CHAPTER 7
Applications of Integration

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Section 7.1 Area of a Region Between Two Curves

1. $A = \int_0^6 [0 - (x^2 - 6x)]\,dx = -\int_0^6 (x^2 - 6x)\,dx$

2. $A = \int_{-2}^{2} [(2x + 5) - (x^2 + 2x + 1)]\,dx$

$= \int_{-2}^{2} (-x^2 + 4)\,dx$

3. $A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)]\,dx$

$= \int_0^3 (-2x^2 + 6x)\,dx$

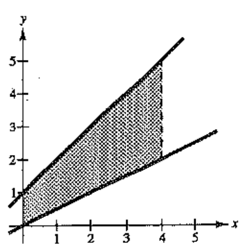
4. $A = \int_0^1 (x^2 - x^3)\,dx$

5. $A = 2\int_{-1}^{0} 3(x^3 - x)\,dx = 6\int_{-1}^{0} (x^3 - x)\,dx$

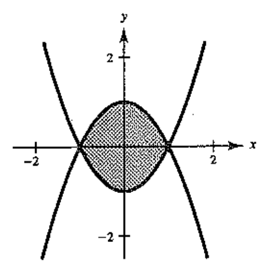
or $-6\int_0^1 (x^3 - x)\,dx$

6. $A = 2\int_0^1 [(x - 1)^3 - (x - 1)]\,dx$

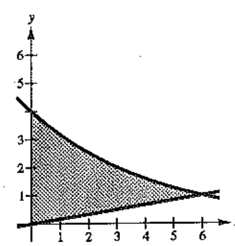
7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$



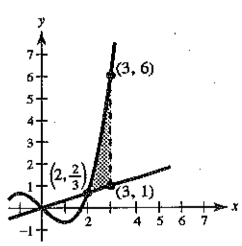
8. $\int_{-1}^{1} [(1 - x^2) - (x^2 - 1)]\,dx$



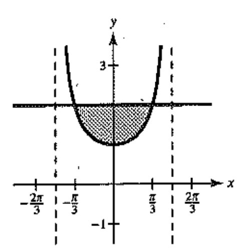
9. $\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6}\right] dx$



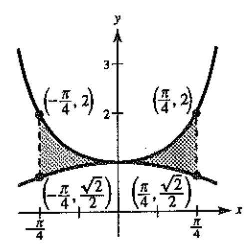
10. $\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$



11. $\int_{-\pi/3}^{\pi/3} (2 - \sec x)\,dx$



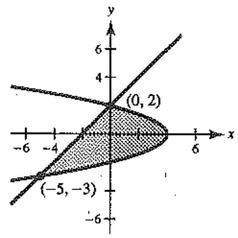
12. $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\,dx$



13. (a)

$$x = 4 - y^2$$
$$x = y - 2$$
$$4 - y^2 = y - 2$$
$$y^2 + y - 6 = 0$$
$$(y + 3)(y - 2) = 0$$

Intersection points: $(0, 2)$ and $(-5, -3)$



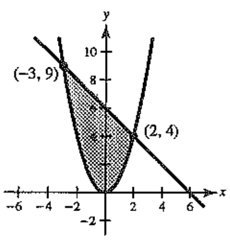
$$A = \int_{-5}^{0} [(x + 2) + \sqrt{4 - x}\, dx + \int_{0}^{4} 2\sqrt{4 - x}\, dx = \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$$

(b) $A = \int_{-3}^{2} [(4 - y^2) - (y - 2)]\, dy = \frac{125}{6}$

14. (a) $y = x^2$ and $y = 6 - x$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$



$$A = \int_{-3}^{2} [(6 - x) - x^2]\, dx = \frac{125}{6}$$

(b) $A = \int_{0}^{4} 2\sqrt{y}\, dy + \int_{4}^{9} [(6 - y) + \sqrt{y}\, dy$

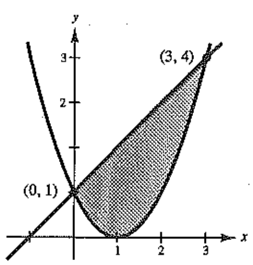
$$= \frac{32}{3} + \frac{61}{6} = \frac{125}{6}$$

15. $f(x) = x + 1$

$g(x) = (x - 1)^2$

$A \approx 4$

Matches (d)

