right)$

$a_1 = \dfrac{2}{1} = 2$

$a_2 = -\dfrac{2}{2} = -1$

$a_3 = \dfrac{2}{3}$

$a_4 = -\dfrac{2}{4} = -\dfrac{1}{2}$

$a_5 = \dfrac{2}{5}$

9. $a_n = 5 - \dfrac{1}{n} + \dfrac{1}{n^2}$

$a_1 = 5 - 1 + 1 = 5$ 5

$a_2 = 5 - \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{19}{4}$ $4\frac{3}{4}$ 4.75

$a_3 = 5 - \dfrac{1}{3} + \dfrac{1}{9} = \dfrac{43}{9}$ $4\frac{7}{9}$ 4.777

$a_4 = 5 - \dfrac{1}{4} + \dfrac{1}{16} = \dfrac{77}{16}$ $4\frac{13}{16}$ 4.81

$a_5 = 5 - \dfrac{1}{5} + \dfrac{1}{25} = \dfrac{121}{25}$ $4\frac{21}{25}$ 4.84

a_{100} $5 - \dfrac{1}{100} + \dfrac{1}{100^2} =$ 4.99

10. $a_n = 10 + \frac{2}{n} + \frac{6}{n^2}$

$a_1 = 10 + 2 + 6 = 18$

$a_2 = 10 + 1 + \frac{3}{2} = \frac{25}{2}$

$a_3 = 10 + \frac{2}{3} + \frac{2}{3} = \frac{34}{3}$

$a_4 = 10 + \frac{1}{2} + \frac{3}{8} = \frac{87}{8}$

$a_5 = 10 + \frac{2}{5} + \frac{6}{25} = \frac{266}{25}$

11. $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$a_2 = 2(a_1 - 1)$

$= 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1)$

$= 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1)$

$= 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1)$

$= 2(10 - 1) = 18$

12. $a_1 = 4, a_{k+1} = \left(\frac{k+1}{2}\right)a_k$

$a_2 = \left(\frac{1+1}{2}\right)a_1 = 4$

$a_3 = \left(\frac{2+1}{2}\right)a_2 = 6$

$a_4 = \left(\frac{3+1}{2}\right)a_3 = 12$

$a_5 = \left(\frac{4+1}{2}\right)a_4 = 30$

13. $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$

$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$

$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$

$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$

$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$

14. $a_1 = 6, a_{k+1} = \frac{1}{3}a_k^2$

$a_2 = \frac{1}{3}a_1^2 = \frac{1}{3}(6^2) = 12$

$a_3 = \frac{1}{3}a_2^2 = \frac{1}{3}(12^2) = 48$

$a_4 = \frac{1}{3}a_3^2 = \frac{1}{3}(48^2) = 768$

$a_5 = \frac{1}{3}a_4^2 = \frac{1}{3}(768)^2 = 196{,}608$

15. $a_n = \frac{8}{n+1}, a_1 = 4, a_2 = \frac{8}{3}$, decreases to 0; matches (f).

16. $a_n = \frac{8n}{n+1}, a_1 = 4, a_2 = \frac{16}{3}$, increases towards 8; matches (a).

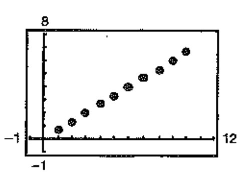
17. $a_n = 4(0.5)^{n-1}, a_1 = 4, a_2 = 2$, decreases to 0; matches (e).

18. $a_n = \frac{4^n}{n!}, a_1 = 4, a_2 = 8$, eventually approaches 0; matches (b).

19. $a_n(-1)^n, a_1 = -1, a_2 = 1, a_3 = -1$, etc.; matches (d).

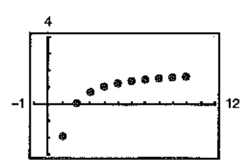
20. $a_n = \frac{(-1)^n}{n}, a_1 = -1, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}$, etc.; matches (c).

21.



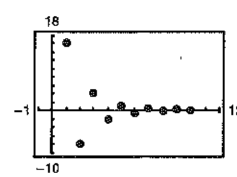
$a_n = \frac{2}{3}n, n = 1, . . . , 10$

22.



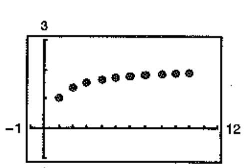
$a_n = 2 - \frac{4}{n}, n = 1, . . . , 10$

23.



$a_n = 16(-0.5)^{n-1}, n = 1, . . . , 10$

24.



$a_n = \frac{2n}{n+1}, n = 1, 2, . . . , 10$

25. $a_n = 3n - 1$

$a_5 = 3(5) - 1 = 14$

$a_6 = 3(6) - 1 = 17$

Add 3 to preceding term.

26. $a_n = \frac{n+6}{2}$

$a_5 = \frac{5+6}{2} = \frac{11}{2}$

$a_6 = \frac{6+6}{2} = 6$

27. $a_{n+1} = 2a_n, a_1 = 5$

$a_5 = 2(40) = 80$

$a_6 = 2(80) = 160$

28. $a_n = -\frac{1}{2}a_{n-1}, a_1 = 1$

$a_5 = \frac{1}{16}, a_6 = -\frac{1}{32}$

29. $a_n = \frac{3}{(-2)^{n-1}}$

$a_5 = \frac{3}{(-2)^4} = \frac{3}{16}$

$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$

Multiply the preceding term by $-\frac{1}{2}$.

30. $a_n = -\frac{3}{2}a_{n-1}, a_1 = 1$

$a_5 = \frac{81}{16}, a_6 = -\frac{243}{32}$

31. $\frac{10!}{8!} = \frac{8!(9)(10)}{8!}$

$= (9)(10) = 90$

32. $\frac{25!}{23!} = \frac{23!(24)(25)}{23!}$

$= (24)(25) = 600$

33. $\frac{(n+1)!}{n!} = \frac{n!(n+1)}{n!} = n + 1$

34. $\frac{(n+2)!}{n!} = \frac{n!(n+1)(n+2)}{n!}$

$= (n+1)(n+2)$

35. $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)}$

$= \frac{1}{2n(2n+1)}$

36. $\frac{(2n+2)!}{(2n)!} = \frac{(2n)!(2n+1)(2n+2)}{(2n)!}$

$= (2n+1)(2n+2)$

37. $\lim_{n\to\infty} \frac{5n^2}{n^2+2} = 5$

38. $\lim_{n\to\infty} \left(5 - \frac{1}{n^2}\right) = 5 - 0 = 5$

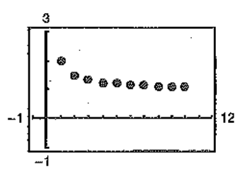
39. $\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+(1/n^2)}} = \frac{2}{1} = 2$

40. $\lim_{n\to\infty} \frac{5n}{\sqrt{n^2+4}} = \lim_{n\to\infty} \frac{5}{\sqrt{1+(4/n^2)}} = \frac{5}{1} = 5$

41. $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

42. $\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = 1$

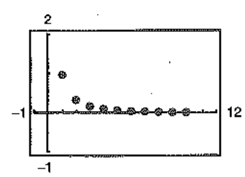
43.



The graph seems to indicate that the sequence converges to 1. Analytically,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} 1 = 1.$

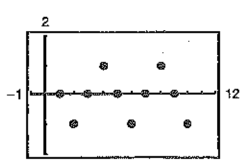
44.



The graph seems to indicate that the sequence converges to 0. Analytically,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{3/2}} = \lim_{x\to\infty} \frac{1}{x^{3/2}} = 0.$

45.

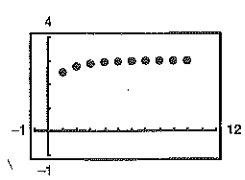


The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$\{a_n\} = \{0, -1, 0, 1, 0, -1, \ldots\}.$

Hence, $\lim_{n\to\infty} a_n$ does not exist.

46.



The graph seems to indicate that the sequence converges to 3. Analytically,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$

47. $\lim_{n\to\infty}(-1)^n\left(\frac{n}{n+1}\right)$

does not exist (oscillates between -1 and 1), diverges.

48. $\lim_{n\to\infty}[1+(-1)^n]$

does not exist, (alternates between 0 and 2), diverges.

49. $\lim_{n\to\infty}\frac{3n^2-n+4}{2n^2+1}=\frac{3}{2}$, converges

50. $\lim_{n\to\infty}\frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}=1$, converges

51. $a_n=\frac{1\cdot3\cdot5\cdots(2n-1)}{(2n)^n}$

$$=\frac{1}{2n}\cdot\frac{3}{2n}\cdot\frac{5}{2n}\cdots\frac{2n-1}{2n}<\frac{1}{2n}$$

Thus, $\lim_{n\to\infty}a_n=0$, converges.

52. The sequence diverges. To prove this analytically, we use mathematical induction to show that $a_n\ge\left(\frac{3}{2}\right)^{n-1}$. Clearly, $a_1=1\ge\left(\frac{3}{2}\right)^0=1$. Assume that $a_k\ge\left(\frac{3}{2}\right)^{k-1}$. Then,

$$a_{k+1}=\frac{1\cdot3\cdot5\cdots(2k-1)(2k+1)}{(k+1)!}=a_k\frac{2k+1}{k+1}\ge\left(\frac{3}{2}\right)^{k-1}\left(\frac{2k+1}{k+1}\right).$$

Since

$$\frac{2k+1}{k+1}\ge\frac{3}{2},$$

we have $a_{k+1}\ge\left(\frac{3}{2}\right)^k$, which shows that $a_n\ge\left(\frac{3}{2}\right)^{n-1}$ for all n.

53. $\lim_{n\to\infty}\frac{1+(-1)^n}{n}=0$, converges

54. $\lim_{n\to\infty}\frac{1+(-1)^n}{n^2}=0$, converges

55. $\lim_{n\to\infty}\frac{\ln(n^3)}{2n}=\lim_{n\to\infty}\frac{3}{2}\frac{\ln(n)}{n}$

$$=\lim_{n\to\infty}\frac{3}{2}\left(\frac{1}{n}\right)=0, \text{ converges}$$

(L'Hôpital's Rule)

56. $\lim_{n\to\infty}\frac{\ln\sqrt{n}}{n}=\lim_{n\to\infty}\frac{1/2\ln n}{n}$

$$=\lim_{n\to\infty}\frac{1}{2n}=0, \text{ converges}$$

(L'Hôpital's Rule)

57. $\lim_{n\to\infty}\left(\frac{3}{4}\right)^n=0$, converges

58. $\lim_{n\to\infty}(0.5)^n=0$, converges

59. $\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}(n+1)$

$=\infty$, diverges

60. $\lim_{n\to\infty}\frac{(n-2)!}{n!}=\lim_{n\to\infty}\frac{1}{n(n-1)}=0$, converges

61. $\lim_{n\to\infty}\left(\frac{n-1}{n}-\frac{n}{n-1}\right)=\lim_{n\to\infty}\frac{(n-1)^2-n^2}{n(n-1)}$

$$=\lim_{n\to\infty}\frac{1-2n}{n^2-n}=0, \text{ converges}$$

62. $\lim_{n\to\infty}\left(\frac{n^2}{2n+1}-\frac{n^2}{2n-1}\right)=\lim_{n\to\infty}\frac{-2n^2}{4n^2-1}=-\frac{1}{2}$,

converges

63. $\lim_{n\to\infty}\frac{n^p}{e^n}=0$, converges

$(p>0, n\ge2)$

64. $a_n = n \sin \frac{1}{n}$

Let $f(x) = x \sin \frac{1}{x}$.

$$\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2} = \lim_{x\to\infty} \cos \frac{1}{x} = \cos 0 = 1 \quad \text{(L'Hôpital's Rule)}$$

or,

$$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1. \text{ Therefore } \lim_{n\to\infty} n \sin \frac{1}{n} = 1.$$

65. $a_n = \left(1 + \frac{k}{n}\right)^n$

$$\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = \lim_{u\to 0} [(1 + u)^{1/u}]^k = e^k$$

where $u = k/n$, converges

66. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, converges

67. $\lim_{n\to\infty} \frac{\sin n}{n} = \lim_{n\to\infty} (\sin n)\frac{1}{n} = 0$,

converges (because $(\sin n)$ is bounded)

68. $\lim_{n\to\infty} \frac{\cos \pi n}{n^2} = 0$, converges

69. $a_n = 3n - 2$

70. $a_n = 4n - 1$

71. $a_n = n^2 - 2$

72. $a_n = \frac{(-1)^{n-1}}{n^2}$

73. $a_n = \frac{n + 1}{n + 2}$

74. $a_n = \frac{(-1)^{n-1}}{2^{n-2}}$

75. $a_n = 1 + \frac{1}{n} = \frac{n + 1}{n}$

76. $a_n = 1 + \frac{2^n - 1}{2^n}$

$= \frac{2^{n+1} - 1}{2^n}$

77. $a_n = \frac{n}{(n + 1)(n + 2)}$

78. $a_n = \frac{1}{n!}$

79. $a_n = \frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n - 1)}$

$= \frac{(-1)^{n-1} 2^n n!}{(2n)!}$

80. $a_n = \frac{x^{n-1}}{(n - 1)!}$

81. $a_n = (2n)!, n = 1, 2, 3, \ldots$

82. $a_n = (2n - 1)!, n = 1, 2, 3, \ldots$

83. $a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n + 1} = a_{n+1}$,

monotonic; $|a_n| < 4$, bounded

84. Let $f(x) = \frac{3x}{x + 2}$. Then $f'(x) = \frac{6}{(x + 2)^2}$.

Thus, f is increasing which implies $\{a_n\}$ is increasing.

$|a_n| < 3$, bounded

85. $\frac{n}{2^{n+2}} \overset{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$

$2^{n+3}n \overset{?}{\geq} 2^{n+2}(n+1)$

$2n \overset{?}{\geq} n+1$

$n \geq 1$

Hence, $n \geq 1$

$2n \geq n+1$

$2^{n+3}n \geq 2^{n+2}(n+1)$

$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$

$a_n \geq a_{n+1}.$

True; monotonic; $|a_n| \leq \frac{1}{8}$, bounded

86. $a_n = ne^{-n/2}$

$a_1 = 0.6065$

$a_2 = 0.7358$

$a_3 = 0.6694$

Not monotonic; $|a_n| \leq 0.7358$, bounded

87. $a_n = (-1)^n\left(\frac{1}{n}\right)$

$a_1 = -1$

$a_2 = \frac{1}{2}$

$a_3 = -\frac{1}{3}$

Not monotonic; $|a_n| \leq 1$, bounded

88. $a_n = \left(-\frac{2}{3}\right)^n$

$a_1 = -\frac{2}{3}$

$a_2 = \frac{4}{9}$

$a_3 = -\frac{8}{27}$

Not monotonic; $|a_n| \leq \frac{2}{3}$, bounded

89. $a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$

Monotonic; $|a_n| \leq \frac{2}{3}$, bounded

90. $a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$

Monotonic; $\lim_{n\to\infty} a_n = \infty$, not bounded

91. $a_n = \sin\left(\frac{n\pi}{6}\right)$

$a_1 = 0.500$

$a_2 = 0.8660$

$a_3 = 1.000$

$a_4 = 0.8660$

Not monotonic; $|a_n| \leq 1$, bounded

92. $a_n = \cos\left(\frac{n\pi}{2}\right)$

$a_1 = \cos\frac{\pi}{2} = 0$

$a_2 = \cos \pi = -1$

$a_3 = \cos\left(\frac{3\pi}{2}\right) = 0$

Not monotonic; $|a_n| \leq 1$, bounded

93. $a_n = \frac{\cos n}{n}$

$a_1 = 0.5403$

$a_2 = -0.2081$

$a_3 = -0.3230$

$a_4 = -0.1634$

Not monotonic; $|a_n| \leq 1$, bounded

94. $a_n = \frac{\sin\sqrt{n}}{n}$

$a_1 = \frac{\sin(1)}{1} \approx 0.8415$ $\quad a_4 = \frac{\sin(4)}{4} \approx -0.1892$

$a_2 = \frac{\sin(2)}{2} \approx 0.4546$ $\quad a_5 = \frac{\sin(5)}{5} \approx -0.1918$

$a_3 = \frac{\sin(3)}{3} \approx 0.0470$ $\quad a_6 = \frac{\sin(6)}{6} \approx -0.0466$

Not monotonic, $|a_n| \leq 1$, bounded

95. (a) $a_n = 5 + \frac{1}{n}$

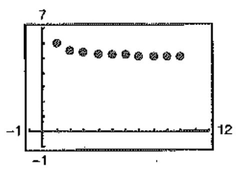
$$\left|5 + \frac{1}{n}\right| \le 6 \implies \{a_n\}, \text{bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \implies \{a_n\}, \text{monotonic}$$

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(5 + \frac{1}{n}\right) = 5$$

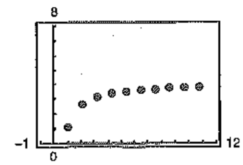
96. (a) $a_n = 4 - \frac{3}{n}$

$$\left|4 - \frac{3}{n}\right| < 4 \implies \text{bounded}$$

$$a_n = 4 - \frac{3}{n} < 3 - \frac{4}{n+1} = a_{n+1} \implies \text{monotonic}$$

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(4 - \frac{3}{n}\right) = 4$$

97. (a) $a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right)$

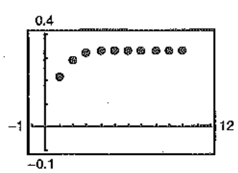
$$\left|\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right| < \frac{1}{3} \implies \{a_n\}, \text{bounded}$$

$$a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right) < \frac{1}{3}\left(1 - \frac{1}{3^{n+1}}\right)$$

$$= a_{n+1} \implies \{a_n\}, \text{monotonic}$$

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left[\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right] = \frac{1}{3}$$

98. (a) $a_n = 4 + \frac{1}{2^n}$

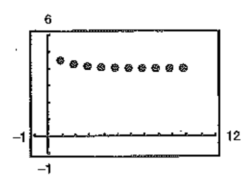
$$\left|4 + \frac{1}{2^n}\right| \le 4.5 \implies \{a_n\}, \text{bounded}$$

$$a_n = 4 + \frac{1}{2^n} > 4 + \frac{1}{2^{n+1}}$$

$$= a_{n+1} \implies \{a_n\}, \text{monotonic}$$

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(4 + \frac{1}{2^n}\right) = 4$$

99. $\{a_n\}$ has a limit because it is a bounded, monotonic sequence. The limit is less than or equal to 4, and greater than or equal to 2.

$$2 \le \lim_{n\to\infty} a_n \le 4$$

100. The sequence $\{a_n\}$ could converge or diverge. If $\{a_n\}$ is increasing, then it converges to a limit less than or equal to 1. If $\{a_n\}$ is decreasing, then it could converge (example: $a_n = 1/n$) or diverge (example: $a_n = -n$).

101. $A_n = P\left(1 + \frac{r}{12}\right)^n$

(a) No, the sequence $\{A_n\}$ diverges, $\lim_{n\to\infty} A_n = \infty$. The amount will grow arbitrarily large over time.

(b) $P = 9000$, $r = 0.055$, $A_n = 9000\left(1 + \frac{0.055}{12}\right)^n$

$A_1 = 9041.25$	$A_6 = 9250.35$
$A_2 = 9082.69$	$A_7 = 9292.75$
$A_3 = 9124.32$	$A_8 = 9335.34$
$A_4 = 9166.14$	$A_9 = 9378.13$
$A_5 = 9208.15$	$A_{10} = 9421.11$

102. (a) $A_n = 100(401)(1.0025^n - 1)$

$A_0 = 0$

$A_1 = 100.25$

$A_2 = 200.75$

$A_3 = 301.50$

$A_4 = 402.51$

$A_5 = 503.76$

$A_6 = 605.27$

(b) $A_{60} = 6480.83$ (c) $A_{240} = 32{,}912.28$

103. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit. See the definition.

(c) A sequence is monotonic if its terms are nondecreasing, or nonincreasing.

(d) A sequence is bounded if it is bounded below ($a_n \geq N$ for some N) and bounded above ($a_n \leq M$ for some M).

104. The first sequence because every other point is below the x-axis.

105. $a_n = 10 - \frac{1}{n}$

106. Impossible. The sequence converges by Theorem 9.5.

107. $a_n = \frac{3n}{4n + 1}$

108. Impossible. An unbounded sequence diverges.

109. (a) $A_n = (0.8)^n(2.5)$ billion

(b) $A_1 = \$2$ billion

$A_2 = \$1.6$ billion

$A_3 = \$1.28$ billion

$A_4 = \$1.024$ billion

(c) $\lim_{n\to\infty} (0.8)^n(2.5) = 0$

110. $P_n = 16{,}000(1.045)^n$

$P_1 = \$16{,}720.00$

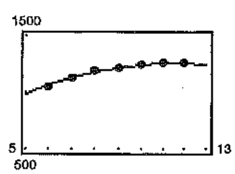
$P_2 = \$17{,}472.40$

$P_3 \approx \$18{,}258.66$

$P_4 \approx \$19{,}080.30$

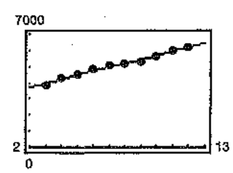
$P_5 \approx \$19{,}938.91$

111. (a) $a_n = -6.60n^2 + 151.7n + 387$



(b) In 2008, $n = 18$ and $a_{18} \approx 979$ endangered species.

112. (a) $a_n = 244.2n + 3250$



(b) For 2008, $n = 18$ and $a_{18} \approx 7646$ million.

113. $a_n = \frac{10^n}{n!}$

(a) $a_9 = a_{10} = \frac{10^9}{9!}$

$= \frac{1{,}000{,}000{,}000}{362{,}880}$

$= \frac{1{,}562{,}500}{567}$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

114. $a_n = \left(1 + \frac{1}{n}\right)^n$

$a_1 = 2.0000$

$a_2 = 2.2500$

$a_3 \approx 2.3704$

$a_4 \approx 2.4414$

$a_5 \approx 2.4883$

$a_6 \approx 2.5216$

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

115. $\{a_n\} = \left\{\sqrt[n]{n}\right\} = \{n^{1/n}\}$

$a_1 = 1^{1/1} = 1$

$a_2 = \sqrt{2} \approx 1.4142$

$a_3 = \sqrt[3]{3} \approx 1.4422$

$a_4 = \sqrt[4]{4} \approx 1.4142$

$a_5 = \sqrt[5]{5} \approx 1.3797$

$a_6 = \sqrt[6]{6} \approx 1.3480$

Let $y = \lim_{n\to\infty} n^{1/n}$.

$$\ln y = \lim_{n\to\infty}\left(\frac{1}{n}\ln n\right) = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{n} = 0$$

Since $\ln y = 0$, we have $y = e^0 = 1$. Therefore, $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

116. Since

$$\lim_{n\to\infty} s_n = L > 0,$$

there exists for each $\varepsilon > 0$,

an integer N such that $|s_n - L| < \varepsilon$ for every $n > N$. Let $\varepsilon = L > 0$ and we have,

$$|s_n - L| < L, -L < s_n - L < L, \text{ or } 0 < s_n < 2L$$

for each $n > N$.

117. True

118. True

119. True

120. True

121. $a_{n+2} = a_n + a_{n+1}$

(a) $a_1 = 1$

$a_2 = 1$

$a_3 = 1 + 1 = 2$

$a_4 = 2 + 1 = 3$

$a_5 = 3 + 2 = 5$

$a_6 = 5 + 3 = 8$

$a_7 = 8 + 5 = 13$

$a_8 = 13 + 8 = 21$

$a_9 = 21 + 13 = 34$

$a_{10} = 34 + 21 = 55$

$a_{11} = 55 + 34 = 89$

$a_{12} = 89 + 55 = 144$

(b) $b_n = \dfrac{a_{n+1}}{a_n}, n \geq 1$

$b_1 = \dfrac{1}{1} = 1$

$b_2 = \dfrac{2}{1} = 2$

$b_3 = \dfrac{3}{2}$

$b_4 = \dfrac{5}{3}$

$b_5 = \dfrac{8}{5}$

$b_6 = \dfrac{13}{8}$

$b_7 = \dfrac{21}{13}$

$b_8 = \dfrac{34}{21}$

$b_9 = \dfrac{55}{34}$

$b_{10} = \dfrac{89}{55}$

(c)
$$1 + \frac{1}{b_{n-1}} = 1 + \frac{1}{a_n/a_{n-1}} = 1 + \frac{a_{n-1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n$$

(d) If $\lim_{n\to\infty} b_n = \rho$, then $\lim_{n\to\infty}\left(1 + \dfrac{1}{b_{n-1}}\right) = \rho$.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n-1}$, we have $1 + (1/\rho) = \rho$.

$$\rho + 1 = \rho^2$$

$$0 = \rho^2 - \rho - 1$$

$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since a_n, and thus b_n, is positive, $\rho = \dfrac{1+\sqrt{5}}{2} \approx 1.6180$.

122. $x_0 = 1, x_n = \dfrac{1}{2}x_{n-1} + \dfrac{1}{x_{n-1}}, n = 1, 2, \ldots$

$x_1 = 1.5$

$x_2 = 1.41667$

$x_3 = 1.414216$

$x_4 = 1.414214$

$x_5 = 1.414214$

$x_6 = 1.414214$

$x_7 = 1.414214$

$x_8 = 1.414114$

$x_9 = 1.414214$

$x_{10} = 1.414214$

The limit of the sequence appears to be $\sqrt{2}$. In fact, this sequence is Newton's Method applied to $f(x) = x^2 - 2$.

123. (a) $a_1 = \sqrt{2} \approx 1.4142$

$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$

$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$

$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$

$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$

(b) $a_n = \sqrt{2 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{2}$

(c) We first use mathematical induction to show that $a_n \leq 2$; clearly $a_1 \leq 2$. So assume $a_k \leq 2$. Then

$$a_k + 2 \leq 4$$
$$\sqrt{a_k + 2} \leq 2$$
$$a_{k+1} \leq 2.$$

Now we show that $\{a_n\}$ is an increasing sequence. Since $a_n \geq 0$ and $a_n \leq 2$,

$$(a_n - 2)(a_n + 1) \leq 0$$
$$a_n^2 - a_n - 2 \leq 0$$
$$a_n^2 \leq a_n + 2$$
$$a_n \leq \sqrt{a_n + 2}$$
$$a_n \leq a_{n+1}.$$

Since $\{a_n\}$ is a bounding increasing sequence, it converges to some number L, by Theorem 9.5.

$$\lim_{n\to\infty} a_n = L \Rightarrow \sqrt{2 + L} = L \Rightarrow 2 + L = L^2 \Rightarrow L^2 - L - 2 = 0$$
$$\Rightarrow (L - 2)(L + 1) = 0 \Rightarrow L = 2 \quad (L \neq -1)$$

124. (a) $a_1 = \sqrt{6} \approx 2.4495$

$a_2 = \sqrt{6 + \sqrt{6}} \approx 2.9068$

$a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}} \approx 2.9844$

$a_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} \approx 2.9974$

$a_5 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}} \approx 2.9996$

(b) $a_n = \sqrt{6 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{6}$

(c) We first use mathematical induction to show that $a_n \leq 3$; clearly $a_1 \leq 3$. So assume $a_k \leq 3$. Then

$$6 + a_k \leq 9$$
$$\sqrt{6 + a_k} \leq 3$$
$$a_{k+1} \leq 3.$$

Now we show that $\{a_n\}$ is an increasing sequence. Since $a_n \geq 0$ and $a_n \leq 3$,

$$(a_n - 3)(a_n + 2) \leq 0$$
$$a_n^2 - a_n - 6 \leq 0$$
$$a_n^2 \leq a_n + 6$$
$$a_n \leq \sqrt{a_n + 6}$$
$$a_n \leq a_{n+1}.$$

Since $\{a_n\}$ is a bounded increasing sequence, it converges to some number L: $\lim_{n\to\infty} a_n = L$. Thus,

$$\sqrt{6 + L} = L \Rightarrow 6 + L = L^2 \Rightarrow L^2 - L - 6 = 0$$
$$\Rightarrow (L - 3)(L + 2) = 0 \Rightarrow L = 3 \quad (L \neq -2)$$

125. (a) We use mathematical induction to show that

$$a_n \le \frac{1+\sqrt{1+4k}}{2}.$$

[Note that if $k = 2$, then $a_n \le 3$, and if $k = 6$, then $a_n \le 3$.] Clearly,

$$a_1 = \sqrt{k} \le \frac{\sqrt{1+4k}}{2} \le \frac{1+\sqrt{1+4k}}{2}.$$

Before proceeding to the induction step, note that

$$2 + 2\sqrt{1+4k} + 4k = 2 + 2\sqrt{1+4k} + 4k$$

$$\frac{1+\sqrt{1+4k}}{2} + k = \frac{1+2\sqrt{1+4k}+1+4k}{4}$$

$$\frac{1+\sqrt{1+4k}}{2} + k = \left[\frac{1+\sqrt{1+4k}}{2}\right]^2$$

$$\sqrt{\frac{1+\sqrt{1+4k}}{2} + k} = \frac{1+\sqrt{1+4k}}{2}.$$

So assume $a_n \le \frac{1+\sqrt{1+4k}}{2}$. Then

$$a_n + k \le \frac{1+\sqrt{1+4k}}{2} + k$$

$$\sqrt{a_n + k} \le \sqrt{\frac{1+\sqrt{1+4k}}{2} + k}$$

$$a_{n+1} \le \frac{1+\sqrt{1+4k}}{2}.$$

$\{a_n\}$ is increasing because

$$\left(a_n - \frac{1+\sqrt{1+4k}}{2}\right)\left(a_n - \frac{1-\sqrt{1+4k}}{2}\right) \le 0$$

$$a_n^2 - a_n - k \le 0$$

$$a_n^2 \le a_n + k$$

$$a_n \le \sqrt{a_n + k}$$

$$a_n \le a_{n+1}.$$

(b) Since $\{a_n\}$ is bounded and increasing, it has a limit L.

(c) $\lim_{n\to\infty} a_n = L$ implies that

$$L = \sqrt{k+L} \Rightarrow L^2 = k + L$$

$$\Rightarrow L^2 - L - k = 0$$

$$\Rightarrow L = \frac{1 \pm \sqrt{1+4k}}{2}.$$

Since $L > 0$, $L = \frac{1+\sqrt{1+4k}}{2}$.

126. (a) $a_0 = 10$, $b_0 = 3$

$$a_1 = \frac{a_0 + b_0}{2} = \frac{10+3}{2} = 6.5 \qquad b_1 = \sqrt{a_0 b_0} = \sqrt{10(3)} \approx 5.4772$$

$$a_2 = \frac{a_1 + b_1}{2} \approx 5.9886 \qquad b_2 = \sqrt{a_1 b_1} \approx 5.9667$$

$$a_3 = \frac{a_2 + b_2}{2} \approx 5.9777 \qquad b_3 = \sqrt{a_2 b_2} \approx 5.9777$$

$$a_4 = \frac{a_3 + b_3}{2} \approx 5.9777 \qquad b_4 = \sqrt{a_3 b_3} \approx 5.9777$$

$$a_5 = \frac{a_4 + b_4}{2} \approx 5.9777 \qquad b_5 = \sqrt{a_4 b_4} \approx 5.9777$$

The terms of $\{a_n\}$ are decreasing, and those of $\{b_n\}$ are increasing. They both seem to approach the same limit.

—CONTINUED—

126. —CONTINUED—

(b) For $n = 0$, we need to show $a_0 > a_1 > b_1 > b_0$. Since $a_0 > b_0$, $2a_0 > a_0 + b_0 \Rightarrow a_0 > \frac{a_0 + b_0}{2} = a_1$.