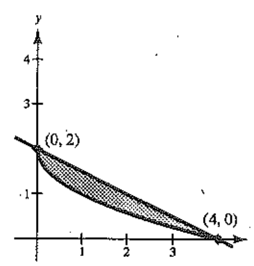


16. $f(x) = 2 - \frac{1}{2}x$

$g(x) = 2 - \sqrt{x}$

$A \approx 1$

Matches (a)

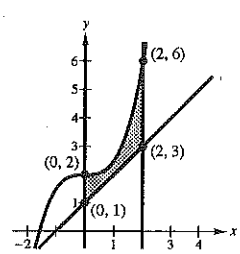


17. $A = \int_{0}^{2} \left[\left(\frac{1}{2}x^3 + 2\right) - (x + 1)\right] dx$

$$= \int_{0}^{2} \left(\frac{1}{2}x^3 - x + 1\right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x\right]_{0}^{2}$$

$$= \left(\frac{16}{8} - \frac{4}{2} + 2\right) - 0 = 2$$

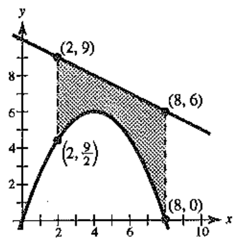


18. $A = \int_2^8 \left[\left(10 - \frac{1}{2}x\right) - \left(-\frac{3}{8}x(x-8)\right)\right] dx$

$= \int_2^8 \left(\frac{3}{8}x^2 - \frac{7}{2}x + 10\right) dx$

$= \left[\frac{x^3}{8} - \frac{7x^2}{4} + 10x\right]_2^8$

$= (64 - 112 + 80) - (1 - 7 + 20) = 18$



19. The points of intersection are given by:

$x^2 - 4x = 0$

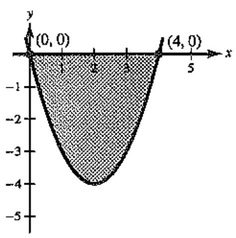
$x(x-4) = 0$ when $x = 0, 4$

$A = \int_0^4 [g(x) - f(x)]\, dx$

$= -\int_0^4 (x^2 - 4x)\, dx$

$= -\left[\frac{x^3}{3} - 2x^2\right]_0^4$

$= \frac{32}{3}$



20. The points of intersection are given by:

$-x^2 + 4x + 1 = x + 1$

$-x^2 + 3x = 0$

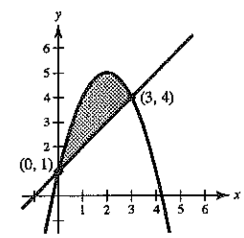
$x^2 = 3x$ when $x = 0, 3$

$A = \int_0^3 [(-x^2 + 4x + 1) - (x + 1)]\, dx$

$= \int_0^3 (-x^2 + 3x)\, dx$

$= \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3$

$= -9 + \frac{27}{2} = \frac{9}{2}$



21. The points of intersection are given by:

$x^2 + 2x + 1 = 3x + 3$

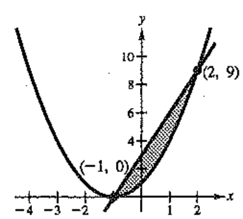
$(x-2)(x+1) = 0$ when $x = -1, 2$

$A = \int_{-1}^2 [g(x) - f(x)]\, dx$

$= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)]\, dx$

$= \int_{-1}^2 (2 + x - x^2)\, dx$

$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$



22. The points of intersection are given by:

$-x^2 + 4x + 2 = x + 2$

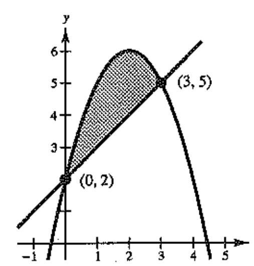
$x(3 - x) = 0$ when $x = 0, 3$

$A = \int_0^3 [f(x) - g(x)]\, dx$

$= \int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\, dx$

$= \int_0^3 (-x^2 + 3x)\, dx$

$= \left[\frac{-x^3}{3} + \frac{3}{2}x^2\right]_0^3 = \frac{9}{2}$



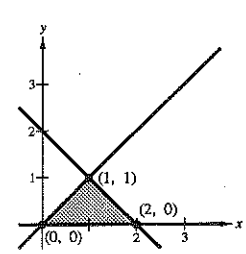
23. The points of intersection are given by:

$x = 2 - x$ and $x = 0$ and $2 - x = 0$

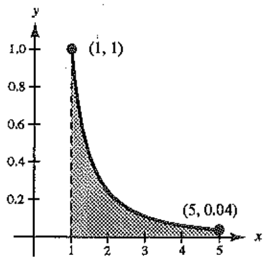
$x = 1$ $\quad x = 0$ $\quad x = 2$

$A = \int_0^1 [(2 - y) - (y)]\, dy = \left[2y - y^2\right]_0^1 = 1$

Note that if we integrate with respect to x, we need two integrals. Also, note that the region is a triangle.



24. $A = \int_1^5 \left(\frac{1}{x^2} - 0\right) dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$



25. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

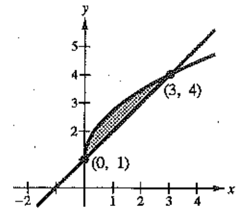
$$\sqrt{3x} = x \quad \text{when } x = 0, 3$$

$$A = \int_0^3 [f(x) - g(x)]\, dx$$

$$= \int_0^3 \left[\left(\sqrt{3x} + 1\right) - (x + 1)\right] dx$$

$$= \int_0^3 [(3x)^{1/2} - x]\, dx$$

$$= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2}$$



26. The points of intersection are given by:

$$\sqrt[3]{x - 1} = x - 1$$

$$x - 1 = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

$$x^3 - 3x^2 + 2x = 0$$

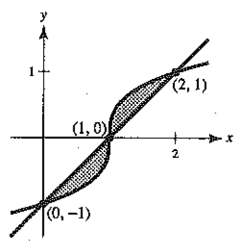
$$x(x^2 - 3x + 2) = 0$$

$$x(x - 2)(x - 1) = 0 \Rightarrow x = 0, 1, 2$$

$$A = 2\int_0^1 \left[(x - 1) - \sqrt[3]{x - 1}\right] dx$$

$$= 2\left[\frac{x^2}{2} - x - \frac{3}{4}(x - 1)^{4/3}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - 1 - 0\right) - \left(-\frac{3}{4}\right)\right] = \frac{1}{2}$$



27. The points of intersection are given by:

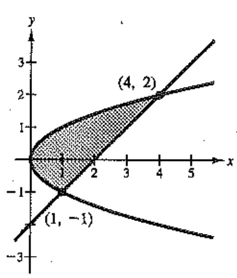
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$

$$A = \int_{-1}^2 [g(y) - f(y)]\, dy$$

$$= \int_{-1}^2 [(y + 2) - y^2]\, dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$$



28. The points of intersection are given by:

$$2y - y^2 = -y$$

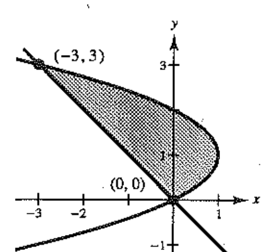
$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$

$$A = \int_0^3 [f(y) - g(y)]\, dy$$

$$= \int_0^3 [(2y - y^2) - (-y)]\, dy$$

$$= \int_0^3 (3y - y^2)\, dy$$

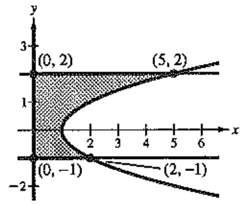
$$= \left[\frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^3 = \frac{9}{2}$$



29. $A = \int_{-1}^{2} [f(y) - g(y)]\, dy$

$= \int_{-1}^{2} [(y^2 + 1) - 0]\, dy$

$= \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6$

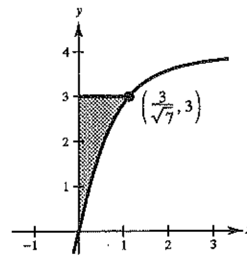


30. $A = \int_{0}^{3} [f(y) - g(y)]\, dy$

$= \int_{0}^{3} \left[\frac{y}{\sqrt{16 - y^2}} - 0\right] dy$

$= -\frac{1}{2}\int_{0}^{3} (16 - y^2)^{-1/2}(-2y)\, dy$

$= \left[-\sqrt{16 - y^2}\right]_{0}^{3} = 4 - \sqrt{7} \approx 1.354$



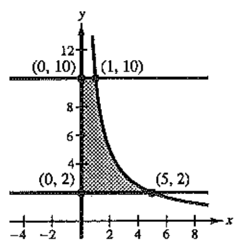
31. $y = \frac{10}{x} \implies x = \frac{10}{y}$