Since $(a_0 - b_0)^2 > 0$,

$$a_0^2 - 2a_0b_0 + b_0^2 > 0 \Rightarrow a_0^2 + 2a_0b_0 + b_0^2 > 4a_0b_0$$
$$\Rightarrow (a_0 + b_0)^2 > 4a_0b_0 \Rightarrow a_0 + b_0 > 2\sqrt{a_0b_0} \Rightarrow a_1 > b_1.$$

Since $a_0 > b_0$, $a_0b_0 > b_0^2 \Rightarrow \sqrt{a_0b_0} > b_0 \Rightarrow b_1 > b_0$. Thus, we have shown that $a_0 > a_1 > b_1 > b_0$.

Now assume $a_k > a_{k+1} > b_{k+1} > b_k$. Since $a_{k+1} > b_{k+1}$,

$$2a_{k+1} > a_{k+1} + b_{k+1} \Rightarrow a_{k+1} > \frac{a_{k+1} + b_{k+1}}{2} = a_{k+2}.$$

Since $(a_{k+1} - b_{k+1})^2 \geq 0$,

$$a_{k+1}^2 - 2a_{k+1}b_{k+1} + b_{k+1}^2 > 0 \Rightarrow a_{k+1}^2 + 2a_{k+1}b_{k+1} + b_{k+1}^2 > 4a_{k+1}b_{k+1}$$
$$\Rightarrow (a_{k+1} + b_{k+1})^2 > 4a_{k+1}b_{k+1}$$
$$\Rightarrow a_{k+1} + b_{k+1} > 2\sqrt{a_{k+1}b_{k+1}}$$
$$\Rightarrow a_{k+2} > b_{k+2}.$$

Since $a_{k+1} > b_{k+1}$, $a_{k+1}b_{k+1} > b_{k+1}^2 \Rightarrow \sqrt{a_{k+1}b_{k+1}} > b_{k+1} \Rightarrow b_{k+2} > b_{k+1}$.

Thus, we have shown that $a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$.

(c) $\{a_n\}$ converges because it is decreasing and bounded below by 0. $\{b_n\}$ converges because it is increasing and bounded above by a_0.

(d) Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then $A = \frac{A + B}{2} \Rightarrow A = B$.

127. (a) $f(x) = \sin x$, $a_n = n \sin \frac{1}{n}$

$f'(x) = \cos x$, $f'(0) = 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)} = 1 = f'(0)$$

(b) $$f'(0) = \lim_{h\to 0^+} \frac{f(0 + h) - f(0)}{h}$$
$$= \lim_{h\to 0^+} \frac{f(h)}{h}$$
$$= \lim_{n\to\infty} \frac{f(1/n)}{(1/n)}$$
$$= \lim_{n\to\infty} nf\left(\frac{1}{n}\right)$$
$$= \lim_{n\to\infty} a_n$$

128. $a_n = nr^n$

(a) $r = \frac{1}{2}$: $a_n = n\left(\frac{1}{2}\right)^n = \frac{n}{2^n} \to 0$

(b) $r = 1$: $a_n = n$, diverges

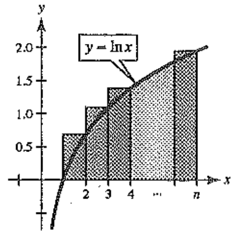
(c) $r = \frac{3}{2}$: $a_n = n\left(\frac{3}{2}\right)^2$, diverges

(d) If $|r| \geq 1$, diverges. If $|r| < 1$,

$$nr^n = \frac{n}{r^{-n}} \qquad \text{“}\frac{\infty}{\infty}\text{”}$$
$$\to \frac{1}{-r^{-n}\ln(r)} = \frac{-r^n}{\ln(r)} \to 0.$$

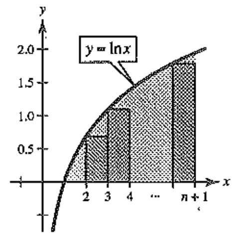
The sequence converges for $|r| < 1$.

129. (a)



$$\int_1^n \ln x\,dx < \ln 2 + \ln 3 + \cdots + \ln n$$
$$= \ln[1 \cdot 2 \cdot 3 \cdots n] = \ln(n!)$$

(b)



$$\int_1^{n+1} \ln x\,dx > \ln 2 + \ln 3 + \cdots + \ln n$$
$$= \ln(n!)$$

(c) $\int \ln x\,dx = x \ln x - x + C$

$$\int_1^n \ln x\,dx = n \ln n - n + 1 = \ln n^n - n + 1$$

From part (a): $\ln n^n - n + 1 < \ln(n!)$

$$e^{\ln n^n - n + 1} < n!$$
$$\frac{n^n}{e^{n-1}} < n!$$

$$\int_1^{n+1} \ln x\,dx = (n+1)\ln(n+1) - (n+1) + 1 = \ln(n+1)^{n+1} - n$$

From part (b): $\ln(n+1)^{n+1} - n > \ln(n!)$

$$e^{\ln(n+1)^{n+1} - n} > n!$$
$$\frac{(n+1)^{n+1}}{e^n} > n!$$

(d) $\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$

$$\frac{n}{e^{1-(1/n)}} < \sqrt[n]{n!} < \frac{(n+1)^{(n+1)/n}}{e}$$

$$\frac{1}{e^{1-(1/n)}} < \frac{\sqrt[n]{n!}}{n} < \frac{(n+1)^{1+(1/n)}}{ne}$$

$$\lim_{n\to\infty} \frac{1}{e^{1-(1/n)}} = \frac{1}{e}$$

$$\lim_{n\to\infty} \frac{(n+1)^{1+(1/n)}}{ne} = \lim_{n\to\infty} \frac{(n+1)}{n}\,\frac{(n+1)^{1/n}}{e}$$
$$= (1)\frac{1}{e} = \frac{1}{e}$$

By the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

(e) $n = 20$: $\frac{\sqrt[20]{20!}}{20} \approx 0.4152$

$n = 50$: $\frac{\sqrt[50]{50!}}{50} \approx 0.3897$

$n = 100$: $\frac{\sqrt[100]{100!}}{100} \approx 0.3799$

$\frac{1}{e} \approx 0.3679$

130. $a_n = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(k/n)}$

(a) $a_1 = \frac{1}{1+1} = \frac{1}{2} = 0.5$

$$a_2 = \frac{1}{2}\left[\frac{1}{1+(1/2)} + \frac{1}{1+1}\right] = \frac{1}{2}\left[\frac{2}{3} + \frac{1}{2}\right] = \frac{7}{12} \approx 0.5833$$

$$a_3 = \frac{1}{3}\left[\frac{1}{1+(1/3)} + \frac{1}{1+(2/3)} + \frac{1}{1+1}\right] = \frac{37}{60} \approx 0.6167$$

$$a_4 = \frac{1}{4}\left[\frac{1}{1+(1/4)} + \frac{1}{1+(2/4)} + \frac{1}{1+(3/4)} + \frac{1}{1+1}\right] = \frac{533}{840} \approx 0.6345$$

$$a_5 = \frac{1}{5}\left[\frac{1}{1+(1/5)} + \frac{1}{1+(2/5)} + \frac{1}{1+(3/5)} + \frac{1}{1+(4/5)} + \frac{1}{1+1}\right] = \frac{1627}{2520} \approx 0.6456$$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(k/n)} = \int_0^1 \frac{1}{1+x}\,dx = \ln|1+x|\Big]_0^1 = \ln 2$

131. For a given $\epsilon > 0$, we must find $M > 0$ such that

$$|a_n - L| = \left|\frac{1}{n^3}\right| < \varepsilon$$

whenever $n > M$. That is,

$$n^3 > \frac{1}{\varepsilon} \text{ or } n > \left(\frac{1}{\varepsilon}\right)^{1/3}.$$

So, let $\varepsilon > 0$ be given. Let M be an integer satisfying $M > (1/\varepsilon)^{1/3}$. For $n > M$, we have

$$n > \left(\frac{1}{\varepsilon}\right)^{1/3}$$

$$n^3 > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{n^3} \Longrightarrow \left|\frac{1}{n^3} - 0\right| < \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

132. For a given $\varepsilon > 0$, we must find $M > 0$ such that $|a_n - L| = |r^n|\varepsilon$ whenever $n > M$. That is, $n \ln|r| < \ln(\varepsilon)$ or

$$n > \frac{\ln(\varepsilon)}{\ln|r|} \text{ (since } \ln|r| < 0 \text{ for } |r| < 1).$$

So, let $\varepsilon > 0$ be given. Let M be an integer satisfying

$$M > \frac{\ln(\varepsilon)}{\ln|r|}.$$

For $n > M$, we have

$$n > \frac{\ln(\varepsilon)}{\ln|r|}$$

$$n \ln|r| < \ln(\varepsilon)$$

$$\ln|r|^n < \ln(\varepsilon)$$

$$|r|^n < \varepsilon$$

$$|r^n - 0| < \varepsilon.$$

Thus, $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

133. If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$. Then $-a_1 \le -a_2 \le -a_3 \le \cdots \le -a_n \le \cdots$ is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Since $\{-a_n\}$ converges, then so does $\{a_n\}$.

134. Define $a_n = \frac{x_{n+1} + x_{n-1}}{x_n}$, $n \ge 1$.

$$x_{n+1}^2 - x_n x_{n+2} = 1 = x_n^2 - x_{n-1}x_{n+1} \Longrightarrow$$

$$x_{n+1}(x_{n+1} + x_{n-1}) = x_n(x_n + x_{n+2})$$

$$\frac{x_{n+1} + x_{n-1}}{x_n} = \frac{x_{n+2} + x_n}{x_{n+1}}$$

$$a_n = a_{n+1}$$

Hence, $a_1 = a_2 = \ldots = a$. Thus,

$$x_{n+1} = a_n x_n - x_{n-1} = a x_n - x_{n-1}.$$

135. $T_n = n! + 2^n$

We use mathematical induction to verify the formula.

$T_0 = 1 + 1 = 2$

$T_1 = 1 + 2 = 3$

$T_2 = 2 + 4 = 6$

Assume $T_k = k! + 2^k$. Then

$$\begin{aligned} T_{k+1} &= (k + 1 + 4)T_k - 4(k + 1)T_{k-1} + (4(k + 1) - 8)T_{k-2} \\ &= (k + 5)[k! + 2^k] - 4(k + 1)((k - 1)! + 2^{k-1}) + (4k - 4)((k - 2)! + 2^{k-2}) \\ &= [(k + 5)(k)(k - 1) - 4(k + 1)(k - 1) + 4(k - 1)](k - 2)! + [(k + 5)4 - 8(k + 1) + 4(k - 1)]2^{k-2} \\ &= [k^2 + 5k - 4k - 4 + 4](k - 1)! + 8 \cdot 2^{k-2} \\ &= (k + 1)! + 2^{k+1}. \end{aligned}$$

By mathematical induction, the formula is valid for all n.

Section 9.2 Series and Convergence

1. $S_1 = 1$

$S_2 = 1 + \frac{1}{4} = 1.2500$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$

$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$

2. $S_1 = \frac{1}{6} \approx 0.1667$

$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$

$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$

$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$

$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$

3. $S_1 = 3$

$S_2 = 3 - \frac{9}{2} = -1.5$

$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$

$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$

$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$

4. $S_1 = 1$

$S_2 = 1 + \frac{1}{3} \approx 1.3333$

$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$

$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \approx 1.6762$

$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \approx 1.7873$

5. $S_1 = 3$

$S_2 = 3 + \frac{3}{2} = 4.5$

$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$

$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$

$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$

6. $S_1 = 1$

$S_2 = 1 - \frac{1}{2} = 0.5$

$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$

$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$

$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$

7. $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$

Geometric series

$r = \frac{3}{2} > 1$

Diverges by Theorem 9.6

8. $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$

Geometric series

$r = \frac{4}{3} > 1$

Diverges by Theorem 9.6

9. $\sum_{n=0}^{\infty} 1000(1.055)^n$

Geometric series

$r = 1.055 > 1$

Diverges by Theorem 9.6

10. $\sum_{n=0}^{\infty} 2(-1.03)^n$

Geometric series

$|r| = 1.03 > 1$

Diverges by Theorem 9.6

11. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Diverges by Theorem 9.9

12. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

13. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

Diverges by Theorem 9.9

14. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$

Diverges by Theorem 9.9

15. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

$\lim_{n\to\infty} \frac{2^n+1}{2^{n+1}} = \lim_{n\to\infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

16. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

Diverges by Theorem 9.9

17. $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right]$

$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \ldots$

Matches graph (c). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

18. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$

$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \ldots$

Matches graph (b). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

19. $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15}{4}\left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$

$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \ldots$

Matches graph (a). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$$

20. $\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n = \frac{17}{3}\left[1 - \frac{8}{9} + \frac{64}{81} - \cdots\right]$

$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \ldots$

Matches (d). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n = \frac{17/3}{1-(-8/9)} = \frac{17/3}{17/9} = 3$$

21. $\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{1}{2}\right)^n = \frac{17}{3}\left[1 - \frac{1}{2} + \frac{1}{4} - \cdots\right]$

$S_0 = \frac{17}{3}, S_1 = \frac{17}{6}, \ldots$

Matches (f). The series is geometric:

$$\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{1}{2}\right)^n = \frac{17}{3}\frac{1}{1-(-1/2)} = \frac{17}{3}\frac{2}{3} = \frac{34}{9}$$

22. $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = 1 + \frac{2}{5} + \frac{4}{25} + \cdots$

$S_0 = 1, S_1 = \frac{7}{5}, \ldots$

Matches (e). The series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-(2/5)} = \frac{5}{3}$$

23. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots, \; S_n = 1 - \frac{1}{n+1}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

24. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

25. $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Geometric series with $r = \frac{3}{4} < 1$

Converges by Theorem 9.6

26. $\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$

Geometric series with $|r| = \left|-\frac{1}{2}\right| < 1$

Converges by Theorem 9.6

27. $\sum_{n=0}^{\infty} (0.9)^n$

Geometric series with $r = 0.9 < 1$

Converges by Theorem 9.6

28. $\sum_{n=0}^{\infty} (-0.6)^n$

Geometric series with $|r| = |-0.6| < 1$

Converges by Theorem 9.6

29. (a) $\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$$= 2\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right]$$

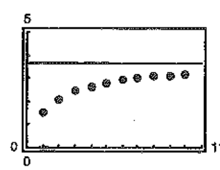
$$\left(S_n = 2\left[1 + \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)\right]\right)$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3}\right] = \frac{11}{3} \approx 3.667$$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

(c)



(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

30. (a) $\sum_{n=1}^{\infty} \frac{4}{n(n+4)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+4}\right)$

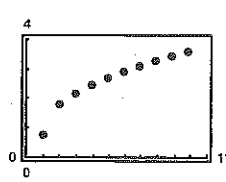
$$= \left(1 - \frac{1}{5}\right) + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$$

(b)

n	5	10	20	50	100
S_n	1.5377	1.7607	1.9051	2.0071	2.0443

(c)



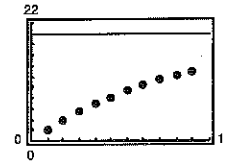
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

31. (a) $\sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

(c)



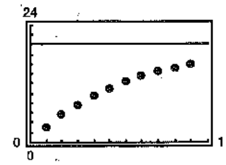
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

32. (a) $\sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1-0.85} = 20$ (Geometric series)

(b)

n	5	10	20	50	100
S_n	11.1259	16.0625	19.2248	19.9941	19.999998

(c)



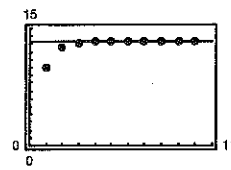
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

33. (a) $\sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333

(c)



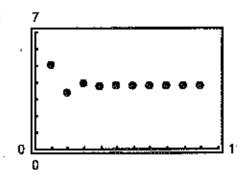
(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

34. (a) $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1-(-1/3)} = \frac{15}{4} = 3.75$

(b)

n	5	10	20	50	100
S_n	3.7654	3.7499	3.7500	3.7500	3.7500

(c)



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

35. $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \cdots\right]$$

$$= \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$$

36. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = 2\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots\right] = 2\left(1+\frac{1}{2}\right) = 3$

37. $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 8\left[\left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots\right] = 8\left(\frac{1}{2}\right) = 4$

38. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{9}\right) + \cdots\right] = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

39. $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-(1/2)} = 2$

40. $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1-(4/5)} = 30$

(Geometric)

41. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$

42. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1-(-2/3)} = \frac{6}{5}$

43. $\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \frac{1}{1-(1/10)} = \frac{10}{9}$

44. $\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1-(3/4)} = 32$

45. $\sum_{n=0}^{\infty} 3\left(-\frac{1}{3}\right)^n = \frac{3}{1-(-1/3)} = \frac{9}{4}$

46. $\sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{4}{1-(-1/2)} = \frac{8}{3}$

47. $\sum_{n=0}^{\infty}\left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)}$

$= 2 - \frac{3}{2} = \frac{1}{2}$

48. $\sum_{n=1}^{\infty}[(0.7)^n + (0.9)^n] = \sum_{n=0}^{\infty}\left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty}\left(\frac{9}{10}\right)^n - 2$

$= \frac{1}{1-(7/10)} + \frac{1}{1-(9/10)} - 2$

$= \frac{10}{3} + 10 - 2 = \frac{34}{3}$

49. Note that $\sin(1) \approx 0.8415 < 1$. The series $\sum_{n=1}^{\infty}[\sin(1)]^n$ is geometric with $r = \sin(1) < 1$. Thus,

$$\sum_{n=1}^{\infty}[\sin(1)]^n = \sin(1)\sum_{n=0}^{\infty}[\sin(1)]^n = \frac{\sin(1)}{1-\sin(1)} \approx 5.3080.$$

50. $S_n = \sum_{k=1}^{n}\frac{1}{9k^2+3k-2} = \sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)}$

$= \sum_{k=1}^{n}\left[\frac{1}{9k-3} - \frac{1}{9k+6}\right] = \frac{1}{3}\sum_{k=1}^{n}\left[\frac{1}{3k-1} - \frac{1}{3k+2}\right]$

$= \frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{8}-\frac{1}{11}\right) + \cdots + \left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)\right]$

$= \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right] = \frac{1}{6}$

51. (a) $0.\overline{4} = \sum_{n=0}^{\infty}\frac{4}{10}\left(\frac{1}{10}\right)^n$

(b) Geometric series with $a = \frac{4}{10}$ and $r = \frac{1}{10}$

$S = \frac{a}{1-r} = \frac{4/10}{1-(1/10)} = \frac{4}{9}$

52. (a) $0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

$= \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = \frac{9}{10}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$

(b) $0.\overline{9} = \frac{9}{10}\,\frac{1}{1-(1/10)} = 1$

53. (a) $0.\overline{81} = \sum_{n=0}^{\infty}\frac{81}{100}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$S = \frac{a}{1-r} = \frac{81/100}{1-(1/100)} = \frac{81}{99} = \frac{9}{11}$

54. (a) $0.\overline{01} = \sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = \frac{1}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n$

(b) $0.\overline{01} = \frac{1}{100}\cdot\frac{1}{1-(1/100)} = \frac{1}{100}\cdot\frac{100}{99} = \frac{1}{99}$

55. (a) $0.0\overline{75} = \sum_{n=0}^{\infty} \frac{3}{40}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{3}{40}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{3/40}{99/100} = \frac{5}{66}$$

56. (a) $0.2\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$$S = \frac{1}{5} + \frac{a}{1-r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$$

57. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

$$\lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

Diverges by Theorem 9.9

58. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$$\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$$

Diverges by Theorem 9.9

59. $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}$, converges

60. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$$= \frac{1}{3}\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{9}\right) + \cdots\right]$$

$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}$, converges

61. $\sum_{n=1}^{\infty} \frac{3n-1}{2n+1}$

$$\lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$$

Diverges by Theorem 9.9

62. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 2)3^n}{3n^2}$$

$$= \lim_{n\to\infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n\to\infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule); diverges by Theorem 9.9

63. $\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

Geometric series with $r = \frac{1}{2}$

Converges by Theorem 9.6

64. $\sum_{n=0}^{\infty} \frac{1}{4^n}$

Geometric series with $r = \frac{1}{4}$

Converges by Theorem 9.6

65. $\sum_{n=0}^{\infty} (1.075)^n$

Geometric series with $r = 1.075$

Diverges by Theorem 9.6

66. $\sum_{n=1}^{\infty} \frac{2^n}{100}$

Geometric series with $r = 2$

Diverges by Theorem 9.6

67. Since $n > \ln(n)$, the terms $a_n = \frac{n}{\ln(n)}$

do not approach 0 as $n \to \infty$. Hence, the series

$\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$ diverges.

68. $S_n = \sum_{k=1}^{n} \ln\left(\frac{1}{k}\right) = \sum_{k=1}^{n} -\ln(k)$

$$= 0 - \ln 2 - \ln 3 - \cdots - \ln(n)$$

Since $\lim_{n\to\infty} S_n$ diverges, $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ diverges.

69. For $k \neq 0$,

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{k}{n}\right)^{n/k}\right]^k = e^k \neq 0.$$

For $k = 0$, $\lim_{n\to\infty}(1+0)^n = 1 \neq 0$.

Hence, $\sum_{n=1}^{\infty}\left[1+\frac{k}{n}\right]^n$ diverges.

70. $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ converges since it is geometric with

$$|r| = \frac{1}{e} < 1.$$

71. $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

Hence, $\sum_{n=1}^{\infty} \arctan n$ diverges.

72.
$$\begin{aligned} S_n &= \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) \\ &= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{n+1}{n}\right) \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) \\ &= \ln(n+1) - \ln(1) = \ln(n+1) \end{aligned}$$

Diverges

73. See definition on page 606.

74. $\lim_{n\to\infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

75. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0$$

is a geometric series with ratio r. When $0 < |r| < 1$, the series converges to $a/(1-r)$. The series diverges if $|r| \geq 1$.

76. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

77. $a_n = \dfrac{n+1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n+1}{n} = 1$ ($\{a_n\}$ converges)

$\sum_{n=1}^{\infty} a_n$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

78. (a) $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

(b) $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$

These are the same. The third series is different, unless $a_1 = a_2 = \cdots = a$ is constant.

(c) $\sum_{n=1}^{\infty} a_k = a_k + a_k + \cdots$

79. $\sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n = \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

Geometric series: converges for $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$

$$\begin{aligned} f(x) &= \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \\ &= \frac{x}{2}\,\frac{1}{1-(x/2)} = \frac{x}{2}\,\frac{2}{2-x} = \frac{x}{2-x}, \quad |x| < 2 \end{aligned}$$

80. $\sum_{n=1}^{\infty} (3x)^n = (3x)\sum_{n=0}^{\infty} (3x)^n$

Geometric series: converges for $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

$$f(x) = (3x)\sum_{n=0}^{\infty}(3x)^n = (3x)\frac{1}{1-3x} = \frac{3x}{1-3x}, \quad |x| < \frac{1}{3}$$

81. $\sum_{n=1}^{\infty}(x-1)^n = (x-1)\sum_{n=0}^{\infty}(x-1)^n$

Geometric series: converges for $|x-1| < 1 \Rightarrow 0 < x < 2$

$$f(x) = (x-1)\sum_{n=0}^{\infty}(x-1)^n$$

$$= (x-1)\frac{1}{1-(x-1)} = \frac{x-1}{2-x}, \quad 0 < x < 2$$

82. $\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n$

Geometric series: converges for

$$\left|\frac{x-3}{4}\right| < 1 \Rightarrow |x-3| < 4 \Rightarrow -1 < x < 7$$

$$f(x) = \sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n = \frac{4}{1-[(x-3)/4]}$$

$$= \frac{4}{(4-x+3)/4} = \frac{16}{7-x}, \quad -1 < x < 7$$

83. $\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n$

Geometric series: converges for

$$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1+x}, \quad -1 < x < 1$$

84. $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n$

Geometric series: converges for

$$|-x^2| < 1 \Rightarrow -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}, \quad -1 < x < 1$$

85. $\sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^n$

Geometric series: converges if $\left|\frac{1}{x}\right| < 1$

$$\Rightarrow |x| > 1 \Rightarrow x < -1 \text{ or } x > 1$$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^n = \frac{1}{1-(1/x)} = \frac{x}{x-1}, \quad x > 1 \text{ or } x < -1$$

86. $\sum_{n=1}^{\infty}\left(\frac{x^2}{x^2+4}\right)^n = \frac{x^2}{x^2+4}\sum_{n=0}^{\infty}\left(\frac{x^2}{x^2+4}\right)^n$

Geometric series: converges for $\left|\frac{x^2}{x^2+4}\right| < 1$

$\Rightarrow x^2 < x^2 + 4 \Rightarrow$ converges for all x

$$f(x) = \frac{x^2}{x^2+4}\cdot\frac{1}{1-\dfrac{x^2}{x^2+4}} = \frac{x^2}{x^2+4}\,\frac{x^2+4}{4} = \frac{x^2}{4}$$

87. (a) Yes, the new series will still diverge.

(b) Yes, the new series will converge.

88. Neither statement is true. The formula

$$\frac{1}{x-1} = 1 + x + x^2 + \cdots$$

holds for $-1 < x < 1$.

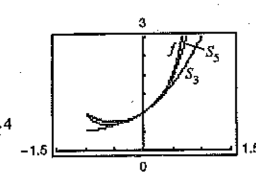
89. (a) x is the common ratio.

(b) $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$

(c) $y_1 = \frac{1}{1-x}$

$y_2 = S_3 = 1 + x + x^2$

$y_3 = S_5 = 1 + x + x^2 + x^3 + x^4$



90. (a) $\left(-\frac{x}{2}\right)$ is the common ratio.

(b) $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$

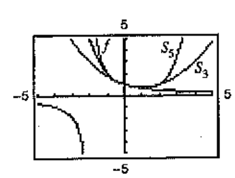
$$= \frac{1}{1-(-x/2)}$$

$$= \frac{2}{2+x}, \quad |x| < 2$$

(c) $y_1 = \frac{2}{2+x}$

$y_2 = S_3 = 1 - \frac{x}{2} + \frac{x^2}{4}$

$y_3 = S_5 = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16}$

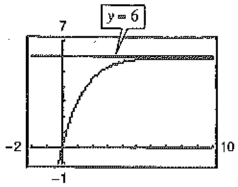


91. $f(x) = 3\left[\dfrac{1 - 0.5^x}{1 - 0.5}\right]$

Horizontal asymptote: $y = 6$

$\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$

$S = \dfrac{3}{1 - (1/2)} = 6$



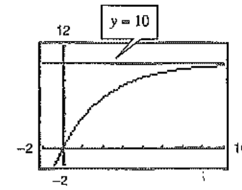
The horizontal asymptote is the sum of the series. $f(n)$ is the nth partial sum.

92. $f(x) = 2\left[\dfrac{1 - 0.8^x}{1 - 0.8}\right]$

Horizontal asymptote: $y = 10$

$\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

$S = \dfrac{2}{1 - (4/5)} = 10$



The horizontal asymptote is the sum of the series. $f(n)$ is the nth partial sum.

93. $\dfrac{1}{n(n+1)} < 0.0001$

$10{,}000 < n^2 + n$

$0 < n^2 + n - 10{,}000$

$n = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-10{,}000)}}{2}$

Choosing the positive value for n we have $n \approx 99.5012$. The first *term* that is less than 0.0001 is $n = 100$.

$\left(\frac{1}{8}\right)^n < 0.0001$

$10{,}000 < 8^n$

This inequality is true when $n = 5$. This series converges at a faster rate.

94. $\dfrac{1}{2^n} < 0.0001$

$10{,}000 < 2^n$

This inequality is true when $n = 14$.

$(0.01)^n < 0.0001$

$10{,}000 < 10^n$

This inequality is true when $n = 5$. This series converges at a faster rate.

95. $\sum_{i=0}^{n-1} 8000(0.9)^i = \dfrac{8000[1 - (0.9)^{(n-1)+1}]}{1 - 0.9}$

$= 80{,}000(1 - 0.9^n),\ n > 0$

96. $V(t) = 225{,}000(1 - 0.3)^n = (0.7)^n(225{,}000)$

$V(5) = (0.7)^5(225{,}000) = \$37{,}815.75$

97. $\sum_{i=0}^{n-1} 100(0.75)^i = \dfrac{100[1 - 0.75^{(n-1)+1}]}{1 - 0.75}$

$= 400(1 - 0.75^n)$ million dollars

Sum = 400 million dollars

98. $\sum_{i=0}^{n-1} 100(0.60)^i = \dfrac{100[1 - 0.6^n]}{1 - 0.6}$

$= 250(1 - 0.6^n)$ million dollars

Sum = 250 million dollars

99. $D_1 = 16$

$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$

$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$

$\vdots$

$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots$

$= -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \dfrac{32}{1 - 0.81}$

≈ 152.42 feet

100. The ball in Exercise 99 takes the following times for each fall.

$s_1 = -16t^2 + 16$	$s_1 = 0$ if $t = 1$
$s_2 = -16t^2 + 16(0.81)$	$s_2 = 0$ if $t = 0.9$
$s_3 = -16t^2 + 16(0.81)^2$	$s_3 = 0$ if $t = (0.9)^2$
$\vdots$	$\vdots$
$s_n = -16t^2 + 16(0.81)^{n-1}$	$s_n = 0$ if $t = (0.9)^{n-1}$

Beginning with s_2, the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n = -1 + 2\sum_{n=0}^{\infty}(0.9)^n$

$= -1 + \dfrac{2}{1 - 0.9} = 19$ seconds.

101. $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$

$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$

102. $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

$P(2) = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$

$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$

103. (a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n$

$= \frac{1}{2}\frac{1}{(1 - (1/2))} = 1$

(b) No, the series is not geometric.

(c) $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$

104. Person 1: $\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{2}\frac{1}{1 - (1/8)} = \frac{4}{7}$

Person 2: $\frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{4}\frac{1}{1 - (1/8)} = \frac{2}{7}$

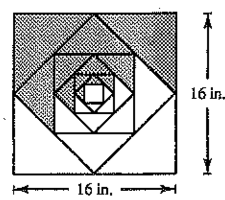
Person 3: $\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{8}\frac{1}{1 - (1/8)} = \frac{1}{7}$

Sum: $\frac{4}{7} + \frac{2}{7} + \frac{1}{7} = 1$

105. (a) $64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$

(b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$

Note: This is one-half of the area of the original square!