$A = \int_{2}^{10} \frac{10}{y}\, dy$

$= \left[10 \ln y\right]_{2}^{10}$

$= 10(\ln 10 - \ln 2)$

$= 10 \ln 5 \approx 16.0944$

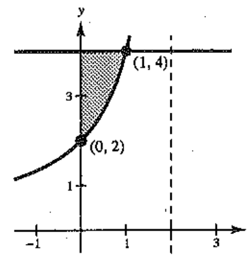


32. $A = \int_{0}^{1} \left(4 - \frac{4}{2 - x}\right) dx$

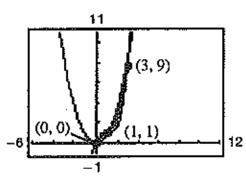
$= \left[4x + 4 \ln|2 - x|\right]_{0}^{1}$

$= 4 - 4 \ln 2$

≈ 1.227



33. (a)



(b) The points of intersection are given by:

$x^3 - 3x^2 + 3x = x^2$

$x(x - 1)(x - 3) = 0$ when $x = 0, 1, 3$

$A = \int_{0}^{1} [f(x) - g(x)]\, dx + \int_{1}^{3} [g(x) - f(x)]\, dx$

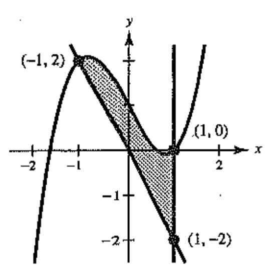
$= \int_{0}^{1} [(x^3 - 3x^2 + 3x) - x^2]\, dx + \int_{1}^{3} [x^2 - (x^3 - 3x^2 + 3x)]\, dx$

$= \int_{0}^{1} (x^3 - 4x^2 + 3x)\, dx + \int_{1}^{3} (-x^3 + 4x^2 - 3x)\, dx$

$= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_{0}^{1} + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_{1}^{3} = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$

(c) Numerical approximation: $0.417 + 2.667 \approx 3.083$

34. (a)



(b) The point of intersection is given by:

$x^3 - 2x + 1 = -2x$

$x^3 + 1 = 0$ when $x = -1$

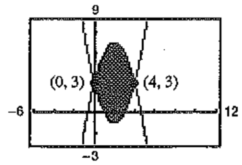
$A = \int_{-1}^{1} [f(x) - g(x)]\, dx$

$= \int_{-1}^{1} [(x^3 - 2x + 1) - (-2x)]\, dx$

$= \int_{-1}^{1} (x^3 + 1)\, dx = \left[\frac{x^4}{4} + x\right]_{-1}^{1} = 2$

(c) Numerical approximation: 2.0

35. (a)



(b) The points of intersection are given by:

$$x^2 - 4x + 3 = 3 + 4x - x^2$$

$$2x(x - 4) = 0 \quad \text{when} \quad x = 0, 4$$

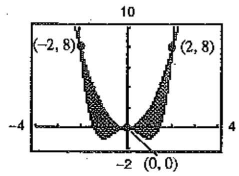
$$A = \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)]\,dx$$

$$= \int_0^4 (-2x^2 + 8x)\,dx$$

$$= \left[-\frac{2x^3}{3} + 4x^2\right]_0^4 = \frac{64}{3}$$

(c) Numerical approximation: 21.333

36. (a)



(b) The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$

$$x^2(x^2 - 4) = 0 \quad \text{when} \quad x = 0, \pm 2$$

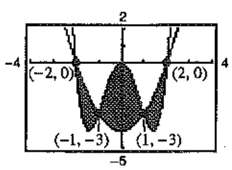
$$A = 2\int_0^2 [2x^2 - (x^4 - 2x^2)]\,dx$$

$$= 2\int_0^2 (4x^2 - x^4)\,dx$$

$$= 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{128}{15}$$

(c) Numerical approximation: 8.533

37. (a) $f(x) = x^4 - 4x^2, \; g(x) = x^2 - 4$



(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$

$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when} \quad x = \pm 2, \pm 1$$

By symmetry:

$$A = 2\int_0^1 [(x^4 - 4x^2) - (x^2 - 4)]\,dx + 2\int_1^2 [(x^2 - 4) - (x^4 - 4x^2)]\,dx$$

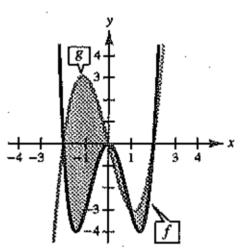
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\,dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$$

$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8$$

(c) Numerical approximation:
5.067 + 2.933 = 8.0

38. (a) $f(x) = x^4 - 4x^2, \; g(x) = x^3 - 4x$



(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^3 - 4x$$

$$x^4 - x^3 - 4x^2 + 4x = 0$$

$$x(x - 1)(x + 2)(x - 2) = 0 \quad \text{when} \quad x = -2, 0, 1, 2$$

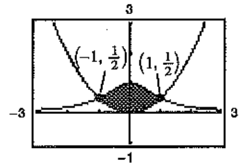
$$A = \int_{-2}^0 [(x^3 - 4x) - (x^4 - 4x^2)]\,dx + \int_0^1 [(x^4 - 4x^2) - (x^3 - 4x)]\,dx$$

$$+ \int_1^2 [(x^3 - 4x) - (x^4 - 4x^2)]\,dx$$

$$= \frac{248}{30} + \frac{37}{60} + \frac{53}{60} = \frac{293}{30}$$

(c) Numerical approximation:
8.267 + 0.617 + 0.883 ≈ 9.767

39. (a)



(b) The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x = \pm 1$$

$$A = 2\int_0^1 [f(x) - g(x)]\,dx$$

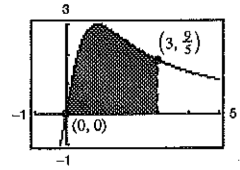
$$= 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2}\right]dx$$

$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

(c) Numerical approximation: 1.237

40. (a)



(b) $A = \displaystyle\int_0^3 \left[\frac{6x}{x^2+1} - 0\right]dx$

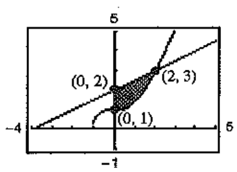
$$= \left[3\ln(x^2+1)\right]_0^3$$

$$= 3\ln 10$$

$$\approx 6.908$$

(c) Numerical approximation: 6.908

41. (a)

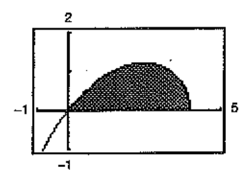


(b) and (c) $\sqrt{1+x^3} \le \frac{1}{2}x + 2$ on $[0, 2]$

You must use numerical integration because $y = \sqrt{1+x^3}$ does not have an elementary antiderivative.

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1+x^3}\right]dx \approx 1.759$$

42. (a)



(b) and (c) You must use numerical integration:

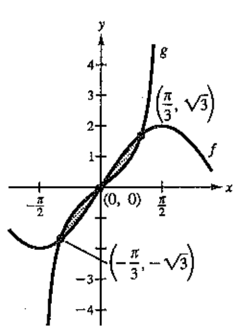
$$A = \int_0^4 x\sqrt{\frac{4-x}{4+x}}\,dx \approx 3.434$$

43. $A = 2\displaystyle\int_0^{\pi/3} [f(x) - g(x)]\,dx$

$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx$$

$$= 2\Big[-2\cos x + \ln|\cos x|\Big]_0^{\pi/3}$$

$$= 2(1 - \ln 2) \approx 0.614$$

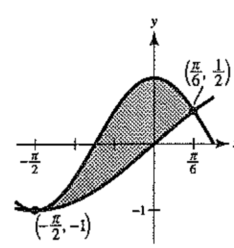


44. $A = \displaystyle\int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x)\,dx$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\pi/2}^{\pi/6}$$

$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0)$$

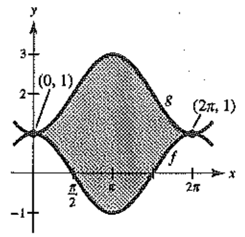
$$= \frac{3\sqrt{3}}{4} \approx 1.299$$



45. $A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\, dx$

$= 2\int_0^{2\pi} (1 - \cos x)\, dx$

$= 2\Big[x - \sin x\Big]_0^{2\pi} = 4\pi \approx 12.566$

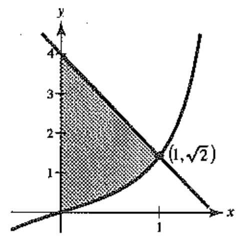


46. $A = \int_0^1 \left[\left(\sqrt{2} - 4\right)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$