106. (a) $\sin\theta = \frac{|Yy_1|}{z} \Longrightarrow |Yy_1| = z\sin\theta$

$\sin\theta = \frac{|x_1y_1|}{|Yy_1|} \Longrightarrow |x_1y_1| = |Yy_1|\sin\theta = z\sin^2\theta$

$\sin\theta = \frac{|x_1y_2|}{|x_1y_1|} \Longrightarrow |x_1y_2| = |x_1y_1|\sin\theta = z\sin^3\theta$

Total: $z\sin\theta + z\sin^2\theta + z\sin^3\theta + \cdots = z\frac{\sin\theta}{1 - \sin\theta}$

(b) If $z = 1$ and $\theta = \frac{\pi}{6}$, then total $= \frac{1/2}{1 - (1/2)} = 1.$

107. $\sum_{n=1}^{20} 50{,}000\left(\frac{1}{1.06}\right)^n = \frac{50{,}000}{1.06}\sum_{i=0}^{19}\left(\frac{1}{1.06}\right)^i$

$= \frac{50{,}000}{1.06}\left[\frac{1 - 1.06^{-20}}{1 - 1.06^{-1}}\right], \quad [n = 20, r = 1.06^{-1}]$

$\approx 573{,}496.06$

108. Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4[\pi + \pi + \cdots] = \infty$

109. $w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$

(a) When $n = 29$: $w = \$5{,}368{,}709.11$

(b) When $n = 30$: $w = \$10{,}737{,}418.23$

(c) When $n = 31$: $w = \$21{,}474{,}836.47$

110. $\sum_{n=0}^{12t-1} P\left(1+\frac{r}{12}\right)^n = \frac{P\left[1-\left(1+\frac{r}{12}\right)^{12t}\right]}{1-\left(1+\frac{r}{12}\right)} = P\left(-\frac{12}{r}\right)\left[\left(1-\left(1+\frac{r}{12}\right)^{12t}\right]\right] = P\left(\frac{12}{r}\right)\left[\left(1+\frac{r}{12}\right)^{12t}-1\right]$

$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1-(e^{r/12})^{12t})}{1-e^{r/12}} = \frac{P(e^{rt}-1)}{e^{r/12}-1}$

111. $P = 50, r = 0.03, t = 20$

(a) $A = 50\left(\frac{12}{0.03}\right)\left[\left(1+\frac{0.03}{12}\right)^{12(20)}-1\right] \approx \$16,415.10$

(b) $A = \frac{50(e^{0.03(20)}-1)}{e^{0.03/12}-1} \approx \$16,421.83$

112. $P = 75, r = 0.05, t = 25$

(a) $A = 75\left(\frac{12}{0.05}\right)\left[\left(1+\frac{0.05}{12}\right)^{12(25)}-1\right] \approx \$44,663.23$

(b) $A = \frac{75(e^{0.05(25)}-1)}{e^{0.05/12}-1} \approx \$44,732.85$

113. $P = 100, r = 0.04, t = 40$

(a) $A = 100\left(\frac{12}{0.04}\right)\left[\left(1+\frac{0.04}{12}\right)^{12(40)}-1\right] \approx \$118,196.13$

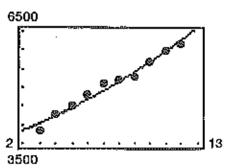
(b) $A = \frac{100(e^{0.04(40)}-1)}{e^{0.04/12}-1} \approx \$118,393.43$

114. $P = 20, r = 0.06, t = 50$

(a) $A = 20\left(\frac{12}{0.06}\right)\left[\left(1+\frac{0.06}{12}\right)^{12(50)}-1\right] \approx \$75,743.82$

(b) $A = \frac{20(e^{0.06(50)}-1)}{e^{0.06/12}-1} \approx \$76,151.45$

115. (a) $a_n = 3484.1363(1.0502)^n = 3484.1363e^{0.04897n}$



(b) Adding the ten values $a_3, a_4, \ldots, a_{12}$ you obtain

$\sum_{n=3}^{12} a_n = 50,809$, or \$50,809,000,000.

(c) $\sum_{n=3}^{12} 3484.1363e^{0.04897n} \approx 50,803$, or \$50,803,000,000

(Answers will vary.)

116. $T = 40,000 + 40,000(1.04) + \cdots + 40,000(1.04)^{39}$

$= \sum_{n=0}^{39} 40,000(1.04)^n = 40,000\left(\frac{1-1.04^{40}}{1-1.04}\right) \approx \$3,801,020$

117. False. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

118. True

119. False; $\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r}\right) - a$

The formula requires that the geometric series begins with $n = 0$.

120. True

$\lim_{n\to\infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$

121. True

$0.74999\ldots = 0.74 + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$

$= 0.74 + \frac{9}{10^3} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$

$= 0.74 + \frac{9}{10^3} \cdot \frac{1}{1-(1/10)}$

$= 0.74 + \frac{9}{10^3} \cdot \frac{10}{9}$

$= 0.74 + \frac{1}{100} = 0.75$

122. True

123. By letting $S_0 = 0$, we have

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}. \text{ Thus,}$$

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (S_n - S_{n-1})\\ &= \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c)\\ &= \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)].\end{aligned}$$

124. Let $\{S_n\}$ be the sequence of partial sums for the convergent series

$\sum_{n=1}^{\infty} a_n = L$. Then $\lim_{n\to\infty} S_n = L$ and since

$$R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

we have

$$\begin{aligned}\lim_{n\to\infty} R_n &= \lim_{n\to\infty} (L - S_n)\\ &= \lim_{n\to\infty} L - \lim_{n\to\infty} S_n = L - L = 0.\end{aligned}$$

125. Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

126. If $\Sigma (a_n + b_n)$ converged, then $\Sigma (a_n + b_n) - \Sigma a_n = \Sigma b_n$ would converge, which is a contradiction. Thus, $\Sigma (a_n + b_n)$ diverges.

127. Suppose, on the contrary, that $\Sigma\, ca_n$ converges. Because $c \neq 0$,

$$\sum \left(\frac{1}{c}\right) ca_n = \sum a_n$$

converges. This is a contradiction since $\Sigma\, ca_n$ diverged. Hence, $\Sigma\, ca_n$ diverges.

128. If $\Sigma\, a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Hence,

$$\lim_{n\to\infty} \frac{1}{a_n} \neq 0,$$

which implies that $\Sigma\, 1/a_n$ diverges.

129. (a)
$$\begin{aligned}\frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}} &= \frac{a_{n+3} - a_{n+1}}{a_{n+1}a_{n+2}a_{n+3}}\\ &= \frac{a_{n+2}}{a_{n+1}a_{n+2}a_{n+3}}\\ &= \frac{1}{a_{n+1}a_{n+3}}\end{aligned}$$

(b)
$$\begin{aligned}S_n &= \sum_{k=0}^{n} \frac{1}{a_{k+1}a_{k+3}}\\ &= \sum_{k=0}^{n} \left[\frac{1}{a_{k+1}a_{k+2}} - \frac{1}{a_{k+2}a_{k+3}}\right]\\ &= \left[\frac{1}{a_1a_2} - \frac{1}{a_2a_3}\right] + \left[\frac{1}{a_2a_3} - \frac{1}{a_3a_4}\right] + \cdots + \left[\frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}}\right]\\ &= \frac{1}{a_1a_2} - \frac{1}{a_{n+2}a_{n+3}}\\ &= 1 - \frac{1}{a_{n+2}a_{n+3}}\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}a_{n+3}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[1 - \frac{1}{a_{n+2}a_{n+3}}\right] = 1$$

130. $S_{2n} = 1 + 2x + x^2 + 2x^3 + \cdots + x^{2n} + 2x^{2n+1}$

$$= (1 + x^2 + x^4 + \cdots + x^{2n}) + (2x + 2x^3 + \cdots + 2x^{2n+1})$$

$$= \sum_{k=0}^{n} (x^2)^k + 2x \sum_{k=0}^{n} (x^2)^k$$

As $n \to \infty$, the two geometric series converge for $|x| < 1$:

$$\lim_{n\to\infty} S_{2n} = \frac{1}{1 - x^2} + 2x\left(\frac{1}{1 - x^2}\right) = \frac{1 + 2x}{1 - x^2}, \quad |x| < 1$$

131. $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \frac{1}{r}\left(\frac{1}{r}\right)^n$

$$= \frac{1/r}{1 - (1/r)}$$

$$= \frac{1}{r - 1} \quad \left(\text{since } \left|\frac{1}{r}\right| < 1\right)$$

This is a geometric series which converges if

$$\left|\frac{1}{r}\right| < 1 \Leftrightarrow |r| > 1.$$

132. The entire rectangle has area 2 because the height is 1 and the base is $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$. The squares all lie inside the rectangle, and the sum of their areas is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

133. Let H represent the half-life of the drug. If a patient receives n equal doses of P units each of this drug, administered at equal time interval of length t, the total amount of the drug in the patient's system at the time the last dose is administered is given by $T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$ where $k = -(\ln 2)/H$. One time interval *after* the last dose is administered is given by $T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}$. Two time intervals *after* the last dose is administered is given by $T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt}$ and so on. Since $k < 0$, $T_{n+s} \to 0$ as $s \to \infty$, where s is an integer.

134. The series is telescoping:

$$S_n = \sum_{k=1}^{n} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

$$= \sum_{k=1}^{n}\left[\frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}}\right]$$

$$= 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}}$$

$$\lim_{n\to\infty} S_n = 3 - 1 = 2$$

135. $f(1) = 0, f(2) = 1, f(3) = 2, f(4) = 4, \ldots$

In general: $f(n) = \begin{cases} n^2/4, & n \text{ even} \\ (n^2 - 1)/4, & n \text{ odd.} \end{cases}$

(See below for a proof of this.)

$x + y$ and $x - y$ are either both odd or both even. If both even, then

$$f(x + y) - f(x - y) = \frac{(x + y)^2}{4} - \frac{(x - y)^2}{4} = xy.$$

If both odd,

$$f(x + y) - f(x - y) = \frac{(x + y)^2 - 1}{4} - \frac{(x - y)^2 - 1}{4}$$

$$= xy.$$

Proof by induction that the formula for $f(n)$ is correct. It is true for $n = 1$. Assume that the formula is valid for k. If k is even, then $f(k) = k^2/4$ and

$$f(k + 1) = f(k) + \frac{k}{2} = \frac{k^2}{4} + \frac{k}{2}$$

$$= \frac{k^2 + 2k}{4} = \frac{(k + 1)^2 - 1}{4}.$$

The argument is similar if k is odd.

Section 9.3 The Integral Test and p-Series

1. $\sum_{n=1}^{\infty} \frac{1}{n+1}$

Let $f(x) = \frac{1}{x+1}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x+1}\,dx = \Big[\ln(x+1)\Big]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

2. $\sum_{n=1}^{\infty} \frac{2}{3n+5}$

Let $f(x) = \frac{2}{3x+5}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x+5}\,dx = \left[\frac{2}{3}\ln(3x+5)\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

3. $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x}\,dx = \Big[-e^{-x}\Big]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 9.10

4. $\sum_{n=1}^{\infty} ne^{-n/2}$

Let $f(x) = xe^{-x/2}$.

f is positive, continuous, and decreasing for $x \geq 3$ since $f'(x) = \frac{2-x}{2e^{x/2}} < 0$ for $x \geq 3$.

$$\int_3^{\infty} xe^{-x/2}\,dx = \Big[-2(x+2)e^{-x/2}\Big]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 9.10

5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 9.10

6. $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let $f(x) = \frac{1}{2x+1}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \Big[\ln\sqrt{2x+1}\Big]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

7. $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \frac{\ln(x+1)}{x+1}$.

f is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{1-\ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \left[\frac{[\ln(x+1)]^2}{2}\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

8. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

Let $f(x) = \frac{\ln x}{\sqrt{x}}$, $f'(x) = \frac{2-\ln x}{2x^{3/2}}$.

f is positive, continuous, and decreasing for $x > e^2 \approx 7.4$.

$$\int_2^{\infty} \frac{\ln x}{\sqrt{x}}\,dx = \Big[2\sqrt{x}(\ln x - 2)\Big]_2^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

9. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

Let $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$, $f'(x) = -\frac{1+2\sqrt{x}}{2x^{3/2}(\sqrt{x}+1)^2} < 0$.

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \left[2\ln(\sqrt{x}+1)\right]_1^{\infty} = \infty$, diverges

Hence, the series diverges by Theorem 9.10.

10. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

Let $f(x) = \frac{x}{x^2+3}$.

$f(x)$ is positive, continuous, and decreasing for $x \geq 2$ since

$f'(x) = \frac{3-x^2}{(x^2+3)} < 0$ for $x \geq 2$.

$\int_1^{\infty} \frac{x}{x^2+3}\,dx = \left[\ln\sqrt{x^2+3}\right]_1^{\infty} = \infty$

Diverges by Theorem 9.10

11. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

Let $f(x) = \frac{1}{\sqrt{x+1}}$, $f'(x) = \frac{-1}{2(x+1)^{3/2}} < 0$.

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{1}{\sqrt{x+1}}\,dx = \left[2\sqrt{x+1}\right]_1^{\infty} = \infty$, diverges

Hence, the series diverges by Theorem 9.10.

12. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \frac{\ln x}{x^3}$, $f'(x) = \frac{1-3\ln x}{x^4}$.

f is positive, continuous, and decreasing for $x > 2$.

$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \left[-\frac{(2\ln x+1)}{4x^4}\right]_2^{\infty}$

$= \frac{2\ln 2+1}{16}$, converges

Hence, the series converges by Theorem 9.10.

13. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Let $f(x) = \frac{\ln x}{x^2}$, $f'(x) = \frac{1-2\ln x}{x^3}$.

f is positive, continuous, and decreasing for $x > e^{1/2} \approx 1.6$.

$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \left[\frac{-(\ln x+1)}{x}\right]_1^{\infty} = 1$, converges

Hence, the series converges by Theorem 9.10.

14. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x) = \frac{1}{x\sqrt{\ln x}}$, $f'(x) = -\frac{2\ln x+1}{2x^2(\ln x)^{3/2}}$.

f is positive, continuous, and decreasing for $x \geq 2$.

$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \left[2\sqrt{\ln x}\right]_2^{\infty} = \infty$, diverges

Hence, the series diverges by Theorem 9.10.

15. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

Let $f(x) = \frac{\arctan x}{x^2+1}$, $f'(x) = \frac{1-2x\arctan x}{(x^2+1)^2} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx = \left[\frac{(\arctan x)^2}{2}\right]_1^{\infty} = \frac{3\pi^2}{32}$, converges

Hence, the series converges by Theorem 9.10.

16. $\sum_{n=3}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$

Let $f(x) = \frac{1}{x\ln x\ln(\ln x)}$,

$f'(x) = \frac{-[(\ln x+1)\ln(\ln x)+1]}{x^2(\ln x)^2(\ln(\ln x))^2}$.

f is positive, continuous, and decreasing for $x \geq 3$.

$\int_3^{\infty} \frac{1}{x\ln x\ln(\ln x)}\,dx = \left[\ln(\ln(\ln x))\right]_3^{\infty} = \infty$, diverges

Hence, the series diverges by Theorem 9.10.

17. $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$

Let $f(x) = \frac{2x}{x^2+1}$, $f'(x) = 2\frac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$.

f is positive, continuous, and decreasing for $x > 1$.

$$\int_1^{\infty} \frac{2x}{x^2+1}\,dx = \left[\ln(x^2+1)\right]_1^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

18. $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

Let $f(x) = \frac{x}{x^4+1}$, $f'(x) = \frac{1-3x^4}{(x^4+1)^2} < 0$ for $x > 1$.

f is positive, continuous, and decreasing for $x > 1$.

$$\int_1^{\infty} \frac{x}{x^4+1}\,dx = \left[\frac{1}{2}\arctan(x^2)\right]_1^{\infty} = \frac{\pi}{8}, \text{ converges}$$

Hence, the series converges by Theorem 9.10.

19. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$

Let $f(x) = \frac{x^{k-1}}{x^k+c}$.

f is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1)-x^k]}{(x^k+c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k+c}\,dx = \left[\frac{1}{k}\ln(x^k+c)\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

20. $\sum_{n=1}^{\infty} n^k e^{-n}$

Let $f(x) = \frac{x^k}{e^x}$.

f is positive, continuous, and decreasing for $x > k$ since

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0 \text{ for } x > k.$$

We use integration by parts.

$$\int_1^{\infty} x^k e^{-x}\,dx = \left[-x^k e^{-x}\right]_1^{\infty} + k\int_1^{\infty} x^{k-1}e^{-x}\,dx$$
$$= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e}$$

Converges by Theorem 9.10

21. Let $f(x) = \frac{(-1)^x}{x}$, $f(n) = a_n$.

The function f is not positive for $x \ge 1$.

22. Let $f(x) = e^{-x}\cos x$, $f(n) = a_n$.

The function f is not positive for $x \ge 1$.

23. Let $f(x) = \frac{2+\sin x}{x}$, $f(n) = a_n$.

The function f is not decreasing for $x \ge 1$.

24. Let $f(x) = \left(\frac{\sin x}{x}\right)^2$, $f(n) = a_n$.

The function f is not decreasing for $x \ge 1$.

25. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

f is positive, continuous, and decreasing for $x \ge 1$.

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 9.10

26. $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

Let $f(x) = \frac{1}{x^{1/3}}$.

f is positive, continuous, and decreasing for $x \ge 1$.

$$\int_1^{\infty} \frac{1}{x^{1/3}}\,dx = \left[\frac{3}{2}x^{2/3}\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

27. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Let $f(x) = \frac{1}{\sqrt{x}}$, $f'(x) = \frac{-1}{2x^{3/2}} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{\infty} = \infty$, diverges

Hence, the series diverges by Theorem 9.10.

28. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Let $f(x) = \frac{1}{x^2}$, $f'(x) = \frac{-2}{x^3} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$, converges

Hence, the series converges by Theorem 9.10.

29. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent p-series with $p = \frac{1}{5} < 1$

30. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Convergent p-series with $p = \frac{5}{3} > 1$

31. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent p-series with $p = \frac{1}{2} < 1$

32. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Convergent p-series with $p = 2 > 1$

33. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p-series with $p = \frac{3}{2} > 1$

34. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Divergent p-series with $p = \frac{2}{3} < 1$

35. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p-series with $p = 1.04 > 1$

36. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Convergent p-series with $p = \pi > 1$

37. $\sum_{n=1}^{\infty} \frac{2}{n^{3/4}}$

$S_1 = 2$

$S_2 \approx 3.1892$

$S_3 \approx 4.0666$

$S_4 \approx 4.7740$

Matches (c), diverges

38. $\sum_{n=1}^{\infty} \frac{2}{n}$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.6667$

$S_4 \approx 4.1667$

Matches (f), diverges

39. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^{\pi}}} = \sum_{n=1}^{\infty} \frac{2}{n^{\pi/2}}$

$S_1 = 2$

$S_2 \approx 2.6732$

$S_3 \approx 3.0293$

$S_4 \approx 3.2560$

Matches (b), converges

Note: The partial sums for 39 and 41 are very similar because $\pi/2 \approx 3/2$.

40. $\sum_{n=1}^{\infty} \frac{2}{n^{2/5}}$

$S_1 = 2$

$S_2 \approx 3.5157$

$S_3 \approx 4.8045$

Matches (a), diverges

41. $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$

$S_1 = 2$

$S_2 \approx 2.7071$

$S_3 \approx 3.0920$

$S_4 \approx 3.3420$

Matches (d), converges

Note: The partial sums for 39 and 41 are very similar because $\pi/2 \approx 3/2$.

42. $\sum_{n=1}^{\infty} \frac{2}{n^2}$

$S_1 = 2$

$S_2 = 2.5$

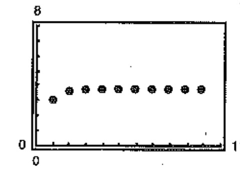
$S_3 \approx 2.7222$

Matches (e), converges

43. (a)

n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75

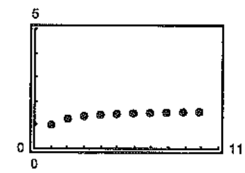
The partial sums approach the sum 3.75 very rapidly.



(b)

n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635

The partial sums approach the sum $\pi^2/6 \approx 1.6449$ slower than the series in part (a).



44. $\sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$

(a)

M	2	4	6	8
N	4	31	227	1674

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

45. Let f be positive, continuous, and decreasing for $x \ge 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge (Theorem 9.10). See Example 1, page 618.

46. A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p-series, $p > 0$.

The p-series converges if $p > 1$ and diverges if $0 < p \le 1$.

47. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

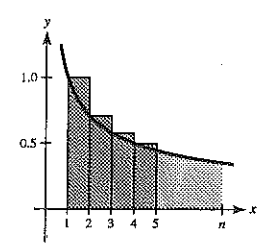
is the harmonic series, starting with the $10,000^{\text{th}}$ term, and hence diverges.

48. No. Theorem 9.9 says that if the series converges, then the terms a_n tend to zero. Some of the series in Exercises 37–42 converge because the terms tend to 0 very rapidly.

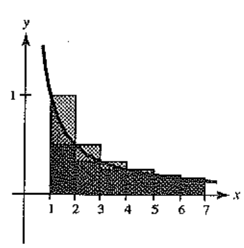
49. The area under the rectangles is greater than the area under the graph of $y = 1/\sqrt{x}, x \ge 1$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$$

Since $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{\infty} = \infty$ diverges, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.



50. $\sum_{n=1}^{6} a_n \ge \int_1^7 f(x)\,dx \ge \sum_{n=2}^{7} a_n$



51. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \ne 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x}\,dx = \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$

52. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{\ln x}{x^p}\,dx = \int_2^{\infty} x^{-p} \ln x\,dx = \left[\frac{x^{-p+1}}{(-p+1)^2}[-1 + (-p+1)\ln x]\right]_2^{\infty}.$$ (Use integration by parts.)

Converges for $-p + 1 < 0$ or $p > 1$

53. $\sum_{n=1}^{\infty} \frac{n}{(1 + n^2)^p}$

If $p = 1$, $\sum_{n=1}^{\infty} \frac{n}{1 + n^2}$ diverges (see Example 1). Let

$$f(x) = \frac{x}{(1 + x^2)^p},\ p \neq 1$$

$$f'(x) = \frac{1 - (2p - 1)x^2}{(1 + x^2)^{p+1}}.$$

For a fixed $p > 0$, $p \neq 1$, $f'(x)$ is eventually negative. f is positive, continuous, and eventually decreasing.

$$\int_1^{\infty} \frac{x}{(1 + x^2)^p}\,dx = \left[\frac{1}{(x^2 + 1)^{p-1}(2 - 2p)}\right]_1^{\infty}$$

For $p > 1$, this integral converges. For $0 < p < 1$, it diverges.

54. $\sum_{n=1}^{\infty} n(1 + n^2)^p$

Since $p > 0$, the series diverges for all values of p.

55. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

Hence, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$, diverges.

56. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

Hence, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{(\ln n)^2}} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}}$, diverges.

57. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

Hence, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, converges.

58. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

Hence, $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)} = \sum_{n=2}^{\infty} \frac{1}{2n \ln n}$

$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$, diverges

59. Since f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

Also,

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x)\,dx$$

$$\leq \int_N^{\infty} f(x)\,dx.$$

Thus, $0 \leq R_N \leq \int_N^{\infty} f(x)\,dx$.

60. From Exercise 59, we have:

$$0 \le S - S_N \le \int_N^\infty f(x)\,dx$$

$$S_N \le S \le S_N + \int_N^\infty f(x)\,dx$$

$$\sum_{n=1}^N a_n \le S \le \sum_{n=1}^N a_n + \int_N^\infty f(x)\,dx$$

61. $S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$

$$R_6 \le \int_6^\infty \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_6^\infty \approx 0.0015$$

$$1.0811 \le \sum_{n=1}^\infty \frac{1}{n^4} \le 1.0811 + 0.0015 = 1.0826$$

62. $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \le \int_4^\infty \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_4^\infty \approx 0.0010$$

$$1.0363 \le \sum_{n=1}^\infty \frac{1}{n^5} \le 1.0363 + 0.0010 = 1.0373$$

63. $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$$R_{10} \le \int_{10}^\infty \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_{10}^\infty = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \le \sum_{n=1}^\infty \frac{1}{n^2+1} \le 0.9818 + 0.0997 = 1.0815$$

64. $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \le \int_{10}^\infty \frac{1}{(x+1)[\ln(x+1)]^3}\,dx = \left[-\frac{1}{2[\ln(x+1)]^2}\right]_{10}^\infty = \frac{1}{2(\ln 11)^3} \approx 0.0870$$

$$1.9821 \le \sum_{n=1}^\infty \frac{1}{(n+1)[\ln(n+1)]^3} \le 1.9821 + 0.0870 = 2.0691$$

65. $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$$R_4 \le \int_4^\infty xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_4^\infty = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$$

$$0.4049 \le \sum_{n=1}^\infty ne^{-n^2} \le 0.4049 + 5.6 \times 10^{-8}$$

66. $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \le \int_4^\infty e^{-x}\,dx = \left[-e^{-x}\right]_4^\infty \approx 0.0183$$

$$0.5713 \le \sum_{n=0}^\infty e^{-n} \le 0.5713 + 0.0183 = 0.5896$$

67. $0 \le R_N \le \int_N^\infty \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_N^\infty = \frac{1}{3N^3} < 0.001$

$\frac{1}{N^3} < 0.003$

$N^3 > 333.33$

$N > 6.93$

$N \ge 7$

68. $0 \le R_N \le \int_N^\infty \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{x^{1/2}}\right]_N^\infty = \frac{2}{\sqrt{N}} < 0.001$

$N^{-1/2} < 0.0005$

$\sqrt{N} > 2000$

$N \ge 4{,}000{,}000$

69. $R_N \le \int_N^{\infty} e^{-5x}\,dx = \left[-\frac{1}{5}e^{-5x}\right]_N^{\infty} = \frac{e^{-5N}}{5} < 0.001$

$\frac{1}{e^{5N}} < 0.005$

$e^{5N} > 200$

$5N > \ln 200$

$N > \frac{\ln 200}{5}$

$N > 1.0597$

$N \ge 2$

70. $R_N \le \int_N^{\infty} e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_N^{\infty} = \frac{2}{e^{N/2}} < 0.001$

$\frac{2}{e^{N/2}} < 0.001$

$e^{N/2} > 2000$

$\frac{N}{2} > \ln 2000$

$N > 2\ln 2000 \approx 15.2$

$N \ge 16$

71. $R_N \le \int_N^{\infty} \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_N^{\infty}$

$= \frac{\pi}{2} - \arctan N < 0.001$

$-\arctan N < -1.5698$

$\arctan N > 1.5698$

$N > \tan 1.5698$

$N \ge 1004$

72. $R_n \le \int_N^{\infty} \frac{2}{x^2+5}\,dx = 2\left[\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right]_N^{\infty}$

$= \frac{2}{\sqrt{5}}\left(\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right)\right) < 0.001$

$\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) < 0.001118$

$1.56968 < \arctan\left(\frac{N}{\sqrt{5}}\right)$

$\frac{N}{\sqrt{5}} > \tan 1.56968$

$N \ge 2004$

73. (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$. This is a convergent p-series with $p = 1.1 > 1$. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a divergent series. Use the Integral Test.

$f(x) = \frac{1}{x\ln x}$ is positive, continuous, and decreasing for $x \ge 2$.

$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \left[\ln|\ln x|\right]_2^{\infty} = \infty$

(b) $\sum_{n=2}^{6} \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$

$\sum_{n=2}^{6} \frac{1}{n\ln n} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \frac{1}{5\ln 5} + \frac{1}{6\ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$

For $n \ge 4$, the terms of the convergent series seem to be larger than those of the divergent series!

(c) $\frac{1}{n^{1.1}} < \frac{1}{n\ln n}$

$n\ln n < n^{1.1}$

$\ln n < n^{0.1}$

This inequality holds when $n \ge 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$.

74. (a) $\int_{10}^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_{10}^{\infty} = \frac{1}{(p-1)10^{p-1}},\ p > 1$

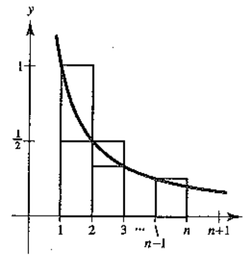
(b) $f(x) = \frac{1}{x^p}$

$R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p}$

$\le$ Area under the graph of f over the interval $[10, \infty)$

(c) The horizontal asymptote is $y = 0$. As n increases, the error decreases.

75. (a) Let $f(x) = 1/x$. f is positive, continuous, and decreasing on $[1, \infty)$.



$$S_n - 1 \le \int_1^n \frac{1}{x}\,dx$$

$$S_n - 1 \le \ln n$$

Hence, $S_n \le 1 + \ln n$. Similarly,

$$S_n \ge \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1).$$

Thus, $\ln(n+1) \le S_n \le 1 + \ln n$.

(b) Since $\ln(n+1) \le S_n \le 1 + \ln n$, we have $\ln(n+1) - \ln n \le S_n - \ln n \le 1$. Also, since $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \ge 1$. Thus, $0 \le S_n - \ln n \le 1$ and the sequence $\{a_n\}$ is bounded.

(c) $a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \displaystyle\int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \ge 0$

Thus, $a_n \ge a_{n+1}$ and the sequence is decreasing.

(d) Since the sequence is bounded and monotonic, it converges to a limit, γ.

(e) $a_{100} = S_{100} - \ln 100 \approx 0.5822$ (Actually $\gamma \approx 0.577216$.)

76. $$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^{\infty} [\ln(n+1) + \ln(n-1) - 2\ln n]$$

$$= \cancel{\ln 3} + \cancel{\ln 1} - 2\ln 2) + (\cancel{\ln 4} + \ln 2 - \cancel{2\ln 3}) + (\cancel{\ln 5} + \cancel{\ln 3} - \cancel{2\ln 4}) + (\cancel{\ln 6} + \cancel{\ln 4} - \cancel{2\ln 5})$$

$$+ (\cancel{\ln 7} + \cancel{\ln 5} - \cancel{2\ln 6}) + (\cancel{\ln 8} + \cancel{\ln 6} - \cancel{2\ln 7}) + (\cancel{\ln 9} + \cancel{\ln 7} - \cancel{2\ln 8}) + \cdots = -\ln 2$$

77. $\displaystyle\sum_{n=2}^{\infty} x^{\ln n}$

(a) $x = 1$: $\displaystyle\sum_{n=2}^{\infty} 1^{\ln n} = \sum_{n=2}^{\infty} 1$, diverges

(b) $x = \dfrac{1}{e}$: $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{e}\right)^{\ln n} = \sum_{n=2}^{\infty} e^{-\ln n} = \sum_{n=2}^{\infty} \frac{1}{n}$, diverges

(c) Let x be given, $x > 0$. Put $x = e^{-p} \Leftrightarrow \ln x = -p$.

$$\sum_{n=2}^{\infty} x^{\ln n} = \sum_{n=2}^{\infty} e^{-p\ln n} = \sum_{n=2}^{\infty} n^{-p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$$

This series converges for $p > 1 \Rightarrow x < \dfrac{1}{e}$.

78. $\xi(x) = \displaystyle\sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

Converges for $x > 1$ by Theorem 9.11

79. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Let $f(x) = \dfrac{1}{2x-1}$.

f is positive, continuous, and decreasing for $x \ge 1$.

$$\int_1^{\infty} \frac{1}{2x-1}\,dx = \Big[\ln\sqrt{2x-1}\Big]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

80. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Let $f(x) = \dfrac{1}{x\sqrt{x^2-1}}$.

f is positive, continuous, and decreasing for $x \ge 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \Big[\operatorname{arcsec} x\Big]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 9.10

81. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

p-series with $p = \frac{5}{4}$

Converges by Theorem 9.11

82. $3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

p-series with $p = 0.95$

Diverges by Theorem 9.11

83. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 9.6

84. $\sum_{n=0}^{\infty} (1.075)^n$

Geometric series with $r = 1.075$

Diverges by Theorem 9.6

85. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$

Diverges by Theorem 9.9

86. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$

Since these are both convergent p-series, the difference is convergent.

87. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$

Fails nth-Term Test

Diverges by Theorem 9.9

88. $\sum_{n=2}^{\infty} \ln(n)$

$\lim_{n\to\infty} \ln(n) = \infty$

Diverges by Theorem 9.9

89. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Let $f(x) = \frac{1}{x(\ln x)^3}$.

f is positive, continuous and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_2^{\infty} (\ln x)^{-3}\frac{1}{x}\,dx = \left[\frac{(\ln x)^{-2}}{-2}\right]_2^{\infty} = \left[-\frac{1}{2(\ln x)^2}\right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

Converges by Theorem 9.10. See Exercise 51.

90. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \frac{\ln x}{x^3}$.

f is positive, continuous, and decreasing for $x \geq 2$ since $f'(x) = \frac{1 - 3\ln x}{x^4} < 0$ for $x \geq 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_2^{\infty} + \frac{1}{2}\int_2^{\infty} \frac{1}{x^3}\,dx = \frac{\ln 2}{8} + \left[-\frac{1}{4x^2}\right]_2^{\infty} = \frac{\ln 2}{8} + \frac{1}{16} \text{ (Use integration by parts.)}$$

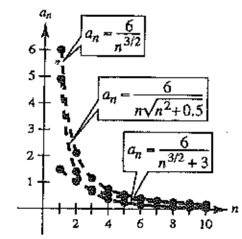
Converges by Theorem 9.10. See Exercise 30.

Section 9.4 Comparisons of Series

1. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots; \; S_1 = 6$

$\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \cdots; \; S_1 = \frac{3}{2}$

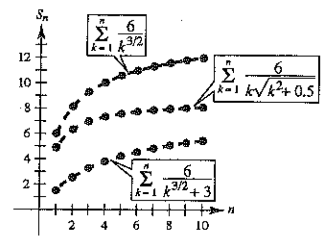
$\displaystyle\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots; \; S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$



(b) The first series is a p-series. It converges $\left(p = \frac{3}{2} > 1\right)$.

(c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent p-series. Hence, the other two series converge.

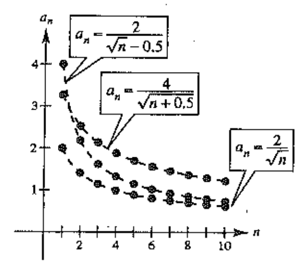
(d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



2. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots \; S_1 = 2$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots \; S_1 = 4$

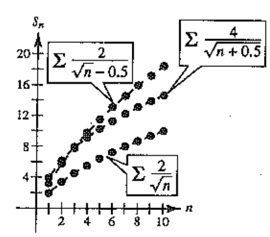
$\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt{n} + 0.5} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots \; S_1 \approx 3.3$



(b) The first series is a p-series. It diverges $\left(p = \frac{1}{2} < 1\right)$.

(c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent p-series. Hence, the other two series diverge.

(d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



3. $\displaystyle 0 < \frac{1}{n^2 + 1} < \frac{1}{n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

4. $\displaystyle \frac{1}{3n^2 + 2} < \frac{1}{3n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent p-series

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. $\displaystyle \frac{1}{n - 1} > \frac{1}{n} > 0$ for $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n - 1}$$

diverges by comparison with the divergent p-series

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

6. $\dfrac{1}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}}$ for $n \ge 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$$

diverges by comparison with the divergent p-series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

7. $0 < \dfrac{1}{3^n+1} < \dfrac{1}{3^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n+1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

8. $\dfrac{3^n}{4^n+5} < \left(\dfrac{3}{4}\right)^n$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n+5}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$

9. For $n \ge 3$, $\dfrac{\ln n}{n+1} > \dfrac{1}{n+1} > 0$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by the Integral Test.

10. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{n^{3/2}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

11. For $n > 3$, $\dfrac{1}{n^2} > \dfrac{1}{n!} > 0$.

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

12. $\dfrac{1}{4\sqrt[3]{n}-1} > \dfrac{1}{4\sqrt[4]{n}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$$

diverges by comparison with the divergent p-series

$$\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

13. $0 < \dfrac{1}{e^{n^2}} \le \dfrac{1}{e^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

14. $\dfrac{4^n}{3^n-1} > \dfrac{4^n}{3^n}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n.$$

15. $\displaystyle\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

16. $\displaystyle\lim_{n\to\infty} \frac{2/(3^n-5)}{1/3^n} = \lim_{n\to\infty} \frac{2 \cdot 3^n}{3^n-5} = 2$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{3^n-5}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

17. $\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

18. $\lim_{n\to\infty} \frac{3/\sqrt{n^2-4}}{1/n} = \lim_{n\to\infty} \frac{3n}{\sqrt{n^2-4}} = 3$

Therefore,

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=3}^{\infty} \frac{1}{n}.$$

19. $\lim_{n\to\infty} \frac{\frac{2n^2-1}{3n^5+2n+1}}{1/n^3} = \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1} = \frac{2}{3}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$$

converges by a limit comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

20. $\lim_{n\to\infty} \frac{\frac{5n-3}{n^2-2n+5}}{1/n} = \lim_{n\to\infty} \frac{5n^2-3n}{n^2-2n+5} = 5$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

21. $\lim_{n\to\infty} \frac{\frac{n+3}{n(n+2)}}{1/n} = \lim_{n\to\infty} \frac{n^2+3n}{n^2+2n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n+3}{n(n+2)}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

22. $\lim_{n\to\infty} \frac{\frac{1}{n(n^2+1)}}{1/n^3} = \lim_{n\to\infty} \frac{n^3}{n^3+n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2+1)}$$

converges by a limit comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

23. $\lim_{n\to\infty} \frac{1/\left(n\sqrt{n^2+1}\right)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

converges by a limit comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

24. $\lim_{n\to\infty} \frac{n/[(n+1)2^{n-1}]}{1/(2^{n-1})} = \lim_{n\to\infty} \frac{n}{n+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

25. $\lim_{n\to\infty} \frac{(n^{k-1})/(n^k+1)}{1/n} = \lim_{n\to\infty} \frac{n^k}{n^k+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

26. $\lim_{n\to\infty} \frac{5/\left(n+\sqrt{n^2+4}\right)}{1/n} = \lim_{n\to\infty} \frac{5n}{n+\sqrt{n^2+4}} = \frac{5}{2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

27. $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\cos(1/n)}{-1/n^2}$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

28. $\lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\sec^2(1/n)}{-1/n^2}$

$$= \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

29. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Diverges

p-series with $p = \frac{1}{2}$

30. $\sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$

Converges

Geometric series with $r = -\frac{1}{5}$

31. $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

Converges

Direct comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

32. $\sum_{n=4}^{\infty} \frac{1}{3n^2-2n-15}$

Converges

Limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

33. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

Diverges; nth-Term Test

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

34. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots = \frac{1}{2}$

Converges; telescoping series

35. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Converges; Integral Test

36. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

37. $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} na_n$. By given conditions $\lim_{n\to\infty} na_n$ is finite and nonzero. Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

38. If $j < k - 1$, then $k - j > 1$. The p-series with $p = k - j$ converges and since

$$\lim_{n\to\infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0, \text{ the series } \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges by the Limit Comparison Test. Similarly, if $j \geq k - 1$, then $k - j \leq 1$ which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the Limit Comparison Test.

39. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$,

which diverges since the degree of the numerator is only one less than the degree of the denominator.

40. $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$,

which converges since the degree of the numerator is two less than the degree of the denominator.

41. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

converges since the degree of the numerator is three less than the degree of the denominator.

42. $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

diverges since the degree of the numerator is only one less than the degree of the denominator.

43. $\lim_{n\to\infty} n\left(\frac{n^3}{5n^4 + 3}\right) = \lim_{n\to\infty} \frac{n^4}{5n^4 + 3} = \frac{1}{5} \neq 0$

Therefore, $\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$ diverges.

44. $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \neq 0$

Therefore, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

45. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$

diverges, (harmonic)

46. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n}$

diverges

47. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200 + n^2}$

converges

48. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3}$

converges

49. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

50. See Theorem 9.12, page 624. One example is

$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges because $\frac{1}{n^2 + 1} < \frac{1}{n^2}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series).

51. See Theorem 9.13, page 626. One example is

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n - 1}}$ diverges because $\lim_{n\to\infty} \frac{1/\sqrt{n - 1}}{1/\sqrt{n}} = 1$ and

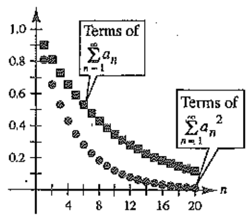
$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series).

52. This is not correct. The beginning terms do not affect the convergence or divergence of a series. In fact,

$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n}$ diverges (harmonic)

and $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series).

53.



For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$. Hence, the lower terms are those of $\sum a_n^2$.

54. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges since the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(b)

n	5	10	20	50	100
S_n	1.1839	1.02087	1.2212	1.2287	1.2312

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

55. False. Let $a_n = \dfrac{1}{n^3}$ and $b_n = \dfrac{1}{n^2}$. $0 < a_n \le b_n$ and both $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

56. True

57. True

58. False. Let $a_n = 1/n$, $b_n = 1/n$, $c_n = 1/n^2$. Then,

$a_n \le b_n + c_n$, but $\displaystyle\sum_{n=1}^{\infty} c_n$ converges.

59. True

60. False. $\displaystyle\sum_{n=1}^{\infty} a_n$ could converge or diverge.

For example, let $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges.

$0 < \dfrac{1}{n} < \dfrac{1}{\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but

$0 < \dfrac{1}{n^2} < \dfrac{1}{\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

61. Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, $\displaystyle\lim_{n\to\infty} b_n = 0$. There exists N such that $b_n < 1$ for $n > N$. Thus, $a_n b_n < a_n$ for $n > N$ and

$\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges by comparison to the convergent

series $\displaystyle\sum_{i=1}^{\infty} a_n$.

62. Since $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$

converges by Exercise 61.

63. $\sum \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^3}$ both converge, and hence so does

$\sum \left(\dfrac{1}{n^2}\right)\left(\dfrac{1}{n^3}\right) = \sum \dfrac{1}{n^5}$.

64. $\sum \dfrac{1}{n^2}$ converge, and hence so does $\sum \left(\dfrac{1}{n^2}\right)^2 = \sum \dfrac{1}{n^4}$.

65. Suppose $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges.

From the definition of limit of a sequence, there exists $M > 0$ such that

$$\left|\frac{a_n}{b_n} - 0\right| < 1$$

whenever $n > M$. Hence, $a_n < b_n$ for $n > M$. From the Comparison Test, $\sum a_n$ converges.

66. Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\Sigma\, b_n$ diverges. From the definition of limit of a sequence, there exists $M > 0$ such that

$$\frac{a_n}{b_n} > 1$$

for $n > M$. Thus, $a_n > b_n$ for $n > M$. By the Comparison Test, $\Sigma\, a_n$ diverges.

67. (a) Let $\sum a_n = \sum \frac{1}{(n+1)^3}$, and $\sum b_n = \sum \frac{1}{n^2}$, converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/[(n+1)^3]}{1/(n^2)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^3} = 0$$

By Exercise 65, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ converges.

(b) Let $\sum a_n = \sum \frac{1}{\sqrt{n}\pi^n}$, and $\sum b_n = \sum \frac{1}{\pi^n}$, converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(\sqrt{n}\pi^n)}{1/(\pi^n)} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

By Exercise 65, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\pi^n}$ converges.

68. (a) Let $\sum a_n = \sum \frac{\ln n}{n}$, and $\sum b_n = \sum \frac{1}{n}$, diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)/n}{1/n} = \lim_{n\to\infty} \ln n = \infty$$

By Exercise 66, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

(b) Let $\sum a_n = \sum \frac{1}{\ln n}$, and $\sum b_n = \sum \frac{1}{n}$, diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty$$

By Exercise 66, $\sum \frac{1}{\ln n}$ diverges.

69. Since $\lim_{n\to\infty} a_n = 0$, the terms of $\Sigma \sin(a_n)$ are positive for sufficiently large n. Since

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1 \text{ and } \sum a_n$$

converges, so does $\Sigma \sin(a_n)$.

70. $$\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n} = \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]/2} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

Since $\Sigma\, 1/n^2$ converges, and

$$\lim_{n\to\infty} \frac{2/[n(n+1)]}{1/(n^2)} = \lim_{n\to\infty} \frac{2n^2}{n(n+1)} = 2,$$

$\sum \frac{1}{1+2+\cdots+n}$ converges.

71. The series diverges. For $n > 1$,

$$n < 2^n$$
$$n^{1/n} < 2$$
$$\frac{1}{n^{1/n}} > \frac{1}{2}$$
$$\frac{1}{n^{(n+1)/n}} > \frac{1}{2n}$$

Since $\sum \frac{1}{2n}$ diverges, so does $\sum \frac{1}{n^{(n+1)/n}}$.

72. Consider two cases:

If $a_n \geq \frac{1}{2^{n+1}}$, then $a_n^{1/(n+1)} \geq \left(\frac{1}{2^{n+1}}\right)^{1/(n+1)} = \frac{1}{2}$, and

$$a_n^{n/(n+1)} = \frac{a_n}{a_n^{1/(n+1)}} \leq 2a_n.$$

If $a_n \leq \frac{1}{2^{n+1}}$, then $a_n^{n/(n+1)} \leq \left(\frac{1}{2^{n+1}}\right)^{n/(n+1)} = \frac{1}{2^n}$, and

combining, $a_n^{n/(n+1)} \leq 2a_n + \frac{1}{2^n}$.

Since $\sum_{n=1}^{\infty}\left(2a_n + \frac{1}{2^n}\right)$ converges, so does $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$ by the Comparison Test.

Section 9.5 Alternating Series

1. $\sum_{n=1}^{\infty} \frac{6}{n^2}$

$S_1 = 6$

$S_2 = 7.5$

$S_3 \approx 8.1667$

Matches (d).

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2}$

$S_1 = 6$

$S_2 = 4.4$

$S_3 \approx 5.1667$

Matches (f).

3. $\sum_{n=1}^{\infty} \frac{3}{n!}$

$S_1 = 3$

$S_2 = 4.5$

$S_3 = 5.0$

Matches (a).

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3}{n!}$

$S_1 = 3$

$S_2 = 1.5$

$S_3 = 2.0$

Matches (b).

5. $\sum_{n=1}^{\infty} \frac{10}{n2^n}$

$S_1 = 5$

$S_2 = 6.25$

Matches (e).

6. $\sum_{n=1}^{\infty} \frac{(-1)^n 10}{n2^n}$

$S_1 = 5$

$S_2 = 3.75$

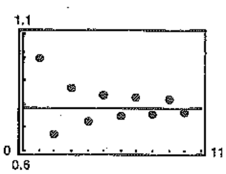
Matches (c).

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

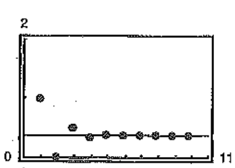
(d) The distance in part (c) is always less than the magnitude of the next term of the series.

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0	0.5	0.3333	0.375	0.3667	0.3681	0.3679	0.3679	0.3679

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

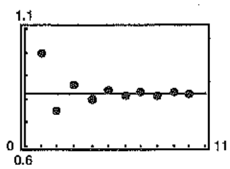
(d) The distance in part (c) is always less than the magnitude of the next series.

9. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

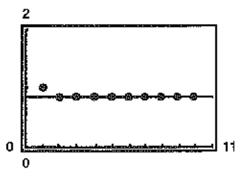
(d) The distance in part (c) is always less than the magnitude of the next term in the series.

10. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.8333	0.8417	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415

(b)



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Converges by Theorem 9.14

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n-1}$

$\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

Diverges by the nth-Term Test

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$

$\lim_{n\to\infty} \frac{1}{2n-1} = 0$

Converges by Theorem 9.14

14. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$

$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

Converges by Theorem 9.14

15. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. Thus, $\lim_{n\to\infty} a_n \neq 0$.

Diverges by the nth-Term Test

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$

$a_{n+1} = \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} = a_n$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

Converges by Theorem 9.14

17. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Converges by Theorem 9.14

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$

$\lim_{n\to\infty} \frac{n^2}{n^2+5} = 1$

Diverges by nth-Term Test

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty$

Diverges by the nth-Term Test

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$

$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1}$ for $n \geq 2$

$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{1/(n+1)}{1} = 0$

Converges by Theorem 9.14

21. $\sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$

Diverges by the nth-Term Test

22. $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Converges; (See Exercise 11.)

23. $\sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$

Diverges by the nth-Term Test

24. $\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Converges; (See Exercise 11.)

25. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$

$\lim_{n\to\infty} \frac{1}{n!} = 0$

Converges by Theorem 9.14

26. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$

$\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$

Converges by Theorem 9.14

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2}$

$< \frac{\sqrt{n}}{n+2}$ for $n \geq 2$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$

Converges by Theorem 9.14

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

$\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$

Diverges by the nth-Term Test

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$$a_{n+1} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n+1}{2n+1} = a_n\left(\frac{n+1}{2n+1}\right) < a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$= \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$= \lim_{n\to\infty} 2\left[\frac{3}{3} \cdot \frac{4}{5} \cdot \frac{5}{7} \cdots \frac{n}{2n-3}\right] \cdot \frac{1}{2n-1} = 0$$

Converges by Theorem 9.14

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$

$$a_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)} = a_n\left(\frac{2n+1}{3n+1}\right) < a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3\left[\frac{5}{4} \cdot \frac{7}{7} \cdot \frac{9}{10} \cdots \frac{2n-1}{3n-5}\right] \frac{1}{3n-2} = 0$$

Converges by Theorem 9.14

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$

Let $f(x) = \dfrac{2e^x}{e^{2x} - 1}$. Then

$$f'(x) = \frac{-2e^x(e^{2x}+1)}{(e^{2x}-1)^2} < 0.$$

Thus, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n}-1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 9.14.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} + 1}$

Let $f(x) = \dfrac{2e^x}{e^{2x} + 1}$. Then

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x}+1)^2} < 0 \text{ for } x > 0.$$

Thus, $f(x)$ is decreasing for $x > 0$ which implies $a_{n+1} < a_n$.

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n}+1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 9.14.

33. $S_6 = \displaystyle\sum_{n=1}^{6} \frac{3(-1)^{n+1}}{n^2} = 2.4325$

$2.4325 - 0.0612 \le S \le 2.4325 + 0.0612$

$|R_6| = |S - S_6| \le a_7 = \dfrac{3}{49} \approx 0.0612$

$2.3713 \le S \le 2.4937$

34. $S_6 = \displaystyle\sum_{n=1}^{6} \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$

$|R_6| = |S - S_6| \le a_7 = \dfrac{4}{\ln 8} \approx 1.9236$

$0.7831 \le S \le 4.6303$

35. $S_6 = \displaystyle\sum_{n=0}^{5} \frac{2(-1)^n}{n!} \approx 0.7333$

$0.7333 - 0.002778 \le S \le 0.7333 + 0.002778$

$|R_6| = |S - S_6| \le a_7 = \dfrac{2}{6!} = 0.002778$

$0.7305 \le S \le 0.7361$

36. $S_6 = \displaystyle\sum_{n=1}^{6} \frac{(-1)^{n+1}n}{2^n} = 0.1875$

$|R_6| = |S - S_6| \le a_7 = \dfrac{7}{2^7} \approx 0.05469$

$0.1328 \le S \le 0.2422$

37. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(a) By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when $N = 6$.

(b) We may approximate the series by

$$\sum_{n=0}^{6} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$
$$\approx 0.368.$$

(7 terms. Note that the sum begins with $n = 0$.)

38. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$

(a) By Theorem 9.15,

$$|R_n| \le a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when $N = 4$.

(b) We may approximate the series by

$$\sum_{n=0}^{4} \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with $n = 0$.)

39. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(a) By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when $N = 2$.

(b) We may approximate the series by

$$\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with $n = 0$.)

40. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

(a) By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=0}^{3} \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with $n = 0$.)

41. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

(a) By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when $N = 1000$.

(b) We may approximate the series by

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000}$$
$$\approx 0.693.$$

(1000 terms)

42. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$

(a) By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=1}^{3} \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^3} < 0.001$$
$$\Rightarrow (N+1)^3 > 1000 \Rightarrow N + 1 > 10.$$

Use 10 terms.

44. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^2} < 0.001$$
$$\Rightarrow (N+1)^2 > 1000 \Rightarrow N = 31.$$

Use 31 terms.

45. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when $N = 7$. Use 7 terms.

46. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

By Theorem 9.15, $|R_N| \le a_{N+1} = \frac{1}{(N+1)^4} < 0.001.$

This inequality is valid when $N = 5$.

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by comparison to the p-series

$\sum_{n=1}^{\infty} \frac{1}{n^2}.$

Therefore, the given series converges absolutely.

48. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$\sum_{n=1}^{\infty} \frac{1}{n+1}$

diverges by the Integral Test. Therefore, the series converges conditionally.

49. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

is a divergent p-series. Therefore, the series converges conditionally.

50. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series.

Therefore, the given series converges absolutely.

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the nth-Term Test.

52. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

$$\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$$

Therefore, the series diverges by the nth-Term Test.

53. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

The given series converges by the Alternating Series Test but does not converge absolutely since the series

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison to the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}.$ Therefore, the series converges conditionally.

54. $\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$ converges by a comparison to

the convergent geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$. Therefore, the given series converges absolutely.

55. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 1}$

$\sum_{n=2}^{\infty} \frac{n}{n^3 - 1}$ converges by a limit comparison to

the convergent p-series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Therefore, the given series converges absolutely.

56. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series.

Therefore, the given series converges absolutely.

57. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

58. $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

59. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$\lim_{n\to\infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1$, therefore the series converges conditionally.

60. $\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

Therefore, the series diverges by the nth-Term Test.

61. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

Therefore, the given series converges absolutely.

62. $\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left|\frac{\sin[(2n-1)\pi/2]}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent p-series. Therefore, the series converges conditionally.

63. An alternating series is a series whose terms alternate in sign. See Theorem 9.14.

64. $|S - S_n| = |R_n| \le a_{n+1}$ (Theorem 9.15)

65. Σa_n is absolutely convergent if $\Sigma|a_n|$ converges. Σa_n is conditionally convergent if $\Sigma|a_n|$ diverges, but Σa_n converges.

66. (b). The partial sums alternate above and below the horizontal line representing the sum.

67. False. Let $a_n = \frac{(-1)^n}{n}$.

Then $\sum a_n$ converges and $\sum (-a_n)$ converges.

But, $\sum |a_n| = \sum \frac{1}{n}$ diverges.

68. True. If $\Sigma\, |a_n|$ converged, then so would $\Sigma\, a_n$ by Theorem 9.16.

69. True. $S_{100} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{100}$

Since the next term $-\frac{1}{101}$ is negative, S_{100} is an overestimate of the sum.

70. False. Let

$$\sum a_n = \sum b_n = \sum \frac{(-1)^n}{\sqrt{n}}.$$

Then both converge by the Alternating Series Test. But,

$$\sum a_n b_n = \sum \frac{1}{n}, \text{ which diverges.}$$

71. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

If $p = 0$, then $\sum_{n=1}^{\infty} (-1)^n$ diverges.

If $p < 0$, then $\sum_{n=1}^{\infty} (-1)^n n^{-p}$ diverges.

If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$ and

$$a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n.$$

Therefore, the series converges for $p > 0$.

72. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+p}$

Assume that $n + p \neq 0$ so that $a_n = 1/(n+p)$ are defined for all n. For all p,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+p} = 0$$

$$a_{n+1} = \frac{1}{n+1+p} < \frac{1}{n+p} < a_n.$$

Therefore, the series converges for all p.

73. Since

$$\sum_{n=1}^{\infty} |a_n|$$

converges we have $\lim_{n\to\infty} |a_n| = 0$. Thus, there must exist an $N > 0$ such that $|a_N| < 1$ for all $n > N$ and it follows that $a_n^2 \leq |a_n|$ for all $n > N$. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let $a_n = 1/n$ to see that the converse is false.

74. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

75. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

76. (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for $-1 < x < 1$, since

$$\left|\frac{x^n}{n}\right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

(b) When $x = -1$, we have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, we have the divergent harmonic series $1/n$. Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges conditionally for } x = -1.$$

77. (a) No, the series does not satisfy $a_{n+1} \leq a_n$ for all n. For example, $\frac{1}{9} < \frac{1}{8}$.

(b) Yes, the series converges.

$$S_{2n} = \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2^n} - \frac{1}{3^n}$$

$$= \left(\frac{1}{2} + \cdots + \frac{1}{2^n}\right) - \left(\frac{1}{3} + \cdots + \frac{1}{3^n}\right)$$

$$= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\right) - \left(1 + \frac{1}{3} + \cdots + \frac{1}{3^n}\right)$$

As $n \to \infty$,

$$S_{2n} \to \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}.$$

78. (a) No, the series does not satisfy $a_{n+1} \le a_n$:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = 1 - \frac{1}{8} + \frac{1}{\sqrt{3}} - \frac{1}{64} + \cdots \text{ and}$$

$$\frac{1}{8} < \frac{1}{\sqrt{3}}.$$

(b) No, the series diverges because $\sum \frac{1}{\sqrt{n}}$ diverges.

79. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$,

convergent p-series

80. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

converges by limit comparison to convergent p-series

$\displaystyle\sum \frac{1}{n^2}.$

81. Diverges by nth-Term Test

$\displaystyle\lim_{n\to\infty} a_n = \infty$

82. Converges by limit comparison to convergent geometric series $\sum \frac{1}{2^n}$.

83. Convergent geometric series

$\left(r = \frac{7}{8} < 1\right)$

84. Diverges by nth-Term Test

$\displaystyle\lim_{n\to\infty} a_n = \frac{3}{2}$

85. Convergent geometric series $\left(r = 1/\sqrt{e}\right)$ or Integral Test

86. Converges (conditionally) by Alternating Series Test

87. Converges (absolutely) by Alternating Series Test

88. Diverges by comparison to Divergent Harmonic Series:

$\displaystyle\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

89. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

90. Rearranging the terms of an alternating series can change the sum.

91. $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

(i) $s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}$

$\frac{1}{2}s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots + \frac{1}{4n-2} - \frac{1}{4n}$

Adding: $s_{4n} + \frac{1}{2}s_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = S_{3n}$

(ii) $\displaystyle\lim_{n\to\infty} s_n = s$ (In fact, $s = \ln 2$.)

$s \ne 0$ since $s > \frac{1}{2}$.

$$S = \lim_{n\to\infty} S_{3n} = s_{4n} + \frac{1}{2}s_{2n} = s + \frac{1}{2}s = \frac{3}{2}s$$

Thus, $S \ne s$.

Section 9.6 The Ratio and Root Tests

1. $\dfrac{(n+1)!}{(n-2)!} = \dfrac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$

$= (n+1)(n)(n-1)$

2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{(2k-2)!}{(2k)(2k-1)(2k-2)!}$

$= \dfrac{1}{(2k)(2k-1)}$

3. Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid since $1 = \dfrac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}.$$

To do this, note that:

$$\begin{aligned}
1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) &= [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1) \\
&= \frac{(2n)!}{2^n n!} \cdot (2n+1) \quad \text{(Induction hypothesis)} \\
&= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)} \\
&= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n!(n+1)} \\
&= \frac{(2n+2)!}{2^{n+1}(n+1)!}
\end{aligned}$$

The formula is valid for all $n \ge 1$.

4. Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid since $\dfrac{1}{1} = \dfrac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n!(2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}.$$

To do this, note that:

$$\begin{aligned}
\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)} \\
&= \frac{2^n n!\cancel{(2n-3)}(2n-1)}{(2n)!} \cdot \frac{1}{\cancel{(2n-3)}} \\
&= \frac{2^n n!(2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)} \\
&= \frac{2^n (2)(n+1)n!(2n-1)(2n+1)}{(2n)!(2n+1)(2n+2)} \\
&= \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}
\end{aligned}$$

The formula is valid for all $n \ge 3$.

5. $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.875$

Matches (d).

6. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.03$

Matches (c).

7. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots$

$S_1 = 9$

Matches (f).

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$

$S_1 = 2$

Matches (b).

9. $\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots$

$S_1 = 2, S_2 = 3.31$

Matches (a).

10. $\sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$

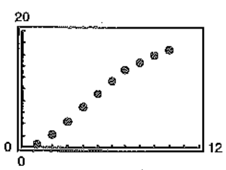
$S_1 = 4$

Matches (e).

11. (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2(5/8)^{n+1}}{n^2(5/8)^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{5}{8} = \frac{5}{8} < 1.$ Converges

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491

(c) 

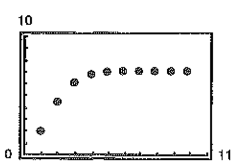
(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

12. (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)^2+1}{(n+1)!}}{\frac{n^2+1}{n!}} = \lim_{n\to\infty}\left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{n+1}\right) = 0 < 1.$ Converges

(b)

n	5	10	15	20	25
S_n	7.0917	7.1548	7.1548	7.1548	7.1548

(c) 

(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

13. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

14. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right|$$

$$= \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

15. $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(3/4)^{n+1}}{n(3/4)^n}\right| = \lim_{n\to\infty}\left|\frac{3(n+1)}{4n}\right| = \frac{3}{4}$$

Therefore, by the Ratio Test, the series converges.

16. $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n3^n}\right| = \lim_{n\to\infty}\frac{3(n+1)}{2n} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

17. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

18. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3/2^{n+1}}{n^3/2^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{2n^3}\right| = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

19. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n}\right| = \lim_{n\to\infty}\frac{2n^2}{(n+1)^2} = 2$$

Therefore, by the Ratio Test, the series diverges.

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \le \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n\to\infty}\frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 9.14, the series converges.

Note: The Ratio Test is inconclusive since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$

The series converges conditionally.

21. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| = \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3/2)^{n+1}}{n^2+2n+1}\cdot\frac{n^2}{(3/2)^n}\right| = \lim_{n\to\infty}\frac{3n^2}{2(n^2+2n+1)} = \frac{3}{2} > 1$$

Therefore, by the Ratio Test, the series diverges.

23. $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)3^{n+1}}\cdot\frac{n3^n}{n!}\right| = \lim_{n\to\infty}\frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

24. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)^5}\cdot\frac{n^5}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty$$

Therefore, by the Ratio Test, the series diverges.

25. $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n}\right|$$
$$= \lim_{n\to\infty} \frac{4}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

26. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right|$$
$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n}$$
$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e > 1$$

Therefore, by the Ratio Test, the series diverges.

27. $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{3^n}\right| = \lim_{n\to\infty} \frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n\to\infty} \frac{3}{n+2}\left(\frac{n+1}{n+2}\right)^n = (0)\left(\frac{1}{e}\right) = 0$$

To find $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$, let $y = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$. Then,

$$\ln y = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} \frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n\to\infty} \frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}.$$

Therefore, by the Ratio Test, the series converges.

28. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{[(n+1)!]^2}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^2}\right|$$
$$= \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

29. $\sum_{n=0}^{\infty} \frac{4^n}{3^n+1}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4^{n+1}}{3^{n+1}+1} \cdot \frac{3^n+1}{4^n}\right|$$
$$= \lim_{n\to\infty} \frac{4(3^n+1)}{3^{n+1}+1}$$
$$= \lim_{n\to\infty} \frac{4(1+1/3^n)}{3+1/3^n} = \frac{4}{3}$$

Therefore, by the Ratio Test, the series diverges.

30. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}}\right| = \lim_{n\to\infty} \frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

31. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n}\right| = \lim_{n\to\infty}\frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = 1$$

Ratio Test is inconclusive.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/2} = 1$$

Ratio Test is inconclusive.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^4} \cdot \frac{n^4}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4 = 1$$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^p} \cdot \frac{n^p}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p = 1$$

37. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \lim_{n\to\infty}\frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

39. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+1}{n-1}\right)^n} = \lim_{n\to\infty}\left(\frac{2n+1}{n-1}\right) = 2$$

Since $2 > 1$, the series diverges.

40. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n-1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{4n+3}{2n-1}\right)^n} = \lim_{n\to\infty}\frac{4n+3}{2n-1} = 2$$

Since $2 > 1$, the series diverges.

41. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

42. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|} = \lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Therefore, by the Root Test, the series diverges.

43. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n} + 1\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(2\sqrt[n]{n} + 1\right)^n} = \lim_{n\to\infty} \left(2\sqrt[n]{n} + 1\right)$$

To find $\lim_{n\to\infty} \sqrt[n]{n}$, let $y = \lim_{n\to\infty} \sqrt[n]{n}$. Then

$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[n]{n}\right) = \lim_{n\to\infty} \frac{1}{n} \ln n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n\to\infty} \left(2\sqrt[n]{n} + 1\right) = 2(1) + 1 = 3$. Therefore, by the Root Test, the series diverges.

44. $\sum_{n=0}^{\infty} e^{-n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

45. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{4^n}}$$

$$= \lim_{n\to\infty} \frac{n^{1/n}}{4} = \frac{1}{4}$$

Since $\frac{1}{4} < 1$, the series converges.

Note: You can use L'Hôpital's Rule to show $\lim_{n\to\infty} n^{1/n} = 1$.

Let $y = n^{1/n} \implies \ln y = \dfrac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Hence, $\ln y \to 0 \implies y = n^{1/n} \to 1$.

46. $\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{500}\right)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{500}\right) = \infty$$

The series diverges.

47. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 - 0 = 0 < 1$$

Hence, the series converges.

48. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n} = 0 < 1$$

By the Root Test, the series converges.

49. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}}$$

$$= \lim_{n\to\infty} \frac{n^{1/n}}{\ln n} = 0$$

Since $0 < 1$, the series converges by the Root Test.

50. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty} \frac{(n!)^n}{(n^2)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{(n!)}{(n^2)^n}}$$

$$= \lim_{n\to\infty} \frac{n!}{n^2} = \infty$$

The series diverges.

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

52. $\sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$

This is the divergent harmonic series.

53. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is convergent p-series.

54. $\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

Since $\pi/4 < 1$,this is convergent geometric series.

55. $\sum_{n=1}^{\infty} \frac{2n}{n+1}$

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the nth-Term Test for Divergence.

56. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

$$\lim_{n\to\infty} \frac{n/(2n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

57. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$

Since $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

58. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

$$\lim_{n\to\infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a Limit Comparison Test with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

59. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

$$\lim_{n\to\infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n\to\infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a Limit Comparison Test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

60. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

$$\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \lim_{n\to\infty} \frac{(\ln 2)2^n}{8n} = \lim_{n\to\infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the nth-Term Test for Divergence.

61. $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

$$\left|\frac{\cos n}{2^n}\right| \leq \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{2^n}\right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

62. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n\ln(n)} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

63. $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n}\right| = \lim_{n\to\infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

64. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

$$\frac{\ln(n)}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

65. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}}\right| = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

(Absolutely)

66. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{3^n}\right| = \lim_{n\to\infty} \frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

67. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n}\right| = \lim_{n\to\infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

68. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right| = \lim_{n\to\infty} \frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{1}{18}$$

Therefore, by the Ratio Test, the series converge.

69. (a) and (c)

$$\sum_{n=1}^{\infty} \frac{n5^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$$

$$= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots$$

70. (b) and (c)

$$\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$$

$$= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots$$

71. (a) and (b) are the same.

72. (a) and (b) are the same.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots$$

73. Replace n with $n + 1$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

74. Replace n with $n + 2$.

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

75. Since

$$\frac{3^{10}}{2^{10}\,10!} \approx 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^{9} \frac{(-3)^k}{2^k\,k!} \approx -0.7769$$

76. $$\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \ldots (2k+1)} = \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} = \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \approx 0.40967$$

(See Exercise 3 and use 10 terms, $k = 9$.)

77. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4n-1)/(3n+2)a_n}{a_n}\right| = \lim_{n\to\infty}\frac{4n-1}{3n+2} = \frac{4}{3} > 1$$

The series diverges by the Ratio Test.

78. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)/(5n-4)a_n}{a_n}\right| = \lim_{n\to\infty}\frac{2n+1}{5n-4} = \frac{2}{5} < 1$$

The series converges by the Ratio Test.

79. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\sin n+1)/(\sqrt{n})a_n}{a_n}\right| = \lim_{n\to\infty}\frac{\sin n+1}{\sqrt{n}} = 0 < 1$$

The series converges by the Ratio Test.

80. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\cos n+1)/(n)a_n}{a_n}\right| = \lim_{n\to\infty}\frac{\cos n+1}{n} = 0 < 1$$

The series converges by the Ratio Test.

81. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(1+(1)/(n))a_n}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

The Ratio Test is inconclusive.

But, $\lim_{n\to\infty} a_n \neq 0$, so the series diverges.

82. The series diverges because $\lim_{n\to\infty} a_n \neq 0$.

$$a_1 = \frac{1}{4}$$

$$a_2 = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}$$

$$a_3 = \left(\frac{1}{2}\right)^{1/3} \approx 0.7937$$

In general, $a_{n+1} > a_n > 0$.

83. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 2\cdots n(n+1)}{1\cdot 3\cdots(2n-1)(2n+1)}}{\dfrac{1\cdot 2\cdots n}{1\cdot 3\cdots(2n-1)}}\right| = \lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{1}{2} < 1$$

The series converges by the Ratio Test.

84. $$\sum_{n=0}^{\infty}\frac{n+1}{3^n}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n+1}{3^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n+1}}{3}$$

Let $y = \lim_{n\to\infty}\sqrt[n]{n+1}$

$$\ln y = \lim_{n\to\infty}\left(\ln\sqrt[n]{n+1}\right) = \lim_{n\to\infty}\frac{1}{n}\ln(n+1) = \lim_{n\to\infty}\frac{\ln(n+1)}{n} = \frac{1}{n+1} = 0.$$

Since $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n\to\infty}\frac{\sqrt{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

85. $$\sum_{n=3}^{\infty}\frac{1}{(\ln n)^n}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty}\frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

86. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{1\cdot 2\cdot 3\cdots(2n-1)(2n)(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(2n-1)}}\right|$$
$$= \lim_{n\to\infty}\frac{2n+1}{(2n)(2n+1)} = 0 < 1$$

The series converges by the Ratio Test.

87. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(x/3)^{n+1}}{2(x/3)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

For the series to converge: $\left|\frac{x}{3}\right| < 1 \Rightarrow -3 < x < 3$.

For $x = 3$, the series diverges.

For $x = -3$, the series diverges.

Answer: $-3 < x < 3$

88. $$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{x+1}{4}\right|^n}$$
$$= \lim_{n\to\infty}\left|\frac{x+1}{4}\right| = \left|\frac{x+1}{4}\right|$$

For the series to converge,

$$\left|\frac{x+1}{4}\right| < 1 \Rightarrow -4 < x+1 < 4$$
$$\Rightarrow -5 < x < 3.$$

For $x = 3$, $\sum_{n=0}^{\infty}(1)^n$, diverges.

For $x = -5$, $\sum_{n=0}^{\infty}(-1)^n$, diverges.

Answer: $-5 < x < 3$

89. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}/(n+1)}{x^n/n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n}{n+1}(x+1)\right| = |x+1|$$

For the series to converge,

$$|x+1| < 1 \Rightarrow -1 < x+1 < 1$$
$$\Rightarrow -2 < x < 0.$$

For $x = 0$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, converges.

For $x = -2$, $\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$, diverges.

Answer: $-2 < x \le 0$

90. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2|x-1|^{n+1}}{2|x-1|^n}\right|$$
$$= \lim_{n\to\infty}|x-1| = |x-1|$$

For the series to converge,

$$|x-1| < 1 \Rightarrow -1 < x-1 < 1$$
$$\Rightarrow 0 < x < 2.$$

For $x = 2$, $\sum_{n=0}^{\infty}2(1)^n$, diverges.

For $x = 0$, $\sum_{n=0}^{\infty}2(-1)^n$, diverges.

Answer: $0 < x < 2$

91. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!\left|\frac{x}{2}\right|^{n+1}}{n!\left|\frac{x}{2}\right|^n}$$
$$= \lim_{n\to\infty}(n+1)\left|\frac{x}{2}\right| = \infty$$

The series converges only at $x = 0$.

92. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+1|^{n+1}}{(n+1)!}\bigg/\frac{|x+1|^n}{n!} = \lim_{n\to\infty}\frac{|x+1|}{n+1} = 0$$

The series converges for all x.

93. See Theorem 9.17, page 597.

94. See Theorem 9.18.

95. No. Let $a_n = \dfrac{1}{n + 10{,}000}$.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.

96. One example is $\displaystyle\sum_{n=1}^{\infty}\left(-100 + \frac{1}{n}\right)$.

97. The series converges absolutely. See Theorem 9.17.

98. Assume that

$\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ or that $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$.

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Longrightarrow \lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n \text{ diverges.}$$

99. First, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$$

and choose R such that $0 \le r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. Thus, for $n > N\, |a_n| < R^n$ and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for infinitely many $n > M$. Thus, for infinitely many $n > M$, we have $|a_n| > R^n > 1$ which implies that $\lim_{n\to\infty} a_n \neq 0$ which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

100. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, p-series

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{n^{p/n}} = 1$$

Hence, the Root Test is inconclusive.

Note: $\lim_{n\to\infty} n^{p/n} = 1$ because if $y = n^{p/n}$, then

$\ln y = \dfrac{p}{n} \ln n$ and $\dfrac{p}{n} \ln n \to 0$ as $n \to \infty$.

Hence $y \to 1$ as $n \to \infty$.

101. Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n(\ln n)^p}{(n+1)(\ln(n+1))^p} = 1, \text{ inconclusive.}$$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n(\ln n)^p}} = \lim_{n\to\infty} \frac{1}{n^{1/n}(\ln n)^{p/n}}$$

$\lim_{n\to\infty} n^{1/n} = 1$. Furthermore, let $y = (\ln n)^{p/n} \Longrightarrow$

$\ln y = \dfrac{p}{n} \ln(\ln n)$. $\displaystyle\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{p \ln(\ln n)}{n}$

$$= \lim_{n\to\infty} \frac{p}{\ln(n)(1/n)} = 0 \Longrightarrow \lim_{n\to\infty} (\ln n)^{p/n} = 1.$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{1}{n^{1/n}(\ln n)^{p/n}} = 1$, inconclusive.

102. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$, x positive integer

(a) $x = 1$: $\displaystyle\sum \frac{(n!)^2}{n!} = \sum n!$, diverges

(b) $x = 2$: $\displaystyle\sum \frac{(n!)^2}{(2n)!}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(2n+2)!} \bigg/ \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

(c) $x = 3$: $\displaystyle\sum \frac{(n!)^2}{(3n)!}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(3n+3)!} \bigg/ \frac{(n!)^2}{(3n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

(d) Use the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{[x(n+1)]!} \bigg/ \frac{(n!)^2}{(xn)!} = \lim_{n\to\infty} (n+1)^2 \frac{(xn)!}{(xn+x)!}$$

The cases $x = 1, 2, 3$ were solved above. For $x > 3$, the limit is 0. Hence, the series converges for all integers $x \geq 2$.

103. For $n = 1, 2, 3, \ldots, -|a_n| \leq a_n \leq |a_n| \Rightarrow -\displaystyle\sum_{n=1}^{k} |a_n| \leq \sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} |a_n|$.

Taking limits as $k \to \infty$, $-\displaystyle\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n| \Rightarrow \left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

104. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and $f(x)$ is a real function such that $f(1/n) = U_n$ for all positive integers n and d^2f/dx^2 exists at $x = 0$, then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise. Below are some examples.

Convergent Series	Divergent Series
$\sum \frac{1}{n^3}, f(x) = x^3$	$\sum \frac{1}{n}, f(x) = x$
$\sum \left(1 - \cos\frac{1}{n}\right), f(x) = 1 - \cos x$	$\sum \sin\frac{1}{n}, f(x) = \sin x$

105. Using the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{n!}{(n+1)^n}\left(\frac{19}{7}\right)^n \bigg/ \frac{(n-1)!}{n^{n-1}}\left(\frac{19}{7}\right)^{n-1}\right]$$

$$= \lim_{n\to\infty} \left[\frac{n \cdot n^{n-1}}{(n+1)^n}\left(\frac{19}{7}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{(1 + (1/n))^n}\left(\frac{19}{7}\right)\right]$$

$$= \frac{19}{7} \cdot \frac{1}{e} < 1$$

Hence, the series converges.

106. We first prove Abel's Summation Theorem:

If the partial sums of $\sum a_n$ are bounded and if $\{b_n\}$ decreases to zero, then $\sum a_n b_n$ converges.

Let $S_k = \sum_{i=1}^{k} a_i$. Let M be a bound for $\{|S_k|\}$.

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n = S_1b_1 + (S_2 - S_1)b_2 + \cdots + (S_n - S_{n-1})b_n$$
$$= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_nb_n$$
$$= \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_nb_n$$

The series $\sum_{i=1}^{\infty} S_i(b_i - b_{i+1})$ is absolutely convergent because $|S_i(b_i - b_{i+1})| \le M(b_i - b_{i+1})$ and $\sum_{i=1}^{\infty}(b_i - b_{i+1})$ converges to b_1.

Also, $\lim_{n\to\infty} S_nb_n = 0$ because $\{S_n\}$ bounded and $b_n \to 0$. Thus, $\sum_{n=1}^{\infty} a_nb_n = \lim_{n\to\infty}\sum_{i=1}^{n} a_ib_i$ converges.

Now let $b_n = \frac{1}{n}$ to finish the problem.

Section 9.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

2. $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

y-axis symmetry

Three relative extrema

Matches (c)

3. $y = e^{-1/2}[(x + 1) + 1]$

Linear

Matches (a)

4. $y = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

Cubic

Matches (b)

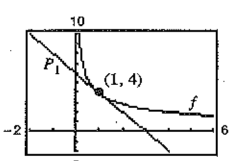
5. $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\quad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\quad f'(1) = -2$

$P_1(x) = f(1) + f'(1)(x - 1)$

$= 4 + (-2)(x - 1)$

$P_1(x) = -2x + 6$



P_1 is called the first degree Taylor polynomial for f at c.

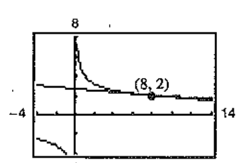
6. $f(x) = \dfrac{4}{\sqrt[3]{x}} = 4x^{-1/3}$ $\quad f(8) = 2$

$f'(x) = -\dfrac{4}{3}x^{-4/3}$ $\quad f'(8) = -\dfrac{1}{12}$

$P_1(x) = f(8) + f'(8)(x - 8)$

$= 2 + \left(-\dfrac{1}{12}\right)(x - 8)$

$P_1(x) = -\dfrac{1}{12}x + \dfrac{8}{3}$



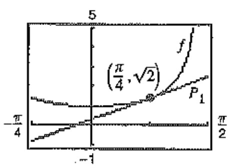
P_1 is called the first degree Taylor polynomial for f at c.

7. $f(x) = \sec x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x \qquad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$



P_1 is called the first degree Taylor polynomial for f at c.

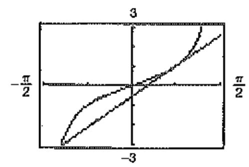
8. $f(x) = \tan x \qquad f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x \qquad f'\left(\frac{\pi}{4}\right) = 2$

$$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = 2x + 1 - \frac{\pi}{2}$$



P_1 is called the first degree Taylor polynomial for f at c.

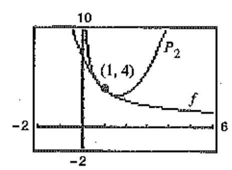
9. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \qquad f(1) = 4$

$f'(x) = -2x^{-3/2} \qquad f'(1) = -2$

$f''(x) = 3x^{-5/2} \qquad f''(1) = 3$

$$P_2 = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2$$

$$= 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$$



x	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

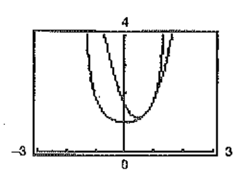
10. $f(x) = \sec x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x \qquad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$



x	-2.15	0.585	0.685	$\pi/4$	0.885	0.985	1.785
$f(x)$	-1.8270	1.1995	1.2913	1.4142	1.5791	1.8088	-4.7043
$P_2(x)$	15.5414	1.2160	1.2936	1.4142	1.5761	1.7810	4.9475

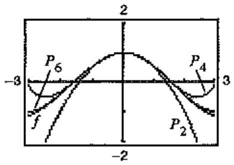
11. $f(x) = \cos x$

$P_2(x) = 1 - \frac{1}{2}x^2$

$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

(a)



(b) $f'(x) = -\sin x \qquad P_2'(x) = -x$

$f''(x) = -\cos x \qquad P_2''(x) = -1$

$f''(0) = P_2''(0) = -1$

$f'''(x) = \sin x \qquad P_4'''(x) = x$

$f^{(4)}(x) = \cos x \qquad P_4^{(4)}(x) = 1$

$f^{(4)}(0) = 1 = P_4^{(4)}(0)$

$f^{(5)}(x) = -\sin x \qquad P_6^{(5)}(x) = -x$

$f^{(6)}(x) = -\cos x \qquad P^{(6)}(x) = -1$

$f^{(6)}(0) = -1 = P_6^{(6)}(0)$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all n.

12. $f(x) = x^2e^x, f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x \qquad f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x \qquad f''(0) = 2$

$f'''(x) = (x^2 + 6x + 6)e^x \qquad f'''(0) = 6$

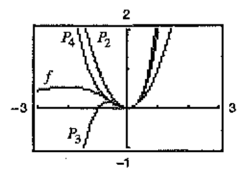
$f^{(4)}(x) = (x^2 + 8x + 12)e^x \qquad f^{(4)}(0) = 12$

$$P_2(x) = \frac{2x^2}{2!} = x^2$$

$$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$$

$$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$$

(b)



(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

13. $f(x) = e^{-x} \qquad f(0) = 1$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f'''(x) = -e^{-x} \qquad f'''(0) = -1$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

14. $f(x) = e^{-x} \qquad f(0) = 1$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f'''(x) = -e^{-x} \qquad f'''(0) = -1$

$f^{(4)}(x) = e^{-x} \qquad f^{(4)}(0) = 1$

$f^{(5)}(x) = -e^{-x} \qquad f^{(5)}(0) = -1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$+ \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

15. $f(x) = e^{2x} \qquad f(0) = 1$

$f'(x) = 2e^{2x} \qquad f'(0) = 2$

$f''(x) = 4e^{2x} \qquad f''(0) = 4$

$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$

$f^{(4)}(x) = 16^{2x} \qquad f^{(4)}(0) = 16$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

16. $f(x) = e^{3x} \qquad f(0) = 1$

$f'(x) = 3e^{3x} \qquad f'(0) = 3$

$f''(x) = 9e^{3x} \qquad f''(0) = 9$

$f'''(x) = 27e^{3x} \qquad f'''(0) = 27$

$f^{(4)}(x) = 81e^{3x} \qquad f^{(4)}(0) = 81$

$$P_4(x) = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4$$

$$= 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

17. $f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

18. $f(x) = \sin \pi x \qquad f(0) = 0$

$f'(x) = \pi \cos \pi x \qquad f'(0) = \pi$

$f''(x) = -\pi^2 \sin \pi x \qquad f''(0) = 0$

$f'''(x) = -\pi^3 \cos \pi x \qquad f'''(0) = -\pi^3$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

19. $f(x) = xe^x \qquad f(0) = 0$

$f'(x) = xe^x + e^x \qquad f'(0) = 1$

$f''(x) = xe^x + 2e^x \qquad f''(0) = 2$

$f'''(x) = xe^x + 3e^x \qquad f'''(0) = 3$

$f^{(4)}(x) = xe^x + 4e^x \qquad f^{(4)}(0) = 4$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

20. $f(x) = x^2e^{-x} \qquad f(0) = 0$

$f'(x) = 2xe^{-x} - x^2e^{-x} \qquad f'(0) = 0$

$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} \qquad f''(0) = 2$

$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \qquad f'''(0) = -6$

$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \qquad f^{(4)}(0) = 12$

$$P_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4$$

$$= x^2 - x^3 + \frac{1}{2}x^4$$

21. $f(x) = \dfrac{1}{x+1} \qquad f(0) = 1$

$f'(x) = -\dfrac{1}{(x+1)^2} \qquad f'(0) = -1$

$f''(x) = \dfrac{2}{(x+1)^3} \qquad f''(0) = 2$

$f'''(x) = \dfrac{-6}{(x+1)^4} \qquad f'''(0) = -6$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5} \qquad f^{(4)}(0) = 24$

$$P_4(x) = 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 - x + x^2 - x^3 + x^4$$

22. $f(x) = \dfrac{x}{x+1} = \dfrac{x+1-1}{x+1} \qquad f(0) = 0$

$= 1 - (x+1)^{-1}$

$f'(x) = (x+1)^{-2} \qquad f'(0) = 1$

$f''(x) = -2(x+1)^{-3} \qquad f''(0) = -2$

$f'''(x) = 6(x+1)^{-4} \qquad f'''(0) = 6$

$f^{(4)}(x) = -24(x+1)^{-5} \qquad f^{(4)}(0) = -24$

$$P_4(x) = 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4$$

$$= x - x^2 + x^3 - x^4$$

23. $f(x) = \sec x \qquad f(0) = 1$

$f'(x) = \sec x \tan x \qquad f'(0) = 0$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''(0) = 1$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

24. $f(x) = \tan x \qquad f(0) = 0$

$f'(x) = \sec^2 x \qquad f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x \qquad f''(0) = 0$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \qquad f'''(0) = 2$

$$P_3(x) = 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3$$

25. $f(x) = \frac{1}{x}$ $f(1) = 1$

$f'(x) = -\frac{1}{x^2}$ $f'(1) = -1$

$f''(x) = \frac{2}{x^3}$ $f''(1) = 2$

$f'''(x) = -\frac{6}{x^4}$ $f'''(1) = -6$

$f^{(4)}(x) = \frac{24}{x^5}$ $f^{(4)}(1) = 24$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

26. $f(x) = 2x^{-2}$ $f(2) = \frac{1}{2}$

$f'(x) = -4x^{-3}$ $f'(2) = -\frac{1}{2}$

$f''(x) = 12x^{-4}$ $f''(2) = \frac{3}{4}$

$f'''(x) = -48x^{-5}$ $f'''(x) = -\frac{3}{2}$

$f^{(4)}(x) = 240x^{-6}$ $f^{(4)}(x) = \frac{15}{4}$

$$P_4(x) = \frac{1}{2} - \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{5}{32}(x-2)^4$$

27. $f(x) = \sqrt{x}$ $f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ $f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x\sqrt{x}}$ $f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8x^2\sqrt{x}}$ $f'''(1) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16x^3\sqrt{x}}$ $f^{(4)}(1) = -\frac{15}{16}$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

28. $f(x) = x^{1/3}$ $f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $f''(8) = -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$$

29. $f(x) = \ln x$ $f(1) = 0$

$f'(x) = \frac{1}{x}$ $f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4}$ $f^{(4)}(1) = -6$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

30. $f(x) = x^2 \cos x$ $f(\pi) = -\pi^2$

$f'(x) = \cos x - x^2 \sin x$ $f'(\pi) = -2\pi$

$f''(x) = 2\cos x - 4x \sin x - x^2 \cos x$ $f''(\pi) = -2 + \pi^2$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2 - 2)}{2}(x-\pi)^2$$

31. $f(x) = \tan x$

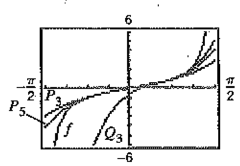
$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{(4)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$f^{(5)}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x$



(a) $n = 3, c = 0$

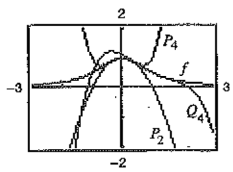
$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

(b) $n = 3, c = \frac{\pi}{4}$

$$Q_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

32. $f(x) = \dfrac{1}{x^2 + 1}$



$f'(x) = \dfrac{-2x}{(x^2 + 1)^2}$

$f''(x) = \dfrac{2(3x^2 - 1)}{(x^2 + 1)^3}$

$f'''(x) = \dfrac{24x(1 - x^2)}{(x^2 + 1)^4}$

$f^{(4)}(x) = \dfrac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$

(a) $n = 4, c = 0$

$$P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$$

(b) $n = 4, c = 1$

$$Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x - 1) + \frac{1/2}{2!}(x - 1)^2 + \frac{0}{3!}(x - 1)^3 + \frac{-3}{4!}(x - 1)^4 = \frac{1}{2} - \frac{1}{2}(x - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{8}(x - 1)^4$$

33. $f(x) = \sin x$

$P_1(x) = x$

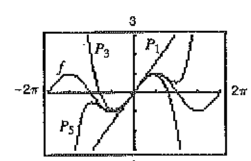
$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

(a)

x	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417

(b)



(c) As the distance increases, the accuracy decreases.

34. $f(x) = \ln x,\ c = 1$

$P_1(x) = x - 1$

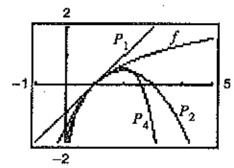
$P_2(x) = (x - 1) - \frac{1}{2}(x - 1)^2$

$P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

(a)

x	1.00	1.25	1.50	1.75	2.00
$\ln x$	0	0.2231	0.4055	0.5596	0.6931
$P_1(x)$	0	0.2500	0.5000	0.7500	1.0000
$P_2(x)$	0	0.2188	0.375	0.4688	0.5000
$P_4(x)$	0	0.2230	0.4010	0.5303	0.5833

(b)



(c) As the degree increases, the accuracy increases. As the distance from x to 1 increases, the accuracy decreases.

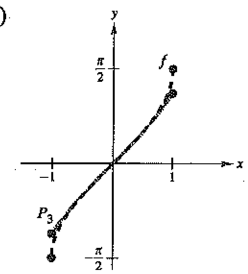
35. $f(x) = \arcsin x$

(a) $P_3(x) = x + \dfrac{x^3}{6}$

(b)

x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$	−0.848	−0.524	−0.253	0	0.253	0.524	0.848
$P_3(x)$	−0.820	−0.521	−0.253	0	0.253	0.521	0.820

(c)



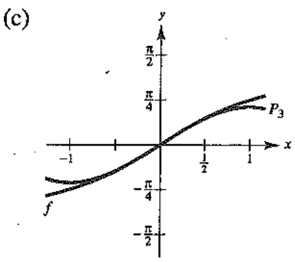
36. (a) $f(x) = \arctan x$

$P_3(x) = x - \dfrac{x^3}{3}$

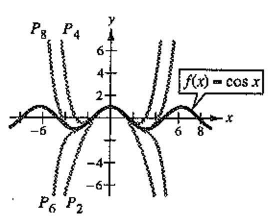
(b)

x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$	−0.6435	−0.4636	−0.2450	0	0.2450	0.4636	0.6435
$P_3(x)$	−0.6094	−0.4583	−0.2448	0	0.2448	0.4583	0.6094

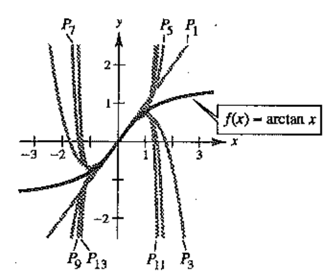
(c)



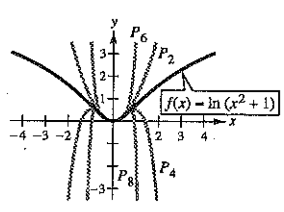
37. $f(x) = \cos x$



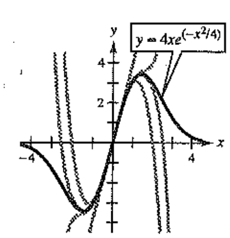
38. $f(x) = \arctan x$



39. $f(x) = \ln(x^2 + 1)$



40. $f(x) = 4xe^{-x^2/4}$



41. $f(x) = e^{-x} \approx 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$

$f\left(\dfrac{1}{2}\right) \approx 0.6042$

42. $f(x) = x^2e^{-x} \approx x^2 - x^3 + \dfrac{1}{2}x^4$

$f\left(\dfrac{1}{5}\right) \approx 0.0328$

43. $f(x) = \ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

$f(1.2) \approx 0.1823$

44. $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x-\pi) + \left(\dfrac{\pi^2 - 2}{2}\right)(x-\pi)^2$

$f\left(\dfrac{7\pi}{8}\right) \approx -6.7954$

45. $f(x) = \cos x; f^{(5)}(x) = -\sin x \implies$ Max on $[0, 0.3]$ is 1.

$R_4(x) \le \dfrac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

Note: you could use $R_5(x)$: $f^{(6)}(x) = -\cos x$, max on $[0, 0.3]$ is 1.

$R_5(x) \le \dfrac{1}{6!}(0.3)^6 = 1.0125 \times 10^{-6}$

Exact error: $0.000001 = 1.0 \times 10^{-6}$

46. $f(x) = e^x; f^{(6)}(x) = e^x \implies$ Max on $[0, 1]$ is e^1.

$R_5(x) \le \dfrac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$

47. $f(x) = \arcsin x; f^{(4)}(x) = \dfrac{x(6x^2+9)}{(1-x^2)^{7/2}} \implies$ Max on $[0, 0.4]$ is $f^{(4)}(0.4) \approx 7.3340$.

$R_3(x) \le \dfrac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$. The exact error is 8.5×10^{-4}. [Note: You could use R_4.]

48. $f(x) = \arctan x; f^{(4)}(x) = \dfrac{24x(x^2+1)}{(1-x^2)^4}$

$\implies$Max on $[0, 0.4]$ is $f^{(4)}(0.4) \approx 22.3672$.

$R_3(x) \le \dfrac{22.3672}{4!}(0.4)^4 \approx 0.0239$

49. $g(x) = \sin x$

$|g^{(n+1)}(x)| \le 1$ for all x.

$R_n(x) \le \dfrac{1}{(n+1)!}(0.3)^{n+1} < 0.001$

By trial and error, $n = 3$.

50. $f(x) = \cos x$

$|f^{(n+1)}(x)| \le 1$ for all x and all n.

$$|R_n(x)| = \left|\frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!}\right| \le \frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 2$.

51. $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

Max on $[0, 0.6]$ is $e^{0.6} \approx 1.8221$.

$R_n \le \dfrac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$

By trial and error, $n = 5$.

52. $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

The maximum value on $[0, 0.3]$ is $e^{0.3} \approx 1.3499$.

$$|R_n| \le \frac{1.3499}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error, $n = 3$.

53. $f(x) = \ln(x + 1)$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \le \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

54. $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \cdots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \cdots$$

$$f(0.6) = 1 - \frac{\pi^2}{2!}(0.6)^4 + \frac{\pi^4}{4!}(0.6)^8 - \frac{\pi^6}{6!}(0.6)^{12} + \cdots$$

Since this is an alternating series,

$$R_n \le a_{n+1} = \frac{\pi^{2n}}{(2n)!}(0.6)^{4n} < 0.0001.$$

By trial and error, $n = 4$. Using 4 terms $f(0.6) \approx 0.4257$.

55. $f(x) = e^{-\pi x}$, $f(1.3)$

$f'(x) = (-\pi)e^{-\pi x}$

$f^{(n+1)}(x) = (-\pi)^{n+1}e^{-\pi x} \le |(-\pi)^{n+1}|$ on $[0, 1.3]$

$$|R_n| \le \frac{(\pi)^{n+1}}{(n+1)!}(1.3)^{n+1} < 0.0001$$

By trial and error, $n = 16$.