$= \left[\frac{\sqrt{2} - 4}{2}x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4}\right]_0^1$

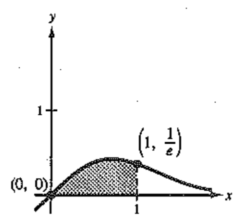
$= \left(\frac{\sqrt{2} - 4}{2} + 4 - \frac{4}{\pi}\sqrt{2}\right) - \left(-\frac{4}{\pi}\right)$

$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}\left(1 - \sqrt{2}\right) \approx 2.1797$



47. $A = \int_0^1 \left[xe^{-x^2} - 0\right] dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



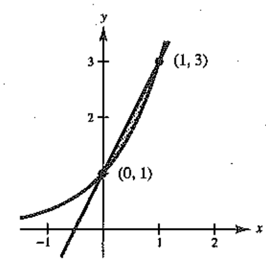
48. From the graph we see that f and g intersect twice at $x = 0$ and $x = 1$.

$A = \int_0^1 [g(x) - f(x)]\, dx$

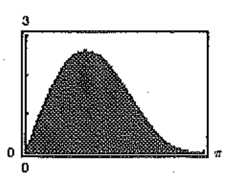
$= \int_0^1 [(2x + 1) - 3^x]\, dx$

$= \left[x^2 + x - \frac{1}{\ln 3}(3^x)\right]_0^1$

$= 2\left(1 - \frac{1}{\ln 3}\right) \approx 0.180$



49. (a)



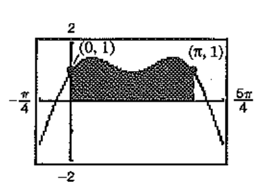
(b) $A = \int_0^{\pi} (2\sin x + \sin 2x)\, dx$

$= \left[-2\cos x - \frac{1}{2}\cos 2x\right]_0^{\pi}$

$= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) = 4$

(c) Numerical approximation: 4.0

50. (a)

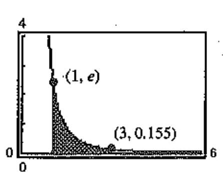


(b) $A = \int_0^{\pi} (2\sin x + \cos 2x)\, dx$

$= \left[-2\cos x + \frac{1}{2}\sin 2x\right]_0^{\pi} = 4$

(c) Numerical approximation: 4

51. (a)

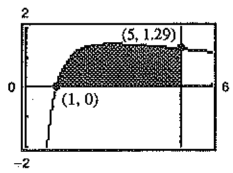


(b) $A = \int_1^3 \frac{1}{x^2}e^{1/x}\, dx$

$= \left[-e^{1/x}\right]_1^3$

$= e - e^{1/3}$

(c) Numerical approximation: 1.323

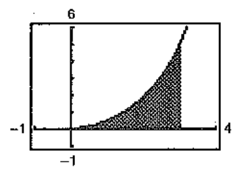
52. (a) 

(b) $A = \int_1^5 \frac{4 \ln x}{x}\,dx$

$= \left[2(\ln x)^2\right]_1^5$

$= 2(\ln 5)^2$

(c) Numerical approximation: 5.181

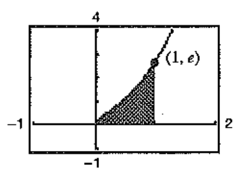
53. (a) 

(b) The integral

$$A = \int_0^3 \sqrt{\frac{x^3}{4-x}}\,dx$$

does not have an elementary antiderivative.

(c) $A \approx 4.7721$

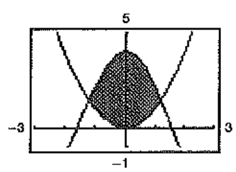
54. (a) 

(b) The integral

$$A = \int_0^1 \sqrt{x}e^x\,dx$$

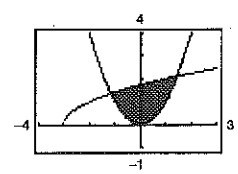
does not have an elementary antiderivative.

(c) 1.2556

55. (a) 

(b) The intersection points are difficult to determine by hand.

(c) Area $= \int_{-c}^{c} [4\cos x - x^2]\,dx \approx 6.3043$ where $c \approx 1.201538$.

56. (a) 

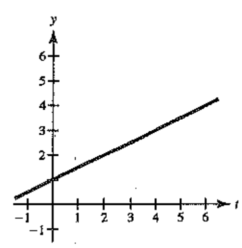
(b) The intersection points are difficult to determine.