56. $f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f^{(n+1)}(x) = (-1)^{n+1}e^{-x} \le 1$ on $[0, 1]$

$$|R_n| \le \frac{1}{(n+1)!}1^{n+1} \le 0.0001$$

By trial and error, $n = 7$.

57. $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6},\ x < 0$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$e^z x^4 < 0.024$

$|xe^{z/4}| < 0.3936$

$$|x| < \frac{0.3936}{e^{z/4}} < 0.3936,\ z < 0$$

$-0.3936 < x < 0$

58. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$$|R_3(x)| = \left|\frac{\sin z}{4!}x^4\right| \le \frac{|x^4|}{4!} < 0.001$$

$x^4 < 0.024$

$|x| < 0.3936$

$-0.3936 < x < 0.3936$

59. $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, fifth degree polynomial

$|f^{(n+1)}(x)| \le 1$ for all x and all n.

$$|R_5(x)| \le \frac{1}{6!}|x|^6 < 0.001$$

$|x|^6 < 0.72$

$|x| < 0.9467$

$-0.9467 < x < 0.9467$

Note: Use a graphing utility to graph $y = \cos x - (1 - x^2/2 + x^4/24)$ in the viewing window $[-0.9467, 0.9467] \times [-0.001, 0.001]$ to verify the answer.

60. $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$

$f'(x) = -2e^{-2x}, f''(x) = 4e^{-2x}, f'''(x) = -8e^{-2x}, f^{(4)}(x) = 16e^{-2x}$

$R_3(x) = \frac{f^4(z)}{4!}(x-0)^4 = \frac{16e^{-2z}}{24}x^4 = \frac{2}{3}e^{-2z}x^4 < 0.001$

$e^{-2z}x^4 < 0.0015$

$x < \left[\frac{0.0015}{e^{-2z}}\right]^{1/4} \approx 0.1970e^{2z} < 0.1970$, for $z < 0$.

Thus $0 < x < 0.1970$.

In fact, by graphing $f(x) = e^{-2x}$ and $y = 1 - 2x + 2x^2 - \frac{4}{3}x^3$, you can verify that $|f(x) - y| < 0.001$ on $(-0.19294, 0.20068)$

61. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$ and the slopes of P and f agree at $(c, f(c))$. Depending on the degree of P, the nth derivatives of P and f agree at $(c, f(c))$.

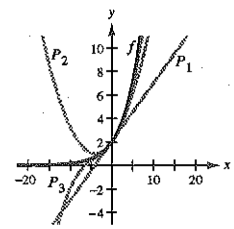
62. $f(c) = P_2(c), f'(c) = P_2'(c)$, and $f''(c) = P_2''(c)$

63. See definition on page 650.

64. See Theorem 9.19, page 654.

65. The accuracy increases as the degree increases (for values within the interval of convergence).

66.



67. (a) $f(x) = e^x$

$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$g(x) = xe^x$

$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$

$Q_5(x) = x\,P_4(x)$

(b) $f(x) = \sin x$

$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$g(x) = x \sin x$

$Q_6(x) = x\,P_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$

(c) $g(x) = \frac{\sin x}{x} = \frac{1}{x}P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$

68. (a) $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $f(x) = \sin x$

$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for $\cos x$

$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$

(c) $R(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

The first four terms are the same!

69. (a) $Q_2(x) = -1 + \dfrac{\pi^2(x+2)^2}{32}$ (b) $R_2(x) = -1 + \dfrac{\pi^2(x-6)^2}{32}$ (c) No. The polynomial will be linear. Translations are possible at $x = -2 + 8n$.

70. Let f be an odd function and P_n be the n^{th} Maclaurin polynomial for f. Since f is odd, f' is even:

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0}\frac{-f(x-h) + f(x)}{h} = \lim_{h\to 0}\frac{f(x+(-h)) - f(x)}{-h} = f'(x).$$

Similarly, f'' is odd, f''' is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \cdots = 0$. Hence, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$ all the coefficients of the even power of x are zero.

71. Let f be an even function and P_n be the nth Maclaurin polynomial for f. Since f is even, f' is odd, f'' is even, f''' is odd, etc. All of the odd derivatives of f are odd and thus, all of the odd powers of x will have coefficients of zero. P_n will only have terms with even powers of x.

72. Let $P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ where $a_i = \dfrac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

For $1 \le k \le n$, $P_n^{(k)}(c) = a_n k! = \left(\dfrac{f^{(k)}(c)}{k!}\right)k! = f^{(k)}(c)$.

73. As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

Section 9.8 Power Series

1. Centered at 0

2. Centered at 0

3. Centered at 2

4. Centered at π

5. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right||x| = |x|$$

$|x| < 1 \Rightarrow R = 1$

6. $\displaystyle\sum_{n=0}^{\infty}(2x)^n$

$$L = \lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right| = 2|x|$$

$2|x| < 1 \Rightarrow R = \frac{1}{2}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(2x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n^2x}{(n+1)^2}\right| = 2|x|$$

$2|x| < 1 \Rightarrow R = \frac{1}{2}$

8. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(-1)^n x^n}\right|$$

$$= \frac{1}{2}|x|$$

$\frac{1}{2}|x| < 1 \Rightarrow R = 2$

9. $\sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2x)^2}{(2n+2)(2n+1)}\right| = 0$$

Thus, the series converges for all x. $R = \infty$.

10. $\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x^2}{(n+1)}\right| = \infty$$

The series only converges at $x = 0$. $R = 0$.

11. $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

Since the series is geometric, it converges only if $|x/2| < 1$ or $-2 < x < 2$.

12. $\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$

is geometric, so it converges if $|x/5| < 1 \Rightarrow |x| < 5 \Rightarrow -5 < x < 5$.

13. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

When $x = -1$, the p-series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Therefore, the interval of convergence is $-1 < x \le 1$.

14. $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\sum_{n=0}^{\infty}-(n+1)$ diverges.

Therefore, the interval of convergence is $-1 < x < 1$.

15. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$

The series converges for all x. Therefore, the interval of convergence is $-\infty < x < \infty$.

16. $\sum_{n=0}^{\infty}\frac{(3x)^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+1)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

17. $\sum_{n=0}^{\infty}(2n)!\left(\frac{x}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(2n)!x^n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x}{2}\right| = \infty$$

Therefore, the series converges only for $x = 0$.

18. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{(-1)^n x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x}{n+3}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore, the interval of convergence is $-1 \le x \le 1$.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{4^n}$

Since the series is geometric, it converges only if $|x/4| < 1$ or $-4 < x < 4$.

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)!(x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n!(x-4)^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-4)}{3}\right| = \infty$$

$R = 0$

Center: $x = 4$

Therefore, the series converges only for $x = 4$.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n}\right| = \lim_{n\to\infty} \left|\frac{n(x-5)}{5(n+1)}\right| = \frac{1}{5}|x-5|$$

$R = 5$

Center: $x = 5$

Interval: $-5 < x - 5 < 5$ or $0 < x < 10$

When $x = 0$, the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges. When $x = 10$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $0 < x \le 10$.

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-2)^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(x-2)(n+1)}{4(n+2)}\right| = \frac{1}{4}|x-2|$$

$R = 4$

Center: $x = 2$

Interval: $-4 < x - 2 < 4$ or $-2 < x < 6$

When $x = -2$, the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges.

When $x = 6$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

Therefore, the interval of convergence is $-2 \le x < 6$.

23. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-1)}{n+2}\right| = |x-1|$$

$R = 1$

Center: $x = 1$

Interval: $-1 < x - 1 < 1$ or $0 < x < 2$

When $x = 0$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $0 < x \le 2$.

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \Big/ \frac{(-1)^{n+1}(x-2)^n}{n2^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x-2}{2}\frac{n}{n+1}\right| = \left|\frac{x-2}{2}\right|$$

$$\left|\frac{x-2}{2}\right| < 1 \Rightarrow -2 < x - 2 < 2 \Rightarrow 0 < x < 4$$

At $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)(2^n)}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)}{n}, \text{ diverges.}$$

At $x = 4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}, \text{ converges.}$$

Interval of convergence: $0 < x \le 4$

25. $\sum_{n=1}^{\infty} \left(\frac{x-3}{3}\right)^{n-1}$ is geometric. It converges if

$$\left|\frac{x-3}{3}\right| < 1 \Rightarrow |x-3| < 3 \Rightarrow 0 < x < 6.$$

Interval convergence: $0 < x < 6$

26. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{(2n+1)}{(2n+3)}x^2\right| = |x^2|$$

$R = 1$

Interval: $-1 < x < 1$

When $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $-1 \le x \le 1$.

27. $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-2x)^n}{n+2}\cdot\frac{n+1}{n(-2x)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2x)(n+1)^2}{n(n+2)}\right| = 2|x|$$

$R = \frac{1}{2}$

Interval: $-\frac{1}{2} < x < \frac{1}{2}$

When $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the nth Term Test.

When $x = \frac{1}{2}$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

28. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

29. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

30. $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

31. $\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!} = \sum_{n=1}^{\infty}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+1}x\right| = |x|$$

Converges if $|x| < 1 \implies -1 < x < 1$.

At $x = \pm 1$, diverges.

Interval of convergence: $-1 < x < 1$

32. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}(x^{2n+1})$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 4\cdots(2n)(2n+2)x^{2n+3}}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)x^2}{(2n+3)}\right| = |x^2|$$

$R = 1$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty}\frac{1}{2n+1}$$

which diverges. Therefore, the interval of convergence is $-1 < x < 1$.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(-1)^{n+1}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4n+3)(x-3)}{4}\right| = \infty$$

$R = 0$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x+1)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\Big/\frac{(n)!(x+1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(x+1)}{2n+1}\right| = \frac{1}{2}|x+1|$$

Converges if $\frac{1}{2}|x+1| < 1 \implies -2 < x+1 < 2 \implies -3 < x < 1$.

At $x = 1$, $a_n = \dfrac{n!2^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \dfrac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)} > 1$, diverges.

At $x = -3$, $a_n = \dfrac{n!(-2)^n}{1\cdot 3\cdots(2n-1)} = (-1)^n\dfrac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}$, diverges.

Interval of convergence: $-3 < x < 1$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-c)^n}{c^n}\cdot\frac{c^{n-1}}{(x-c)^{n-1}}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}$ diverges.

When $x = 2c$, the series $\displaystyle\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $0 < x < 2c$.

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$, k is a positive integer.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^k x^{n+1}}{[k(n+1)]!}\Big/\frac{(n!)^k x^n}{(kn)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^k x}{(k+nk)(k-1+nk)\cdots(1+nk)}\right|$$

$$= \frac{|x|}{k^k}$$

Converges if $\dfrac{|x|}{k^k} < 1 \implies R = k^k$.

37. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{k}\right)^n$

Since the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}}\cdot\frac{nc^n}{(-1)^{n+1}(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-c)}{c(n+1)}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges. Therefore, the interval of convergence is $0 < x \le 2c$.

39. $\displaystyle\sum_{n=1}^{\infty}\frac{k(k+1)\cdots(k+n-1)x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{k(k+1)\cdots(k+n-1)(k+n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k+1)\cdots(k+n-1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(k+n)x}{n+1}\right| = |x|$$

$R = 1$

When $x = \pm 1$, the series diverges and the interval of convergence is $-1 < x < 1$.

$$\left[\frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n} \ge 1\right]$$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{n!(x-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-c)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!(x-c)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-c)}{2n+1}\right| = \frac{1}{2}|x-c|$$

$R = 2$

Interval: $-2 < x - c < 2$ or $c - 2 < x < c + 2$

The series diverges at the endpoints. Therefore, the interval of convergence is $c - 2 < x < c + 2$.

$$\left[\frac{n!(c+2-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{n!\,2^2}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} > 1\right]$$

41. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$

42. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n = \sum_{n=1}^{\infty}(-1)^n(n)x^{n-1}$

43. $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$

Replace n with $n - 1$.

44. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1}$

45. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n, -2 < x < 2$ (Geometric)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{2}\right)\left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty}\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty}\frac{2}{n+1}\left(\frac{x}{2}\right)^{n+1}, -2 \le x < 2$

46. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n5^n}, 0 < x \le 10$

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}, 0 < x < 10$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}, 0 < x < 10$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}, 0 \le x \le 10$

47. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \le 2$

(b) $f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, \ 0 < x < 2$

(c) $f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n(x-1)^{n-1}, 0 < x < 2$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \le x \le 2$

48. (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}, 1 < x \le 3$

(b) $f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}, 1 < x < 3$

(c) $f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}, 1 < x < 3$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}, 1 \le x \le 3$

49. $g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

$S_1 = 1, S_2 = 1.33$. Matches (c).

50. $g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$

$S_1 = 1, S_2 = 1.67$. Matches (a).

51. $g(3.1) = \sum_{n=0}^{\infty} \left(\frac{3.1}{3}\right)^n$ diverges. Matches (b).

52. $g(-2) = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$ alternating. Matches (d).

53. $g\left(\frac{1}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{1}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = 1 + \frac{1}{4} + \frac{1}{16} + \cdots$, converges

$S_1 = 1, S_2 = 1.25, S_3 = 1.3125$ Matches (b).

54. $g\left(-\frac{1}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(-\frac{1}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = 1 - \frac{1}{4} + \frac{1}{16} - \cdots$, converges

$S_1 = 1, S_2 = 0.75, S_3 = 0.8125$ Matches (c).

55. $g\left(\frac{9}{16}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{9}{16}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{9}{8}\right)^n$, diverges

$S_1 = 1, S_2 = \frac{17}{8}$ Matches (d).

56. $g\left(\frac{3}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{3}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$, converges

$S_1 = 1, S_2 = 1.75$ Matches (a).

57. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at c.

58. The set of all values of x for which the power series converges is the interval of convergence. If the power series converges for all x, then the radius of convergence is $R = \infty$. If the power series converges at only c, then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

59. A single point, an interval, or the entire real line.

60. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

61. Answers will vary.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $-1 \le x < 1$. At $x = -1$, the convergence is conditional because $\sum \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges for $-1 \le x \le 1$. At $x = \pm 1$, the convergence is absolute.

62. Many answers possible.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n$ Geometric: $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$ converges for $-1 < x \le 1$

(c) $\displaystyle\sum_{n=1}^{\infty}(2x+1)^n$ Geometric: $|2x+1| < 1 \Rightarrow -1 < x < 0$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n4^n}$ converges for $-2 \le x < 6$

63. (a) $\displaystyle f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty$

(See Exercise 29.)

$$g(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty$$

(b) $\displaystyle f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $\displaystyle g'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$$

(d) $f(x) = \sin x$ and $g(x) = \cos x$

64. (a) $\displaystyle f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}, -\infty < x < \infty$ (See Exercise 11)

(b) $\displaystyle f'(x) = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = f(x)$

(c) $\displaystyle f(x) = \sum_{n=1}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f(0) = 1$

(d) $f(x) = e^x$

65.
$$y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$$
$$y' = \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$
$$y'' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)x^{2n-1}}{(2n-1)!}$$
$$y'' + y = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} = 0$$

66.
$$y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!}$$
$$y' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!}$$
$$y'' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}$$
$$y'' + y = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} = 0$$

67.
$$y = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$$
$$y' = \sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$
$$y'' = \sum_{n=1}^{\infty}\frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!} = y$$
$$y'' - y = 0$$

68.
$$y = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!}$$
$$y' = \sum_{n=1}^{\infty}\frac{(2n-2)x^{2n-1}}{(2n-2)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$$
$$y'' = \sum_{n=1}^{\infty}\frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!} = y$$
$$y'' - y = 0$$

69.
$$y = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!} \qquad y' = \sum_{n=1}^{\infty}\frac{2nx^{2n-1}}{2^n n!} \qquad y'' = \sum_{n=1}^{\infty}\frac{2n(2n-1)x^{2n-2}}{2^n n!}$$
$$y'' - xy' - y = \sum_{n=1}^{\infty}\frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty}\frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}$$
$$= \sum_{n=1}^{\infty}\frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{2^n n!}$$
$$= \sum_{n=0}^{\infty}\left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!}\cdot\frac{2(n+1)}{2(n+1)}\right]$$
$$= \sum_{n=0}^{\infty}\frac{2(n+1)x^{2n}[(2n+1)-(2n+1)]}{2^{n+1}(n+1)!} = 0$$

70. $$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y'' = \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)}$$

$$y'' + x^2 y = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0$$

71. $$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$$

(a) $$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2k+2}}{2^{2k+2}[(k+1)!]^2}\cdot\frac{2^{2k}(k!)^2}{(-1)^k x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{(-1)x^2}{2^2(k+1)^2}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $$J_0 = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{4^k(k!)^2}$$

$$J_0' = \sum_{k=1}^{\infty}(-1)^k \frac{2kx^{2k-1}}{4^k(k!)^2} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k+2)x^{2k+1}}{4^{k+1}[(k+1)!]^2}$$

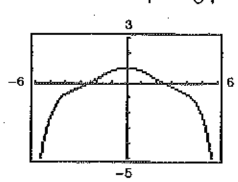
$$J_0'' = \sum_{k=1}^{\infty}(-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k(k!)^2} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k+2)(2k+1)x^{2k}}{4^{k+1}[(k+1)!]^2}$$

$$x^2J_0'' + xJ_0' + x^2J_0 = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+2}}{4^k(k!)^2}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2}}{4^k(k!)^2}\left[(-1)\frac{2(2k+1)}{4(k+1)} + (-1)\frac{2}{4(k+1)} + 1\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2}}{4^k(k!)^2}\left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4}\right] = 0$$

(c) $P_6(x) = 1 - \dfrac{x^2}{4} + \dfrac{x^4}{64} - \dfrac{x^6}{2304}$



(d) $$\int_0^1 J_0\,dx = \int_0^1 \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{4^k(k!)^2}\,dx$$

$$= \left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{4^k(k!)^2(2k+1)}\right]_0^1$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(k!)^2(2k+1)}$$

$$= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92$$

(integral is approximately 0.9197304101)

72. $$J_1(x) = x\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

(a) $$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2k+3}}{2^{2k+3}(k+1)!(k+2)!}\cdot\frac{2^{2k+1}k!(k+1)!}{(-1)^k x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{(-1)x^2}{2^2(k+2)(k+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

—CONTINUED—

72. —CONTINUED

(b) $J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1!)}$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k}}{2^{2k+1}k!(k+1)!}$$

$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k-1}}{2^{2k+1}k!(k+1)!}$$

$$x^2J_1'' + xJ_1' + (x^2-1)J_1 = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

$$+ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

$$= \left[\sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)(2k)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}k!(k+1)!}\right.$$

$$\left. - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right] + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}[(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

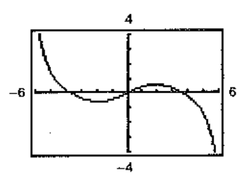
$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1}(k-1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}[(-1)+1]}{2^{2k+1}k!(k+1)!} = 0$$

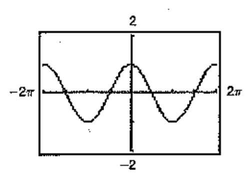
(c) $P_7(x) = \frac{x}{2} - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18{,}432}x^7$



(d) $J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}2(k+1)x^{2k+1}}{2^{2k+2}(k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k+1}}{2^{2k+1}k!(k+1)!}$

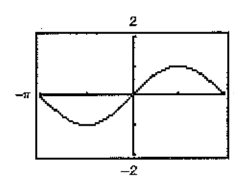
$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1}k!(k+1)!}$ **Note:** $J_0'(x) = -J_1(x)$

73. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$



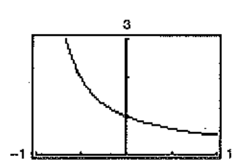
74. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$= \sin x$

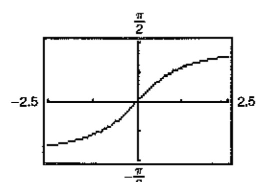


75. $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$ Geometric

$= \frac{1}{1-(-x)} = \frac{1}{1+x}$ for $-1 < x < 1$

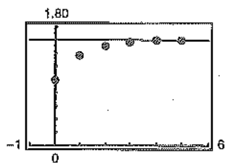


76. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, -1 \le x \le 1$



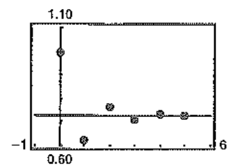
77. $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

(a) $\sum_{n=0}^{\infty}\left(\frac{3/4}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{3}{8}\right)^n = \frac{1}{1-(3/8)} = \frac{8}{5} = 1.6$



(b) $\sum_{n=0}^{\infty}\left(\frac{-3/4}{2}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{3}{8}\right)^n$

$= \frac{1}{1-(-3/8)} = \frac{8}{11} \approx 0.7272$



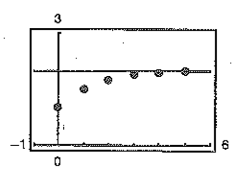
(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d) $\sum_{n=0}^{N}\left(\frac{3}{2}\right)^n > M$

M	10	100	1000	10,000
N	4	9	15	21

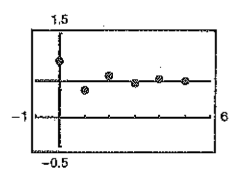
78. $\sum_{n=0}^{\infty}(3x)^n$ converges on $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) $x = \frac{1}{6}$: $\sum_{n=0}^{\infty}\left(3\left(\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-(1/2)} = 2$



(b) $x = -\frac{1}{6}$: $\sum_{n=0}^{\infty}\left(3\left(-\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$

$= \frac{1}{1+(1/2)} = \frac{2}{3}$



(c) The alternating series converges more rapidly. The partial sums in (a) approach the sum 2 from below. The partial sums in (b) alternate sides of the horizontal line $y = \frac{2}{3}$.

(d) $\sum_{n=0}^{N}\left(3 \cdot \frac{2}{3}\right)^n = \sum_{n=0}^{N} 2^n > M$

M	10	100	1000	10,000
N	3	6	9	13

79. False;

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n2^n}$$

converges for $x = 2$ but diverges for $x = -2$.

80. True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = 2$, then we know that it must converge on $(-2, 2]$.

81. True; the radius of convergence is $R = 1$ for both series.

82. True

$$\int_0^1 f(x)\,dx = \int_0^1\left(\sum_{n=0}^{\infty} a_n x^n\right)dx = \left[\sum_{n=0}^{\infty}\frac{a_n x^{n+1}}{n+1}\right]_0^1 = \sum_{n=0}^{\infty}\frac{a_n}{n+1}$$

83. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1+p)!}{(n+1)!(n+1+q)!}x^{n+1} \Big/ \frac{(n+p)!}{n!(n+q)!}x^n\right| = \lim_{n\to\infty}\left|\frac{(n+1+p)x}{(n+1)(n+1+q)}\right| = 0$

Thus, the series converges for all x: $R = \infty$.

84. (a) $g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$

$$S_{2n} = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots + x^{2n} + 2x^{2n+1}$$
$$= (1 + x^2 + x^4 + \cdots + x^{2n}) + 2x(1 + x^2 + x^4 + \cdots + x^{2n})$$

$$\lim_{n\to\infty} S_{2n} = \sum_{n=0}^{\infty} x^{2n} + 2x\sum_{n=0}^{\infty} x^{2n}$$

Since each series is geometric, $R = 1$.

(b) For $|x| < 1$, $g(x) = \frac{1}{1-x^2} + 2x\frac{1}{1-x^2} = \frac{1+2x}{1-x^2}$.

85. (a) $f(x) = \sum_{n=0}^{\infty} c_n x^n,\ c_{n+3} = c_n$

$$= c_0 + c_1x + c_2x^2 + c_0x^3 + c_1x^4 + c_2x^5 + c_0x^6 + \cdots$$

$$S_{3n} = c_0(1 + x^3 + \cdots + x^{3n}) + c_1x(1 + x^3 + \cdots + x^{3n}) + c_2x^2(1 + x^3 + \cdots + x^{3n})$$

$$\lim_{n\to\infty} S_{3n} = c_0\sum_{n=0}^{\infty} x^{3n} + c_1x\sum_{n=0}^{\infty} x^{3n} + c_2x^2\sum_{n=0}^{\infty} x^{3n}$$

Each series is geometric, $R = 1$, and the interval of convergence is $(-1, 1)$.

(b) For $|x| < 1$, $f(x) = c_0\frac{1}{1-x^3} + c_1x\frac{1}{1-x^3} + c_2x^2\frac{1}{1-x^3} = \frac{c_0 + c_1x + c_2x^2}{1-x^3}$.

86. For the series $\sum c_n x^n$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_nx^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}x\right| < 1 \Rightarrow |x| < \left|\frac{c_n}{c_{n+1}}\right| = R$$

For the series $\sum c_n x^{2n}$,

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{2n+2}}{c_nx^{2n}}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}x^2\right| < 1 \Rightarrow |x^2| < \left|\frac{c_n}{c_{n+1}}\right| = R \Rightarrow |x| < \sqrt{R}.$$

87. At $x_0 + R$, the series is $\sum_{n=0}^{\infty} c_n(x_0 - x_0 - R)^n = \sum_{n=0}^{\infty} c_n(-R)^n$ which converges.

At $x_0 - R$, the series is $\sum_{n=0}^{\infty} c_nR^n$, which diverges.

Hence, at $x_0 + R$, we have $\sum_{n=0}^{\infty} c_n(-R)^n$ converges, $\sum_{n=0}^{\infty} |c_n(-R)^n| = \sum_{n=0}^{\infty} c_nR^n$ diverges.

$\Rightarrow$ The series converges conditionally.

Section 9.9 Representation of Functions by Power Series

1. (a) $$\frac{1}{2-x} = \frac{1/2}{1-(x/2)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b) $$\begin{array}{r l} & \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots \\ 2 - x \,\big) & 1 \\ & \underline{1 - \frac{x}{2}} \\ & \quad \frac{x}{2} \\ & \quad \underline{\frac{x}{2} - \frac{x^2}{4}} \\ & \qquad \frac{x^2}{4} \\ & \qquad \underline{\frac{x^2}{4} - \frac{x^3}{8}} \\ & \qquad\quad \frac{x^3}{8} \\ & \qquad\quad \underline{\frac{x^3}{8} - \frac{x^4}{16}} \\ & \qquad\qquad \vdots \end{array}$$

2. (a) $$f(x) = \frac{4}{5-x} = \frac{4/5}{1-x/5} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

(b) $$\begin{array}{r l} & \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\ 5 - x \,\big) & 4 \\ & \underline{4 - \frac{4}{5}x} \\ & \quad \frac{4}{5}x \\ & \quad \underline{\frac{4}{5}x - \frac{4}{25}x^2} \\ & \qquad \frac{4}{25}x^2 \\ & \qquad \underline{\frac{4}{25}x^2 - \frac{4x^3}{125}} \\ & \qquad\quad \frac{4x^3}{125} \\ & \qquad\quad \underline{\frac{4x^3}{125} - \frac{4x^4}{625}} \\ & \qquad\qquad \vdots \end{array}$$

3. (a) $$\frac{1}{2+x} = \frac{1/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b) $$\begin{array}{r l} & \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\ 2 + x \,\big) & 1 \\ & \underline{1 + \frac{x}{2}} \\ & \quad -\frac{x}{2} \\ & \quad \underline{-\frac{x}{2} - \frac{x^2}{4}} \\ & \qquad \frac{x^2}{4} \\ & \qquad \underline{\frac{x^2}{4} + \frac{x^3}{8}} \\ & \qquad\quad -\frac{x^3}{8} \\ & \qquad\quad \underline{-\frac{x^3}{8} - \frac{x^4}{16}} \\ & \qquad\qquad \vdots \end{array}$$

4. (a) $$\frac{1}{1+x} = \frac{1}{1-(-x)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series converges on $(-1, 1)$.

(b) $$\begin{array}{r l} & 1 - x + x^2 - x^3 + \cdots \\ 1 + x \,\big) & 1 \\ & \underline{1 + x} \\ & \quad -x \\ & \quad \underline{-x - x^2} \\ & \qquad x^2 \\ & \qquad \underline{x^2 + x^3} \\ & \qquad\quad -x^3 \\ & \qquad\quad \underline{-x^3 - x^4} \\ & \qquad\qquad \vdots \end{array}$$

5. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (-1/3)(x-5)$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3}\left[-\frac{1}{3}(x-5)\right]^n$$

$$= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}},\ |x-5| < 3 \text{ or } 2 < x < 8.$$

6. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for $f(x)$ is given by

$$\frac{4}{5-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7}\left(\frac{1}{7}(x+2)\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}.$$

$|x+2| < 7$ or $-9 < x < 5$

7. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n$$

$$= -3\sum_{n=0}^{\infty}(2x)^n,\ |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

8. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = (-2/3)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left[-\frac{2}{3}(x-2)\right]^n,$$

$$= \sum_{n=0}^{\infty} \frac{(-2)^n(x-2)^n}{3^n},$$

$$|x-2| < \frac{3}{2} \text{ or } \frac{1}{2} < x < \frac{7}{2}.$$

9. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{-1}{11-2(x+3)}$$

$$= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{11}\right)\left[\frac{2}{11}(x+3)\right]^n$$

$$= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}},$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

10. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that $a = -1/5$ and $r = (2/5)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)\left(\frac{2}{5}x\right)^n = -\sum_{n=0}^{\infty} \frac{2^n x^n}{5^{n+1}},$$

$$|x| < \frac{5}{2} \text{ or } -\frac{5}{2} < x < \frac{5}{2}.$$

11. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

$|x| < 2$ or $-2 < x < 2$.

12. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1/2$ and $r = (-3/8)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{4}{3x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left[-\frac{3}{8}(x-2)\right]^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-3)^n(x-2)^n}{8^n},$$

$$|x-2| < \frac{8}{3} \text{ or } -\frac{2}{3} < x < \frac{14}{3}.$$

13. $\dfrac{3x}{x^2 + x - 2} = \dfrac{2}{x+2} + \dfrac{1}{x-1} = \dfrac{2}{2+x} + \dfrac{1}{-1+x} = \dfrac{1}{1+(1/2)x} + \dfrac{-1}{1-x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right| = |x|.$$

14. $\dfrac{4x-7}{2x^2+3x-2} = \dfrac{3}{x+2} - \dfrac{2}{2x-1} = \dfrac{3}{2+x} - \dfrac{2}{-1+2x} = \dfrac{3/2}{1+(1/2)x} + \dfrac{2}{1-2x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{4x-7}{2x^2+3x-2} = \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}2(2x)^n = \sum_{n=0}^{\infty}\left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1}\right]x^n,\ |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

15. $\dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1-x^2} = \sum_{n=0}^{\infty}x^n + \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(1+(-1)^n)x^n = \sum_{n=0}^{\infty}2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2x^{2n+2}}{2x^{2n}}\right| = |x^2|$.

16. First finding the power series for $4/(4+x)$, we have

$$\frac{1}{1+(1/4)x} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n}$$

Now replace x with x^2.

$$\frac{4}{4+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is $|x^2| < 4$ or $-2 < x < 2$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n x^{2n}}\right| = \left|-\frac{x^2}{4}\right| = \frac{|x^2|}{4}.$$

17. $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty}(-1)^{2n}x^n = \sum_{n=0}^{\infty}x^n$$

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}[(-1)^n + 1]x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty}2x^{2n},\ -1 < x < 1 \text{ (See Exercise 15.)}$$

18. $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1+x)} - \dfrac{1}{2(1-x)} = \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n - \dfrac{1}{2}\sum_{n=0}^{\infty} x^n$

$$= \frac{1}{2}\sum_{n=0}^{\infty}[(-1)^n - 1]x^n = \frac{1}{2}[0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2)x^{2n+1} = -\sum_{n=0}^{\infty} x^{2n+1}, 1 < x < 1$$

19. By taking the first derivative, we have $\dfrac{d}{dx}\left[\dfrac{1}{x+1}\right] = \dfrac{-1}{(x+1)^2}$. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n nx^{n-1}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n, -1 < x < 1.$$

20. By taking the second derivative, we have $\dfrac{d^2}{dx^2}\left[\dfrac{1}{x+1}\right] = \dfrac{2}{(x+1)^3}$. Therefore,

$$\frac{2}{(x+1)^3} = \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$

$$= \frac{d}{dx}\left[\sum_{n=1}^{\infty}(-1)^n nx^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n-1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n, -1 < x < 1.$$

21. By integrating, we have $\displaystyle\int \frac{1}{x+1}\,dx = \ln(x+1)$. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}, -1 < x \le 1.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}, -1 < x \le 1.$$

22. By integrating, we have

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x}\,dx = -\ln(1-x) + C_2.$$

$f(x) = \ln(1 - x^2) = \ln(1+x) - [-\ln(1-x)]$. Therefore,

$$\ln(1 - x^2) = \int \frac{1}{1+x}\,dx - \int \frac{1}{1-x}\,dx$$

$$= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx - \int\left[\sum_{n=0}^{\infty} x^n\right]dx = \left[C_1 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}\right] - \left[C_2 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right]$$

$$= C + \sum_{n=0}^{\infty}\frac{[(-1)^n - 1]x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n+1}$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1 - x^2) = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1}, -1 < x < 1.$$

23. $\dfrac{1}{x^2+1} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}, -1 < x < 1$

24. $\dfrac{2x}{x^2+1} = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n}$ (See Exercise 23.)

$= \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$

Since $\dfrac{d}{dx}(\ln(x^2+1)) = \dfrac{2x}{x^2+1}$, we have

$$\ln(x^2+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, \ -1 \le x \le 1.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, \ -1 \le x \le 1.$$

25. Since, $\dfrac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$, we have $\dfrac{1}{4x^2+1} = \sum_{n=0}^{\infty}(-1)^n(4x^2)^n = \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n(2x)^{2n}, \ -\dfrac{1}{2} < x < \dfrac{1}{2}.$

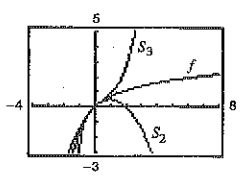
26. Since $\displaystyle\int\frac{1}{4x^2+1}\,dx = \frac{1}{2}\arctan(2x)$, we can use the result of Exercise 25 to obtain

$$\arctan(2x) = 2\int\frac{1}{4x^2+1}\,dx = 2\int\left[\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\right]dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \ -\frac{1}{2} < x \le \frac{1}{2}.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \ -\frac{1}{2} < x \le \frac{1}{2}.$$

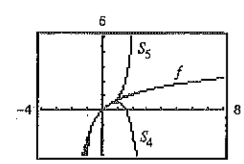
27. $x - \dfrac{x^2}{2} \le \ln(x+1) \le x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \dfrac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.182	0.336	0.470	0.588	0.693
$x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

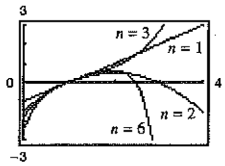
28. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} \le \ln(x+1)$

$\le x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x+1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

29. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n} = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

(a) 

(b) From Example 4,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1}$$

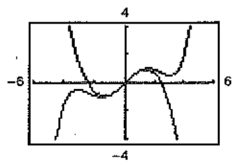
$$= \ln x, \quad 0 < x \le 2, \quad R = 1.$$

(c) $x = 0.5$:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1/2)^n}{n} = \sum_{n=1}^{\infty}\frac{-(1/2)^n}{n} \approx -0.693147$$

(d) This is an approximation of $\ln\left(\frac{1}{2}\right)$. The error is approximately 0. [The error is less than the first omitted term, $1/(51 \cdot 2^{51}) \approx 8.7 \times 10^{-18}$.]

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(a) 

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x, R = \infty$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(1/2)^{2n+1}}{(2n+1)!} \approx 0.4794255386$

(d) This is an approximation of $\sin\left(\frac{1}{2}\right)$.

The error is approximately 0.

31. $g(x) = x$ line

Matches (c)

32. $g(x) = x - \dfrac{x^3}{3}$, cubic with 3 zeros.

Matches (d)

33. $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$

Matches (a)

34. $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$,

Matches (b)

In Exercises 35-38, $\arctan x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

35. $\displaystyle\arctan\frac{1}{4} = \sum_{n=0}^{\infty}(-1)^n\frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \cdots$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan\frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

36.

$$\arctan x^2 = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\int_0^{3/4}\arctan x^2\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{(3/4)^{4n+3}}{(4n+3)(2n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344{,}064} + \frac{177{,}147}{230{,}686{,}720}$$

Since $177{,}147/230{,}686{,}720 < 0.001$, we can approximate the series by its first two terms: 0.134.

37. $$\frac{\arctan x^2}{x} = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^2\frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(4n+2)(2n+1)} + C \text{ (Note: } C = 0)$$

$$\int_0^{1/2}\frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2}\frac{\arctan x^2}{x}\,dx \approx 0.125.$

38. $$x^2\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{2n+1}$$

$$\int x^2\arctan x\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2}x^2\arctan x\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2}x^2\arctan x\,dx \approx 0.016.$

In Exercises 39–43, use $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$, $|x| < 1$.

39. $$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}x^n\right] = \sum_{n=1}^{\infty}nx^{n-1}, |x| < 1$$

40. $$\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty}nx^{n-1} = \sum_{n=1}^{\infty}nx^n, |x| < 1$$

41. $$\frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2}$$

$$= \sum_{n=1}^{\infty}n(x^{n-1} + x^n), \quad |x| < 1$$

$$= \sum_{n=0}^{\infty}(2n+1)x^n, \quad |x| < 1$$

42. $$\frac{x(1+x)}{(1-x)^2} = x\sum_{n=0}^{\infty}(2n+1)x^n = \sum_{n=0}^{\infty}(2n+1)x^{n+1}, |x| < 1$$

(See Exercise 41.)

43. $$P(n) = \left(\frac{1}{2}\right)^n$$

$$E(n) = \sum_{n=1}^{\infty}nP(n) = \sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{2}\frac{1}{[1-(1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

In Exercise 44, $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$, $|x|1$.

44. (a) $$\frac{1}{3}\sum_{n=1}^{\infty}n\left(\frac{2}{3}\right)^n = \frac{2}{9}\sum_{n=1}^{\infty}n\left(\frac{2}{3}\right)^{n-1} = \frac{2}{9}\frac{1}{[1-(2/3)]^2} = 2$$

(b) $$\frac{1}{10}\sum_{n=1}^{\infty}n\left(\frac{9}{10}\right)^n = \frac{9}{100}\sum_{n=1}^{\infty}n\left(\frac{9}{10}\right)^{n-1} = \frac{9}{100}\cdot\frac{1}{[1-(9/10)]^2} = 9$$

45. Replace x with $(-x)$.

46. Replace x with x^2.

47. Replace x with $(-x)$ and multiply the series by 5.

48. Integrate the series and multiply by (-1).

49. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan \theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \tan \theta$$

$$\frac{x + y}{1 - xy} = \tan \theta$$

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \theta.$$ Therefore, $\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right)$ for $xy \neq 1$.

50. (a) From Exercise 49, we have

$$\arctan \frac{120}{119} - \arctan \frac{1}{239} = \arctan \frac{120}{119} + \arctan\left(-\frac{1}{239}\right)$$

$$= \arctan\left[\frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)}\right] = \arctan\left(\frac{28{,}561}{28{,}561}\right) = \arctan 1 = \frac{\pi}{4}$$

(b) $2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan\left[\frac{2(1/5)}{1 - (1/5)^2}\right] = \arctan \frac{10}{24} = \arctan \frac{5}{12}$

$$4 \arctan \frac{1}{5} = 2 \arctan \frac{1}{5} + 2 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan\left[\frac{2(5/12)}{1 - (5/12)^2}\right] = \arctan \frac{120}{119}$$

$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$ (see part (a).)

51. (a) $2 \arctan \frac{1}{2} = \arctan \frac{1}{2} + \arctan \frac{1}{2} = \arctan\left[\frac{\frac{1}{2} + \frac{1}{2}}{1 - (1/2)^2}\right] = \arctan \frac{4}{3}$

$$2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \arctan \frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3) - (1/7)}{1 + (4/3)(1/7)}\right] = \arctan \frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

(b) $\pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$

52. (a) $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan\left[\frac{(1/2) + (1/3)}{1 - (1/2)(1/3)}\right] = \arctan\left(\frac{5/6}{5/6}\right) = \frac{\pi}{4}$

(b) $\pi = 4\left[\arctan \frac{1}{2} + \arctan \frac{1}{3}\right]$

$$= 4\left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7}\right] + 4\left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right] \approx 4(0.4635) + 4(0.3217) \approx 3.14$$

53. From Exercise 21, we have

$$\ln(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Thus, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1/2)^n}{n}$

$$= \ln\left(\frac{1}{2} + 1\right) = \ln \frac{3}{2} \approx 0.4055.$$

54. From Exercise 53, we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1/3)^n}{n}$$

$$= \ln\left(\frac{1}{3} + 1\right) = \ln \frac{4}{3} \approx 0.2877.$$

55. From Exercise 53, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{5^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2/5)^n}{n}$$

$$= \ln\left(\frac{2}{5}+1\right) = \ln\frac{7}{5} \approx 0.3365.$$

56. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}$$

$$= \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

57. From Exercise 56, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{2} \approx 0.4636.$$

58. From Exercise 56, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(1/3)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{3} \approx 0.3218.$$

59. $f(x) = \arctan x$ is an odd function (symmetric to the origin).

60. The approximations of degree 3, 7, 11, . . . , ($4n-1$, $n = 1, 2, \ldots$) have relative extrema.

61. The series in Exercise 56 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

62. Because $\frac{d}{dx}\left[\sum_{n=0}^{\infty}a_n x^n\right] = \sum_{n=1}^{\infty}na_n x^{n-1}$, the radius of convergence is the same, 3.

63. Because the first series is the derivative of the second series, the second series converges for $|x+1| < 4$ (and perhaps at the endpoints, $x = 3$ and $x = -5$.)

64. Using a graphing utility, you obtain the following partial sums for the left hand side. Note that $1/\pi \approx 0.3183098862$.

$n = 0$: $S_0 \approx 0.3183098784$

$n = 1$: $S_1 \approx 0.3183098862$

65. $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)}$

From Example 5 we have $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)}\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n(1/\sqrt{3})^{2n+1}}{2n+1}$$

$$= \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$= \sqrt{3}\left(\frac{\pi}{6}\right) \approx 0.9068997$$

66. $$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{3^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(\pi/3)^{2n+1}}{(2n+1)!}$$

$$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866025$$

Section 9.10 Taylor and Maclaurin Series

1. For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}.$$

2. For $c = 0$, we have:

$$f(x) = e^{3x}$$
$$f^{(n)}(x) = 3^n e^{3x} \Longrightarrow f^{(n)}(0) = 3^n$$
$$e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}.$$

3. For $c = \pi/4$, we have:

$$f(x) = \cos(x) \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f'(x) = -\sin(x) \qquad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, we have:

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!}$$
$$= \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \cdots\right]$$
$$= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^n}{n!}.$$

[**Note:** $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \ldots$]

4. For $c = \pi/4$, we have:

$$f(x) = \sin x \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore we have:

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!}$$
$$= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \cdots\right]$$
$$= \frac{\sqrt{2}}{2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}[x - (\pi/4)]^{n+1}}{(n+1)!} + 1\right\}.$$

5. For $c = 1$, we have,

$$f(x) = \ln x \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \qquad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}\\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \cdots\\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.\end{aligned}$$

6. For $c = 1$, we have:

$$f(x) = e^x$$

$$f^{(n)}(x) = e^x \Longrightarrow f^{(n)}(1) = e$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.$$

7. For $c = 0$, we have:

$$f(x) = \sin 2x \qquad f(0) = 0$$

$$f'(x) = 2\cos 2x \qquad f'(0) = 2$$

$$f''(x) = -4\sin 2x \qquad f''(0) = 0$$

$$f'''(x) = -8\cos 2x \qquad f'''(0) = -8$$

$$f^{(4)}(x) = 16\sin 2x \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 32\cos 2x \qquad f^{(5)}(0) = 32$$

$$f^{(6)}(x) = -64\sin 2x \qquad f^{(6)}(0) = 0$$

$$f^{(7)}(x) = -128\cos 2x \qquad f^{(7)}(0) = -128$$

and so on. Therefore, we have:

$$\begin{aligned}\sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots\\ &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}.\end{aligned}$$

8. For $c = 0$, we have:

$$f(x) = \ln(x^2+1) \qquad f(0) = 0$$

$$f'(x) = \frac{2x}{x^2+1} \qquad f'(0) = 0$$

$$f''(x) = \frac{2-2x^2}{(x^2+1)^2} \qquad f''(0) = 2$$

$$f'''(x) = \frac{4x(x^2-3)}{(x^2+1)^3} \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \frac{12(-x^4+6x^2-1)}{(x^2+1)^4} \qquad f^{(4)}(0) = -12$$

$$f^{(5)}(x) = \frac{48x(x^4-10x^2+5)}{(x^2+1)^5} \qquad f^{(5)}(0) = 0$$

$$f^{(6)}(x) = \frac{-240(5x^6-15x^4+15x^2-1)}{(x^2+1)^6} \qquad f^{(6)}(0) = 240$$

and so on. Therefore, we have:

$$\ln(x^2+1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}.$$

9. For $c = 0$, we have:

$$f(x) = \sec(x) \qquad f(0) = 1$$

$$f'(x) = \sec(x)\tan(x) \qquad f'(0) = 0$$

$$f''(x) = \sec^3(x) + \sec(x)\tan^2(x) \qquad f''(0) = 1$$

$$f'''(x) = 5\sec^3(x)\tan(x) + \sec(x)\tan^3(x) \qquad f'''(0) = 0$$

$$f^{(4)}(x) = 5\sec^5(x) + 18\sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) \qquad f^{(4)}(0) = 5$$

$$\sec(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots.$$

10. For $c = 0$, we have;

$$f(x) = \tan(x) \qquad f(0) = 0$$

$$f'(x) = \sec^2(x) \qquad f'(0) = 1$$

$$f''(x) = 2\sec^2(x)\tan(x) \qquad f''(0) = 0$$

$$f'''(x) = 2[\sec^4(x) + 2\sec^2(x)\tan^2(x)] \qquad f'''(0) = 2$$

$$f^{(4)}(x) = 8[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)] \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 8[2\sec^6(x) + 11\sec^4(x)\tan^2(x) + 2\sec^2(x)\tan^4(x)] \qquad f^{(5)}(0) = 16$$

$$\tan(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots.$$

11. The Maclaurin series for $f(x) = \cos x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, we have $|f^{(n+1)}(z)| \le 1$ for all z. Hence by Taylor's Theorem,

$$0 \le |R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $R_n(x) \to 0$ as $n \to \infty$. Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all x.

12. The Maclaurin series for $f(x) = e^{-2x}$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. Hence, by Taylor's Theorem,

$$0 \le |Rn(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{(-2)^{n+1}e^{-2z}}{(n+1)!}x^{n+1}\right|.$$

Since $\displaystyle\lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)!}\right| = 0$, it follows that $R_n(x) \to 0$ as $n \to \infty$.

Hence, the Maclaurin Series for e^{-2x} converges to e^{-2x} for all x.

13. The Maclaurin series for $f(x) = \sinh x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

$f^{(n+1)}(x) = \sinh x$ (or $\cosh x$). For fixed x,

$$0 \le |R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{\sinh(z)}{(n+1)!}x^{n+1}\right| \to 0 \text{ as } n \to \infty.$$

(The argument is the same if $f^{(n+1)}(x) = \cosh x$). Hence, the Maclaurin series for $\sinh x$ converges to $\sinh x$ for all x.

14. The Maclaurin series for $f(x) = \cosh x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

$f^{(n+1)}(x) = \sinh x$ (or $\cosh x$). For fixed x,

$$0 \le |R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{\sinh(z)}{(n+1)!}x^{n+1}\right| \to 0 \text{ as } n \to \infty.$$

(The argument is the same if $f^{(n+1)}(x) = \cosh x$). Hence, the Maclaurin series for $\cosh x$ converges to $\cosh x$ for all x.

15. Since $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$(1+x)^{-2} = 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{4!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n.$$

16. Since $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\left[1+(-x)\right]^{-1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \cdots$$
$$= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}.$$

17. $\dfrac{1}{\sqrt{4+x^2}} = \left(\dfrac{1}{2}\right)\left[1+\left(\dfrac{x}{2}\right)^2\right]^{-1/2}$ and since $(1+x)^{-1/2} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5\cdots(2n-1)x^n}{2^n n!}$, we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdots(2n-1)(x/2)^{2n}}{2^n\,n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdots(2n-1)x^{2n}}{2^{3n+1}n!}.$$

18. $$(1+x)^{1/4} = 1 + \frac{1}{4}x + \frac{(1/4)(-3/4)}{2!}x^2 + \frac{(1/4)(-3/4)(-7/4)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{4}x - \frac{3}{4^2 2!}x^2 + \frac{3\cdot 7}{4^3 3!}x^3 - \frac{3\cdot 7\cdot 11}{4^4 4!}x^4 + \cdots$$

$$= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-5)}{4^n n!}x^n$$

19. Since $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\,1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n\,n!}$

we have $(1+x^2)^{1/2} = 1 + \dfrac{x^2}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\,1\cdot 3\cdot 5\cdots(2n-3)x^{2n}}{2^n\,n!}$.

20. Since $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\,1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n\,n!}$

we have $(1+x^3)^{1/2} = 1 + \dfrac{x^3}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\,1\cdot 3\cdot 5\cdots(2n-3)x^{3n}}{2^n\,n!}$.

21. $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{x^2/2} = \sum_{n=0}^{\infty}\frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

22. $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-3x} = \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots$$

23. $$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 3x = \sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{(2n+1)!}$$

24. $$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 4x = \sum_{n=0}^{\infty}\frac{(-1)^n(4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n}x^{2n}}{(2n)!}$$

$$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots$$

25. $$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos x^{3/2} = \sum_{n=0}^{\infty}\frac{(-1)^n(x^{3/2})^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{(2n)!}$$

$$= 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

26. $$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$2\sin x^3 = 2\sum_{n=0}^{\infty}\frac{(-1)^n(x^3)^{2n+1}}{(2n+1)!}$$

$$= 2\left[x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right]$$

$$= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \cdots$$

27. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$

$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$

$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

28. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$

$2 \cos h(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!}$

29. $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - \cdots\right] = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n}}{(2n)!}\right]$

30. The formula for the binomial series gives $(1 + x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, which implies that

$(1 + x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!}$

$\ln\left(x + \sqrt{x^2+1}\right) = \int \frac{1}{\sqrt{x^2+1}}\,dx$

$= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \ldots (2n-1)x^{2n+1}}{2^n(2n+1)n!}$

$= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$

31. $x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$

$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

32. $x \cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$

33. $\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$

34. $\frac{\arcsin x}{x} = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} \cdot \frac{1}{x}$

$= \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{(2^n n!)^2(2n+1)}, x \neq 0$

35. $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$

$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$

$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$

$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$

36. $e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots$ (See Exercise 35.)

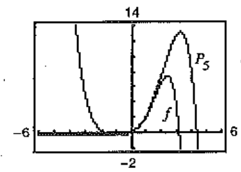
$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

37. $f(x) = e^x \sin x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

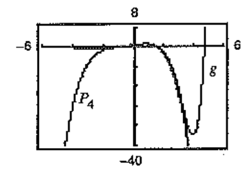


38. $g(x) = e^x \cos x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

$$= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

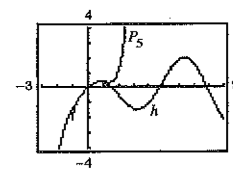


39. $h(x) = \cos x \ln(1 + x)$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$

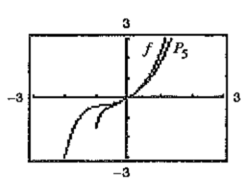


40. $f(x) = e^x \ln(1 + x)$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

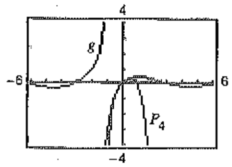
$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$$



41. $g(x) = \dfrac{\sin x}{1 + x}$. Divide the series for $\sin x$ by $(1 + x)$.

$$
\begin{array}{l}
 x - x^2 + \dfrac{5x^3}{6} - \dfrac{5x^4}{6} + \\
1 + x \overline{\smash{)}\, x + 0x^2 - \dfrac{x^3}{6} + 0x^4 + \dfrac{x^5}{120} + \cdots} \\
 \underline{x + x^2} \\
 -x^2 - \dfrac{x^3}{6} \\
 \underline{-x^2 - x^3} \\
 \dfrac{5x^3}{6} + 0x^4 \\
 \underline{\dfrac{5x^3}{6} + \dfrac{5x^4}{6}} \\
 -\dfrac{5x^4}{6} + \dfrac{x^5}{120} \\
 \underline{-\dfrac{5x^4}{6} - \dfrac{5x^5}{6}} \\
 \vdots
\end{array}
$$

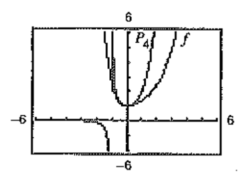
$g(x) = x - x^2 + \dfrac{5x^3}{6} - \dfrac{5x^4}{6} + \cdots$



42. $f(x) = \dfrac{e^x}{1 + x}$. Divide the series for e^x by $(1 + x)$.

$$
\begin{array}{l}
 1 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{3x^4}{8} + \cdots \\
1 + x \overline{\smash{)}\, 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + \cdots} \\
 \underline{1 + x} \\
 0 + \dfrac{x^2}{2} + \dfrac{x^3}{6} \\
 \underline{\dfrac{x^2}{2} + \dfrac{x^3}{2}} \\
 -\dfrac{x^3}{3} + \dfrac{x^4}{24} \\
 \underline{-\dfrac{x^3}{3} - \dfrac{x^4}{3}} \\
 \dfrac{3x^4}{8} + \dfrac{x^5}{120} \\
 \underline{\dfrac{3x^4}{8} + \dfrac{3x^5}{8}} \\
 \vdots
\end{array}
$$

$f(x) = 1 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{3x^4}{8} - \cdots$



43. $y = x^2 - \dfrac{x^4}{3!} = x\left(x - \dfrac{x^3}{3!}\right)$

$f(x) = x \sin x$

Matches (c)

44. $y = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} = x\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\right)$

$f(x) = x \cos x$

Matches (d)

45. $y = x + x^2 + \dfrac{x^3}{2!} = x\left(1 + x + \dfrac{x^2}{2!}\right)$

$f(x) = xe^x$

Matches (a)

46. $y = x^2 - x^3 + x^4 = x^2(1 - x + x^2)$

$f(x) = x^2 \dfrac{1}{1 + x}$

Matches (b)

47. $$\int_0^x (e^{-t^2} - 1)\,dt = \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right) - 1\right] dt$$

$$= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!}\right] dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}$$

48. $$\int_0^x \sqrt{1+t^3}\,dt = \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n}}{2^n n!}\right] dt$$

$$= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n+1}}{(3n+1)2^n n!}\right]_0^x$$

$$= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^{3n+1}}{(3n+1)2^n n!}$$

49. Since $\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots, \quad (0 < x \le 2)$

we have $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931.$ (10,001 terms)

50. Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415.$ (4 terms)

51. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$

we have $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891.$ (12 terms)

52. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$, we have $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

and $\frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321.$ (6 terms)

53. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$\frac{1 - \cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

we have $\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n+2)!} = 0.$

54. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

we have $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$

55. $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

Since $1/(7 \cdot 7!) < 0.0001$, we need three terms:

$$\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \approx 0.9461. \quad \text{(using three nonzero terms)}$$

Note: We are using $\displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$

56. $\displaystyle\int_0^{1/2} \frac{\arctan x}{x}\,dx = \int_0^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right) dx = \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$

Since $1/(9^2 2^9) < 0.0001$, we have

$$\int_0^{1/2} \frac{\arctan x}{x}\,dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right) \approx 0.4872.$$

Note: We are using $\displaystyle\lim_{x \to 0^+} \frac{\arctan x}{x} = 1.$

57. $\displaystyle\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots\right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots\right]_{0.1}^{0.3}$

Since $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, we need two terms.

$$\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4)\right] \approx 0.201.$$

58. $\displaystyle\int_0^{1/4} x \ln(x + 1)\,dx = \int_0^{1/4} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4 \cdot 2} + \frac{x^5}{5 \cdot 3} - \frac{x^6}{6 \cdot 4} + \cdots\right]_0^{1/4}$

Since $\dfrac{(1/4)^5}{15} < 0.0001$, $\displaystyle\int_0^{1/4} x \ln(x + 1)\,dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$

59. $\displaystyle\int_0^{\pi/2} \sqrt{x} \cos x\,dx = \int_0^{\pi/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!}\right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!}\right]_0^{\pi/2}$

Since $2(\pi/2)^{23/2}/(23 \cdot 10!) < 0.0001$, we need five terms.

$$\int_0^1 \sqrt{x} \cos x\,dx = 2\left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10{,}800} + \frac{(\pi/2)^{19/2}}{766{,}080}\right] \approx 0.7040.$$

60. $\displaystyle\int_{0.5}^1 \cos \sqrt{x}\,dx = \int_{0.5}^1 \left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right) dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^1$

Since $\frac{1}{201{,}600}(1 - 0.5^5) < 0.0001$, we have

$$\int_{0.5}^1 \cos \sqrt{x}\,dx \approx \left[(1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201{,}600}(1 - 0.5)^5\right] \approx 0.3243.$$

61. From Exercise 21, we have

$$\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_0^1 = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left[1 - \frac{1}{2 \cdot 1 \cdot 3} + \frac{1}{2^2 \cdot 2! \cdot 5} - \frac{1}{2^3 \cdot 3! \cdot 7}\right] \approx 0.3414.$$

62. From Exercise 21, we have

$$\frac{1}{\sqrt{2\pi}}\int_1^2 e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_1^2 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_1^2$$

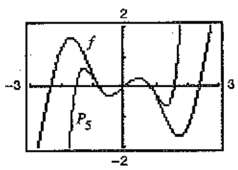
$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2^{n+1}-1)}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left[1 - \frac{7}{2\cdot 1\cdot 3} + \frac{31}{2^2\cdot 2!\cdot 5} - \frac{127}{2^3\cdot 3!\cdot 7} + \frac{511}{2^4\cdot 4!\cdot 9} - \frac{2047}{2^5\cdot 5!\cdot 11}\right.$$

$$\left. + \frac{8191}{2^6\cdot 6!\cdot 13} - \frac{32{,}767}{2^7\cdot 7!\cdot 15} + \frac{131{,}071}{2^8\cdot 8!\cdot 17} - \frac{524{,}287}{2^9\cdot 9!\cdot 19}\right] \approx 0.1359.$$

63. $f(x) = x\cos 2x = \sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

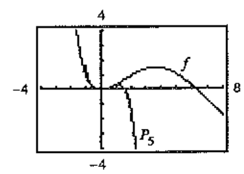
$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$



The polynomial is a reasonable approximation on the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

64. $f(x) = \sin\frac{x}{2}\ln(1+x)$

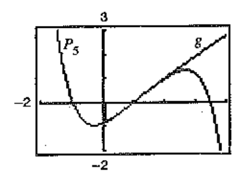
$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$



The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.

65. $f(x) = \sqrt{x}\ln x,\ c = 1$

$P_5(x) = (x-1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$

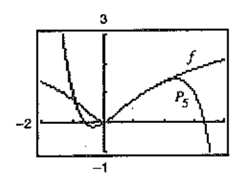


The polynomial is a reasonable approximation on the interval $\left[\frac{1}{4}, 2\right]$.

66. $f(x) = \sqrt[3]{x}\cdot\arctan x,\ c = 1$

$$P_5(x) \approx 0.7854 + 0.7618(x-1) - 0.3412\left[\frac{(x-1)^2}{2!}\right] - 0.0424\left[\frac{(x-1)^3}{3!}\right]$$

$$+ 1.3025\left[\frac{(x-1)^4}{4!}\right] - 5.5913\left[\frac{(x-1)^5}{5!}\right]$$



The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.

67. See Guidelines, page 680.

68. $a_{2n+1} = 0$ (odd coefficients are zero)

69. (a) Replace x with $(-x)$.

(b) Replace x with $3x$.

(c) Multiply series by x.

(d) Replace x with $2x$, then replace x with $-2x$, and add the two together.

70. The binomial series is $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$. The radius of convergence is $R = 1$.

71. $y = \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)$

$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} - \frac{g}{k^2}\left[-\frac{kx}{v_0\cos\theta} - \frac{1}{2}\left(\frac{kx}{v_0\cos\theta}\right)^2 - \frac{1}{3}\left(\frac{kx}{v_0\cos\theta}\right)^3 - \frac{1}{4}\left(\frac{kx}{v_0\cos\theta}\right)^4 - \cdots\right]$

$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} + \frac{gx}{kv_0\cos\theta} + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{gkx^3}{3v_0^3\cos^3\theta} + \frac{gk^2x^4}{4v_0^4\cos^4\theta} + \cdots\Bigg]$

$= (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots$

72. $\theta = 60°,\ v_0 = 64,\ k = \frac{1}{16},\ g = -32$

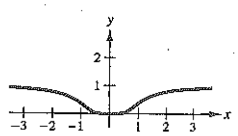
$y = \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots$

$= \sqrt{3}x - 32\left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^3 16} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots\right]$

$= \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{2^nx^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{x^n}{n(32)^n(16)^{n-2}}$

73. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b) $f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{e^{-1/x^2} - 0}{x}$

Let $y = \lim_{x\to 0}\frac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x\to 0}\ln\left(\frac{e^{-1/x^2}}{x}\right) = \lim_{x\to 0^+}\left[-\frac{1}{x^2} - \ln x\right] = \lim_{x\to 0^+}\left[\frac{-1 - x^2\ln x}{x^2}\right] = -\infty.$$

Thus, $y = e^{-\infty} = 0$ and we have $f'(0) = 0$.

(c) $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots = 0 \neq f(x)$ This series converges to f at $x = 0$ only.

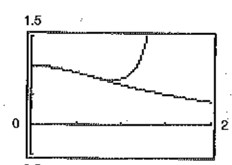
74. (a) $f(x) = \frac{\ln(x^2 + 1)}{x^2}$.

From Exercise 8, you obtain:

$P = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+2}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{n+1}$

$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$.

(b)



(c) $F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2}\,dt$

$G(x) = \int_0^x P_8(t)\,dt$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6924	0.8865	1.6878	9.6063

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.

75. By the Ratio Test: $\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$ which shows that $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x.

76. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right)$$

$$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots) = 2x\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}, R = 1$$

$$\ln 3 = \ln\left(\frac{1+1/2}{1-1/2}\right) \approx 2\left(\frac{1}{2}\right)\left[1 + \frac{(1/2)^2}{3} + \frac{(1/2)^4}{5} + \frac{(1/2)^6}{7}\right] = 1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448} \approx 1.098065$$

($\ln 3 \approx 1.098612$)

77. $\binom{5}{3} = \frac{5\cdot 4\cdot 3}{3!} = \frac{60}{6} = 10$

78. $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$

79. $\binom{0.5}{4} = \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!}$

$$= -0.0390625 = -\frac{5}{128}$$

80. $\binom{-1/3}{5} = \frac{(-1/3)(-4/3)(-7/3)(-10/3)(-13/3)}{5!}$

$$= \frac{-91}{729} \approx -0.12483$$

81. $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$

Example: $(1+x)^2 = \sum_{n=0}^{\infty}\binom{2}{n}x^n = 1 + 2x + x^2$

82. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N!\left[e - \left(1 + 1 + \ldots + \frac{1}{N!}\right)\right]$, a positive integer. But,

$$a = N!\left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots\right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1}\left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right] = \frac{1}{N+1}\left[\frac{1}{1-\left(\frac{1}{N+1}\right)}\right] = \frac{1}{N}, \text{ a contradiction.}$$

83. $g(x) = \frac{x}{1-x-x^2} = a_0 + a_1x + a_2x^2 + \cdots$

$x = (1 - x - x^2)(a_0 + a_1x + a_2x^2 + \cdots)$

$x = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \cdots$

Equating coefficients,

$a_0 = 0$

$a_1 - a_0 = 1 \implies a_1 = 1$

$a_2 - a_1 - a_0 = 0 \implies a_2 = 1$

$a_3 - a_2 - a_1 = 0 \implies a_3 = 2$

$a_4 = a_3 + a_2 = 3$, etc.

In general, $a_n = a_{n-1} + a_{n-2}$. The coefficients are the Fibonacci numbers.

84. Assume the interval is $[-1, 1]$. Let $x \in [-1, 1]$,

$$f(1) = f(x) + (1 - x)f'(x) + \tfrac{1}{2}(1 - x)^2 f''(c),\ c \in (x, 1)$$

$$f(-1) = f(x) + (-1 - x)f'(x) + \tfrac{1}{2}(-1 - x)^2 f''(d),\ d \in (-1, x).$$

Hence, $f(1) - f(-1) = 2f'(x) + \tfrac{1}{2}(1 - x)^2 f''(c) - \tfrac{1}{2}(1 + x)^2 f''(d)$

$$2f'(x) = f(1) - f(-1) - \tfrac{1}{2}(1 - x)^2 f''(c) + \tfrac{1}{2}(1 + x)^2 f''(d).$$

Since $|f(x)| \le 1$ and $|f''(x)| \le 1$,

$$2|f'(x)| \le |f(1)| + |f(-1)| + \tfrac{1}{2}(1 - x)^2 |f''(c)| + \tfrac{1}{2}(1 + x)^2 |f''(d)|$$

$$\le 1 + 1 + \tfrac{1}{2}(1 - x^2) + \tfrac{1}{2}(1 + x)^2$$

$$= 3 + x^2 \le 4.$$

Thus, $|f'(x)| \le 2$.

Note: Let $f(x) = \tfrac{1}{2}(x + 1)^2 - 1$. Then $|f'(x)| \le 1$, $|f''(x)| = 1$ and $f'(1) = 2$.

Review Exercises for Chapter 9

1. $a_n = \dfrac{1}{n!}$

2. $a_n = \dfrac{n}{n^2 + 1}$

3. $a_n = 4 + \dfrac{2}{n}$: 6, 5, 4.67, . . .

Matches (a)

4. $a_n = 4 - \dfrac{n}{2}$: 3.5, 3, . . .

Matches (c)

5. $a_n = 10(0.3)^{n-1}$: 10, 3, . . .

Matches (d)

6. $a_n = 6\left(-\dfrac{2}{3}\right)^{n-1}$: 6, −4, . . .

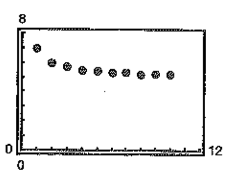
Matches (b)

7. $a_n = \dfrac{5n + 2}{n}$

The sequence seems to converge to 5.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n + 2}{n}$$

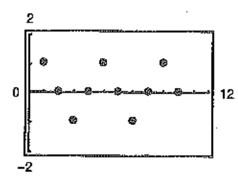
$$= \lim_{n\to\infty}\left(5 + \frac{2}{n}\right) = 5$$



8. $a_n = \sin\dfrac{n\pi}{2}$

The sequence seems to diverge (oscillates).

$\sin\dfrac{n\pi}{2}$: 1, 0, −1, 0, 1, 0, . . .



9. $\displaystyle\lim_{n\to\infty} \frac{n + 1}{n^2} = 0$

Converges

10. $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Converges

11. $\displaystyle\lim_{n\to\infty} \frac{n^3}{n^2 + 1} = \infty$

12. $\displaystyle\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{n\to\infty} \frac{1}{1/n}$

$= \infty$

Diverges

13. $\displaystyle\lim_{n\to\infty}\left(\sqrt{n + 1} - \sqrt{n}\right) = \lim_{n\to\infty}\left(\sqrt{n + 1} - \sqrt{n}\right)\frac{\sqrt{n + 1} + \sqrt{n}}{\sqrt{n + 1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n + 1} + \sqrt{n}} = 0$ Converges

14. $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^n = \lim_{k\to\infty}\left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges; $k = 2n$

15. $\displaystyle\lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} = 0$

Converges

16. Let $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{b^n + c^n}(b^n \ln b + c^n \ln c).$$

Assume $b \geq c$ and note that the terms

$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \to \infty$. Hence a_n converges.

17. $A_n = 5000\left(1 + \frac{0.05}{4}\right)^n = 5000(1.0125)^n$

$n = 1, 2, 3$

(a) $A_1 = 5062.50$ $\quad A_5 \approx 5320.41$

$A_2 \approx 5125.78$ $\quad A_6 \approx 5386.92$

$A_3 \approx 5189.85$ $\quad A_7 \approx 5454.25$

$A_4 \approx 5254.73$ $\quad A_8 \approx 5522.43$

(b) $A_{40} \approx 8218.10$

18. (a) $V_n = 120{,}000(0.70)^n$, $n = 1, 2, 3, 4, 5$

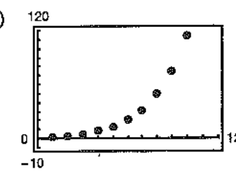
(b) $V_5 = 120{,}000(0.70)^5 = \$20{,}168.40$

19. (a)

k	5	10	15	20	25
S_k	13.2	113.3	873.8	6448.5	50,500.3

The series diverges (geometric $r = \frac{3}{2} > 1$).

(b)

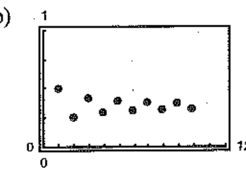


20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

The series converges by the Alternating Series Test.

(b)

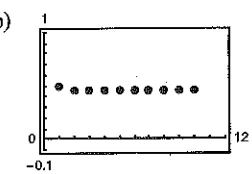


21. (a)

k	5	10	15	20	25
S_k	0.4597	0.4597	0.4597	0.4597	0.4597

The series converges by the Alternating Series Test.

(b)

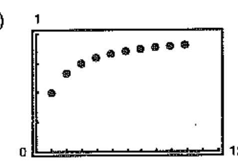


22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

The series converges, by the Limit Comparison Test with $\sum \frac{1}{n^2}$.

(b)



23. Converges. Geometric series, $r = 0.82$, $|r| < 1$.

24. Diverges. Geometric series, $r = 1.82 > 1$.

25. Diverges. nth-Term Test. $\lim_{n\to\infty} a_n \neq 0$.

26. Diverges. nth-Term Test, $\lim_{n\to\infty} a_n = \frac{2}{3}$.

27. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ Geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$S = \frac{a}{1 - r} = \frac{1}{1 - (2/3)} = \frac{1}{1/3} = 3$$

28. $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

29. $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n=\frac{1}{1-(1/2)}-\frac{1}{1-(1/3)}=2-\frac{3}{2}=\frac{1}{2}$

30. $\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n-\frac{1}{(n+1)(n+2)}\right]=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

$$=\frac{1}{1-(2/3)}-\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\right]=3-1=2$$

31. (a) $0.\overline{09}=0.09+0.0009+0.000009+\cdots=0.09(1+0.01+0.0001+\cdots)=\sum_{n=0}^{\infty}(0.09)(0.01)^n$

(b) $0.\overline{09}=\frac{0.09}{1-0.01}=\frac{1}{11}$

32. (a) $0.\overline{923076}=0.923076[1+0.000001+(0.000001)^2+\cdots]=\sum_{n=0}^{\infty}(0.923076)(0.000001)^n$

(b) $0.\overline{923076}=\frac{0.923076}{1-0.000001}=\frac{923{,}076}{999{,}999}=\frac{12(76{,}923)}{13(76{,}923)}=\frac{12}{13}$

33. $D_1=8$

$D_2=0.7(8)+0.7(8)=16(0.7)$

$\vdots$

$D=8+16(0.7)+16(0.7)^2+\cdots+16(0.7)^n+\cdots$

$=-8+\sum_{n=0}^{\infty}16(0.7)^n=-8+\frac{16}{1-0.7}=45\frac{1}{3}$ meters

34. $S=\sum_{n=0}^{39}32{,}000(1.055)^n=\frac{32{,}000(1-1.055^{40})}{1-1.055}$

$\approx \$4{,}371{,}379.65$

35. See Exercise 110 in Section 9.2.

$A=\frac{P(e^{rt}-1)}{e^{r/12}-1}$

$=\frac{200(e^{(0.06)(2)}-1)}{e^{0.06/12}-1}$

$\approx \$5087.14$

36. See Exercise 110 in Section 9.2.

$A=P\left(\frac{12}{r}\right)\left[\left(1+\frac{r}{12}\right)^{12t}-1\right]$

$=100\left(\frac{12}{0.035}\right)\left[\left(1+\frac{0.035}{12}\right)^{120}-1\right]$

$\approx \$14{,}343.25$

37. $\int_1^{\infty}x^{-4}\ln(x)\,dx=\lim_{b\to\infty}\left[-\frac{\ln x}{3x^3}-\frac{1}{9x^3}\right]_1^b=0+\frac{1}{9}=\frac{1}{9}$

By the Integral Test, the series converges.

38. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$

Divergent p-series, $p=\frac{3}{4}<1$

39. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{n}$

Since the second series is a divergent p-series while the first series is a convergent p-series, the difference diverges.

40. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{2^n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{2^n}$

The first series is a convergent p-series and the second series is a convergent geometric series. Therefore, their difference converges.

41. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2n}}$

$\lim_{n\to\infty}\frac{1/\sqrt{n^3+2n}}{1/(n^{3/2})}=\lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+2n}}=1$

By a limit comparison test with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, the series converges.

42. $\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}$

$\lim_{n\to\infty}\frac{(n+1)/n(n+2)}{1/n}=\lim_{n\to\infty}\frac{n+1}{n+2}=1$

By a limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$, the series diverges.

43. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$= \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2}\right)\frac{1}{2n} > \frac{1}{2n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does the original series.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

45. Converges by the Alternating Series Test. (Conditional convergence)

46. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \le \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

47. Diverges by the nth-Term Test.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3\ln(n+1)}{n+1} < \frac{3\ln n}{n} = a_n, \lim_{n\to\infty} \frac{3\ln n}{n} = 0$$

49. $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{e^{2n+1}}\right)\left(\frac{n+1}{n}\right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

50. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

51. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n}\right| = \lim_{n\to\infty} \frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

52. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)}\right| = \lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

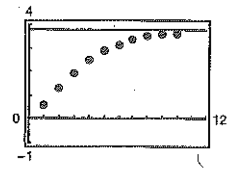
By the Ratio Test, the series converges.

53. (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(3/5)^{n+1}}{n(3/5)^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{3}{5}\right) = \frac{3}{5} < 1$, Converges

(b)

x	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499

(c)



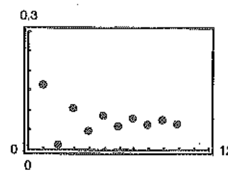
(d) The sum is approximately 3.75.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721

(c)



(d) The sum is approximately 0.0714.

55. (a) $\int_N^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_N^\infty = \frac{1}{N}$

N	5	10	20	30	40
$\sum_{n=1}^{N}\frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^\infty \frac{1}{x^2}\,dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b) $\int_N^\infty \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_N^\infty = \frac{1}{4N^4}$

N	5	10	20	30	40
$\sum_{n=1}^{N}\frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^\infty \frac{1}{x^5}\,dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

56. No. Let $a_n = \frac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\sum_{n=1}^{\infty}\frac{3937.5}{n^2}$ is a convergent p-series.

57. $f(x) = e^{-x/2}$ $\quad f(0) = 1$

$f'(x) = -\frac{1}{2}e^{-x/2}$ $\quad f'(0) = -\frac{1}{2}$

$f''(x) = \frac{1}{4}e^{-x/2}$ $\quad f''(0) = \frac{1}{4}$

$f'''(x) = -\frac{1}{8}e^{-x/2}$ $\quad f'''(0) = -\frac{1}{8}$

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} = 1 - \frac{1}{2}x + \frac{1}{4}\frac{x^2}{2!} - \frac{1}{8}\frac{x^3}{3!} = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$$

58. $f(x) = \tan x$ $\quad f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x$ $\quad f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x$ $\quad f'''\left(-\frac{\pi}{4}\right) = 16$

$$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$$

59. Since $\dfrac{(95\pi)^9}{180^9 \cdot 9!} < 0.001$, use four terms.

$$\sin 95° = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} \approx 0.99594$$

60. $\cos(0.75) \approx 1 - \dfrac{(0.75)^2}{2!} + \dfrac{(0.75)^4}{4!} - \dfrac{(0.75)^6}{6!} \approx 0.7317$

61. $\ln(1.75) \approx (0.75) - \dfrac{(0.75)^2}{2} + \dfrac{(0.75)^3}{3} - \dfrac{(0.75)^4}{4} + \dfrac{(0.75)^5}{5} - \dfrac{(0.75)^6}{6} + \cdots - \dfrac{(0.75)^{14}}{14} \approx 0.559062$

62. $e^{-0.25} \approx 1 - 0.25 + \dfrac{(0.25)^2}{2!} - \dfrac{(0.25)^3}{3!} + \dfrac{(0.25)^4}{4!} \approx 0.779$

63. $f(x) = \cos x,\ c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

$$|f^{(n+1)}(z)| \le 1 \implies R_n(x) \le \frac{x^{n+1}}{(n+1)!}$$

(a) $R_n(x) \le \dfrac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 4$.

(b) $R_n(x) \le \dfrac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 6$.

(c) $R_n(x) \le \dfrac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 5$.

(d) $R_n(x) \le \dfrac{2^{n+1}}{(n+1)!} < 0.0001$

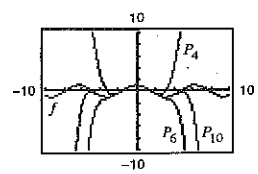
This inequality is true for $n = 10$.

64. $f(x) = \cos x$

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$



65. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

66. $\displaystyle\sum_{n=0}^{\infty}(2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

67. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right| = |x-2|$$

$R = 1$

Center: 2

Since the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $1 \le x \le 3$.

68. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{3^n(x-2)^n}\right| = 3|x-2|$$

$R = \dfrac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \le x < \frac{7}{3}$.

69. $\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right| = \infty$$

which implies that the series converges only at the center $x = 2$.

70. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty}\left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left|\frac{x-2}{2}\right| < 1 \quad \text{or} \quad 0 < x < 4.$$

71.

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n(n!)^2}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n(2n)x^{2n-1}}{4^n(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{4^{n+1}[(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{4^{n+1}[(n+1)!]^2}$$

$$x^2y'' + xy' + x^2y = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n(n!)^2}$$

$$= \sum_{n=0}^{\infty}\left[(-1)^{n+1}\frac{(2n+2)(2n+1)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^{n+1}(2n+2)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^n}{4^n(n!)^2}\right]x^{2n+2}$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}(2n+2)(2n+1+1)}{4^{n+1}[(n+1)!]^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2}$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}4(n+1)^2}{4^{n+1}[(n+1)!]^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2}$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}1}{4^n(n!)^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2} = 0$$

72.

$$y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n(2n)x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)x^{2n+1}}{2^{n+1}(n+1)!}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!}$$

$$y'' + 3xy' + 3y = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}(2n+2)x^{2n+2}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}(2n+2)x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}[-(2n+1)+1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}(-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n+1}x^{2n}}{2^n n!}(2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^{n-1}(n-1)!}\cdot\frac{2n}{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}x^{2n}}{2^n n!}[-2n+2n] = 0$$

73. $\frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

74. $\frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

75. $g(x) = \dfrac{2}{3-x}$. Power series $\displaystyle\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n$

Derivative: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3} n \left(\frac{x}{3}\right)^{n-1}\left(\frac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{2}{9} n \left(\frac{x}{3}\right)^{n-1}$

$$= \sum_{n=0}^{\infty} \frac{2}{9}(n+1)\left(\frac{x}{3}\right)^n$$

76. Integral: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$

77. $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n = \frac{1}{1-(2x/3)} = \frac{3}{3-2x}, \quad -\frac{3}{2} < x < \frac{3}{2}$

78. $8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} 8\left[\frac{-(x-3)}{4}\right]^n = \frac{8}{1-[-(x-3)/4]}$

$$= \frac{32}{4+(x-3)} = \frac{32}{1+x}, \quad -1 < x < 7$$

79. $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x, \cdots$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[x-(3\pi/4)]^n}{n!}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x-(3\pi/4)]^n}{n!}$$

80. $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x+(\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2\cdot 4!}\left(x+\frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2}[x+(\pi/4)]^{n+1}}{(n+1)!}\right]$$

81. $3^x = (e^{\ln(3)})^x = e^{x\ln(3)}$ and since $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, we have $3^x = \displaystyle\sum_{n=0}^{\infty}\frac{(x\ln 3)^n}{n!} = 1 + x\ln 3 + \frac{x^2[\ln 3]^2}{2!} + \frac{x^3[\ln 3]^3}{3!} + \frac{x^4[\ln 3]^4}{4!} + \cdots$.

82. $f(x) = \csc(x)$

$f'(x) = -\csc(x)\cot(x)$

$f''(x) = \csc^3(x) + \csc(x)\cot^2(x)$

$f'''(x) = -5\csc^3(x)\cot(x) - \csc(x)\cot^3(x)$

$f^{(4)}(x) = 5\csc^5(x) + 15\csc^3(x)\cot^2(x) + \csc(x)\cot^4(x)$

$$\csc(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/2)[x-(\pi/2)]^n}{n!} = 1 + \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{5}{4!}\left(x-\frac{\pi}{2}\right)^4 + \cdots$$

83. $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = -\frac{6}{x^4}, \ldots$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty}(x+1)^n, \ -2 < x < 0$$

84. $f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$

$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$

$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \ldots$

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} = 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \cdots$$

$$= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}$$

85. $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$

$$(1+x)^{1/5} = 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots$$

$$= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \cdots$$

$$= 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!}$$

$$= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots$$

86. $h(x) = (1+x)^{-3}$

$h'(x) = -3(1+x)^{-4}$

$h''(x) = 12(1+x)^{-5}$

$h'''(x) = -60(1+x)^{-6}$

$h^{(4)}(x) = 360(1+x)^{-7}$

$h^{(5)}(x) = -2520(1+x)^{-8}$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)x^n}{2}$$

87. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$$\ln\left(\frac{5}{4}\right) = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{(5/4)-1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n} \approx 0.2231$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$$\ln\left(\frac{6}{5}\right) = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{(6/5)-1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823$$

89. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \approx 1.6487$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

91. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

93. The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

94. $\frac{1}{\sqrt{1+x^3}} = (1+x^3)^{-1/2}, k = -\frac{1}{2}$

$$= 1 - \frac{1}{2}(x^3) + \frac{(-1/2)(-3/2)}{2!}x^6 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^9 + \cdots$$

$$= 1 - \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{2^n n!} x^{3n}$$

95. (a) $f(x) = e^{2x}$ $\quad f(0) = 1$

$f'(x) = 2e^{2x}$ $\quad f'(0) = 2$

$f''(x) = 4e^{2x}$ $\quad f''(0) = 4$

$f'''(x) = 8e^{2x}$ $\quad f'''(0) = 8$

$$P(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

(b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$

$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

(c) $e^x \cdot e^x = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)$

$$P = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

96. (a) $f(x) = \sin 2x$ $\quad f(0) = 0$

$f'(x) = 2\cos 2x$ $\quad f'(0) = 2$

$f''(x) = -4\sin 2x$ $\quad f''(0) = 0$

$f'''(x) = -8\cos 2x$ $\quad f'''(0) = -8$

$f^{(4)}(x) = 16\sin 2x$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = 32\cos 2x$ $\quad f^{(5)}(0) = 32$

$f^{(6)}(x) = -64\sin 2x$ $\quad f^{(6)}(0) = 0$

$f^{(7)}(x) = -128\cos 2x$ $\quad f^{(7)}(0) = -128$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

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96. —CONTINUED—

(b) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(c) $\sin 2x = 2 \sin x \cos x$

$$= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 2\left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120}\right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040}\right) + \cdots\right]$$

$$= 2\left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

97.

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$\int_0^x \frac{\sin t}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

98.

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\cos \frac{\sqrt{t}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!}$$

$$\int_0^x \cos \frac{\sqrt{t}}{2}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}$$

99.

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\ln(1+t) = \int \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$

$$\int_0^x \frac{\ln(t+1)}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

100.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$$

$$\frac{e^t - 1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$$

$$\int_0^x \frac{e^t - 1}{t}\,dt = \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!}\right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

101.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$

$$\lim_{x\to 0^+} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule, $\lim_{x\to 0^+} \frac{\arctan x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x\to 0^+} \frac{2\sqrt{x}}{1+x^2} = 0.$

102. $\arcsin x = x + \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \dfrac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x \to 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule, $\displaystyle\lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{\left(\frac{1}{\sqrt{1 - x^2}}\right)}{1} = 1.$

Problem Solving for Chapter 9

1. (a) $1\left(\dfrac{1}{3}\right) + 2\left(\dfrac{1}{9}\right) + 4\left(\dfrac{1}{27}\right) + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$

$$= \frac{1/3}{1 - (2/3)} = 1$$

(b) $0, \dfrac{1}{3}, \dfrac{2}{3}, 1$, etc.

(c) $\displaystyle\lim_{n \to \infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - 1 = 0$

2. Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

Then $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$

$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots$$

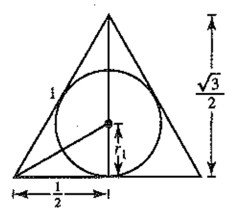
$$= S + \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = S + \frac{1}{2^2}\left(\frac{\pi^2}{6}\right).$$

Thus, $S = \dfrac{\pi^2}{6} - \dfrac{1}{4}\dfrac{\pi^2}{6} = \dfrac{\pi^2}{6}\left(\dfrac{3}{4}\right) = \dfrac{\pi^2}{8}$.

3. If there are n rows, then $a_n = \dfrac{n(n + 1)}{2}$.

For one circle,

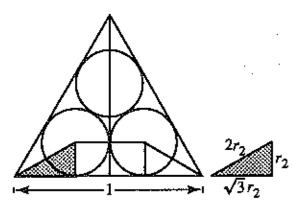
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}.$$



For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

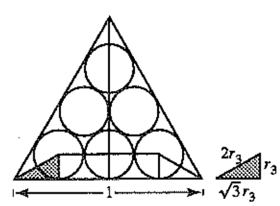
$$r_2 = \frac{1}{2 + 2\sqrt{3}}.$$



For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}.$$



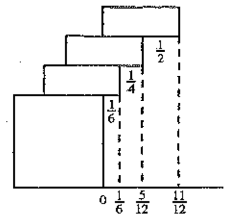
Continuing this pattern, $r_n = \dfrac{1}{2\sqrt{3} + 2(n - 1)}$.

$$\text{Total Area} = (\pi r_n^2)a_n = \pi\left(\frac{1}{2\sqrt{3} + 2(n - 1)}\right)^2 \frac{n(n + 1)}{2}$$

$$A_n = \frac{\pi}{2} \frac{n(n + 1)}{\left[2\sqrt{3} + 2(n - 1)\right]^2}$$

$$\lim_{n \to \infty} A_n = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

4. (a) Position the three blocks as indicated in the figure. The bottom block extends $1/6$ over the edge of the table, the middle block extends $1/4$ over the edge of the bottom block, and the top block extends $1/2$ over the edge of the middle block.



The centers of gravity are located at

bottom block: $\frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

middle block: $\frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$

top block: $\frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}$.

The center of gravity of the top 2 blocks is

$$\left(-\frac{1}{12} + \frac{5}{12}\right)/2 = \frac{1}{6},$$

which lies over the bottom block. The center of gravity of the 3 blocks is

$$\left(-\frac{1}{3} - \frac{1}{12} + \frac{5}{12}\right)/3 = 0$$

which lies over the table. Hence, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

beyond the edge of the table.

(b) Yes. If there are n blocks, then the edge of the top block lies $\sum_{i=1}^{n} \frac{1}{2i}$ from the edge of the table. Using 4 blocks,

$$\sum_{i=1}^{4} \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

(c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{i=1}^{\infty} \frac{1}{2i} = \frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

5. (a) $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \cdots$

$$= (1 + x^3 + x^6 + \cdots) + 2(x + x^4 + x^7 + \cdots) + 3(x^2 + x^5 + x^8 + \cdots)$$
$$= (1 + x^3 + x^6 + \cdots)[1 + 2x + 3x^2]$$
$$= (1 + 2x + 3x^2)\frac{1}{1 - x^3}$$

$R = 1$ because each series in the second line has $R = 1$.

(b) $\sum a_n x^n = (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \cdots) + \cdots$

$$= a_0(1 + x^p + \cdots) + a_1 x(1 + x^p + \cdots) + \cdots + a_{p-1}x^{p-1}(1 + x^p + \cdots)$$
$$= (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})(1 + x^p + \cdots)$$
$$= (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})\frac{1}{1 - x^p}$$

$R = 1$

(Assume all $a_n > 0$.)

6. $a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$

If $a = b$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

If $a \neq b$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n}$ diverges.

No values of a and b give absolute convergence. $a = b$ implies conditional convergence.

7. (a)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x\,dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting $x = 0$, you have $C = 1$. Letting $x = 1$,

$$e - e + 1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}$.

(b) Differentiating,

$$xe^x + e^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Letting $x = 1$,

$$2e = \sum_{n=0}^{\infty} \frac{n+1}{n!} \approx 5.4366.$$

8.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \Rightarrow f^{(12)}(0) = \frac{12!}{6!} = 665{,}280$$

9. Let $a_1 = \int_0^{\pi} \frac{\sin x}{x}\,dx$, $a_2 = -\int_{\pi}^{2\pi} \frac{\sin x}{x}\,dx$, $a_3 = \int_{2\pi}^{3\pi} \frac{\sin x}{x}\,dx$, etc.

Then,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = a_1 - a_2 + a_3 - a_4 + \cdots.$$

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

10. (a) If $p = 1$, $\int_2^{\infty} \frac{1}{x\ln x}\,dx = \ln\ln x\Big]_2^{\infty}$ diverges.

If $p > 1$, $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty}\left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$ converges.

If $p < 1$, diverges.

(b) $\sum_{n=4}^{\infty} \frac{1}{n\ln(n^2)} = \frac{1}{2}\sum_{n=4}^{\infty} \frac{1}{n\ln n}$ diverges by part (a).

11. (a) $a_1 = 3.0$

$a_2 \approx 1.73205$

$a_3 \approx 2.17533$

$a_4 \approx 2.27493$

$a_5 \approx 2.29672$

$a_6 \approx 2.30146$

$\lim_{n\to\infty} a_n = \dfrac{1+\sqrt{13}}{2}$ [See part (b) for proof.]

(b) Use mathematical induction to show the sequence is increasing. Clearly, $a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1$.

Now assume $a_n > a_{n-1}$. Then

$$a_n + a > a_{n-1} + a$$
$$\sqrt{a_n + a} > \sqrt{a_{n-1} + a}$$
$$a_{n+1} > a_n.$$

Use mathematical induction to show that the sequence is bounded above by a. Clearly, $a_1 = \sqrt{a} < a$.

Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$$a(a-1) > a_n(1)$$
$$a^2 - a > a_n$$
$$a^2 > a_n + a$$
$$a > \sqrt{a_n + a} = a_{n+1}.$$

Hence, the sequence converges to some number L. To find L, assume $a_{n+1} \approx a_n \approx L$:

$$L = \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1+4a}}{2}.$$

Hence, $L = \dfrac{1 + \sqrt{1+4a}}{2}$.

12. Let $b_n = a_n r^n$.

$(b_n)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \to Lr$ as $n \to \infty$.

$Lr < \dfrac{1}{r} r = 1.$

By the Root Test, $\sum b_n$ converges $\Rightarrow \sum a_n r^n$ converges.

13. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \cdots$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_1 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

—CONTINUED—

13. —CONTINUED—

(b) $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \dfrac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \ldots$ which diverges.

(c) $\sqrt[n]{\dfrac{1}{2^{n+(-1)^n}}} = \left(\dfrac{1}{2^n \cdot 2^{(-1)^n}}\right)^{1/n}$

$= \dfrac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \to \dfrac{1}{2} < 1$ converges because $\{2^{(-1)^n}\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \ldots$ and $\sqrt[n]{1/2} \to 1$ and $\sqrt[n]{2} \to 1$.

14. (a) $\dfrac{1}{0.99} = \dfrac{1}{1 - 0.01} = \sum_{n=0}^{\infty} (0.01)^n$

$= 1 + 0.01 + (0.01)^2 + \cdots$

$= 1.010101\ldots$

(b) $\dfrac{1}{0.98} = \dfrac{1}{1 - 0.02} = \sum_{n=0}^{\infty} (0.02)^n$

$= 1 + 0.02 + (0.02)^2 + \cdots$

$= 1 + 0.02 + 0.0004 + \cdots$

$= 1.0204081632\ldots$

15. $S_6 = 130 + 70 + 40 = 240$

$S_7 = 240 + 130 + 70 = 440$

$S_8 = 440 + 240 + 130 = 810$

$S_9 = 810 + 440 + 240 = 1490$

$S_{10} = 1490 + 810 + 440 = 2740$

16. (a) Height $= 2\left[1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots\right]$

$= 2\sum_{n=1}^{\infty} \dfrac{1}{n^{1/2}} = \infty \quad \left(p\text{-series}, p = \dfrac{1}{2} < 1\right)$

(b) $S = 4\pi\left[1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots\right] = 4\pi\sum_{n=1}^{\infty} \dfrac{1}{n} = \infty$

(c) $W = \dfrac{4}{3}\pi\left[1 + \dfrac{1}{2^{3/2}} + \dfrac{1}{3^{3/2}} + \cdots\right]$

$= \dfrac{4}{3}\pi\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges.

17. (a) $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2}\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic)

(b) Let $f(x) = \sin x$. By the Mean Value Theorem,

$|f(x) - f(y)| = f'(c)|x - y| = \cos(c)|x - y| \le |x - y|,$

where c is between x and y. Thus,

$0 \le \left|\sin\left(\dfrac{1}{2n}\right) - \sin\left(\dfrac{1}{2n+1}\right)\right|$

$\le \left|\dfrac{1}{2n} - \dfrac{1}{2n+1}\right|$

$= \dfrac{1}{2n(2n+1)}$

Because $\sum_{n=1}^{\infty} \dfrac{1}{2n(2n+1)}$ converges, the Comparison Theorem

tells us that $\sum_{n=1}^{\infty}\left[\sin\left(\dfrac{1}{2n}\right) - \sin\left(\dfrac{1}{2n+1}\right)\right]$ converges.