(c) Intersection points: $(-1.164035, 1.3549778)$ and $(1.4526269, 2.1101248)$

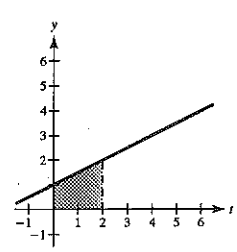
$$A = \int_{-1.164035}^{1.4526269} \left[\sqrt{3+x} - x^2\right] dx \approx 3.0578$$

57. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

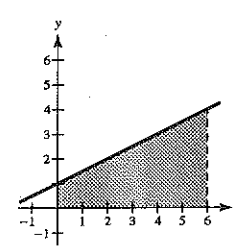
(a) $F(0) = 0$



(b) $F(2) = \frac{2^2}{4} + 2 = 3$

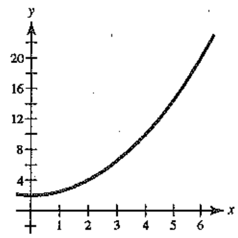


(c) $F(6) = \frac{6^2}{4} + 6 = 15$

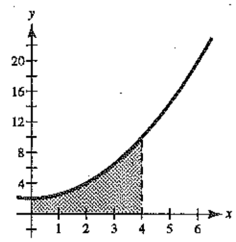


58. $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt = \left[\frac{1}{6}t^3 + 2t\right]_0^x = \frac{x^3}{6} + 2x$

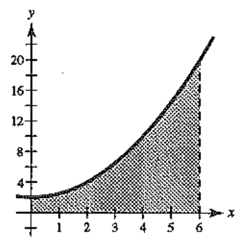
(a) $F(0) = 0$



(b) $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

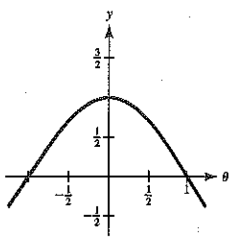


(c) $F(6) = 36 + 12 = 48$

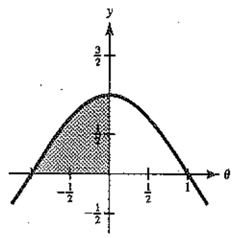


59. $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\, d\theta = \left[\frac{2}{\pi}\sin\frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2} + \frac{2}{\pi}$

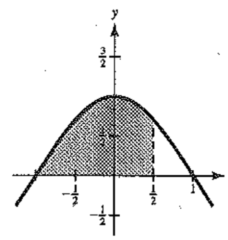
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$

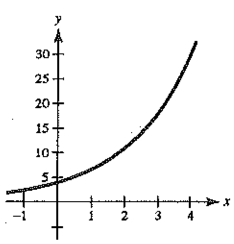


(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

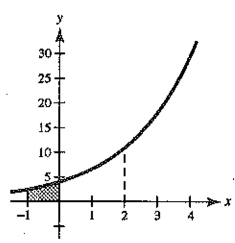


60. $F(y) = \int_{-1}^{y} 4e^{x/2}\, dx = \left[8e^{x/2}\right]_{-1}^{y} = 8e^{y/2} - 8e^{-1/2}$

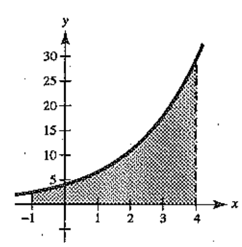
(a) $F(-1) = 0$



(b) $F(0) = 8 - 8e^{-1/2} \approx 3.1478$



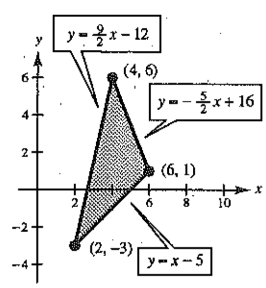
(c) $F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$



61. $A = \int_2^4 \left[\left(\frac{9}{2}x - 12\right) - (x - 5)\right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16\right) - (x - 5)\right] dx$

$= \int_2^4 \left(\frac{7}{2}x - 7\right) dx + \int_4^6 \left(-\frac{7}{2}x + 21\right) dx$

$= \left[\frac{7}{4}x^2 - 7x\right]_2^4 + \left[-\frac{7}{4}x^2 + 21x\right]_4^6 = 7 + 7 = 14$

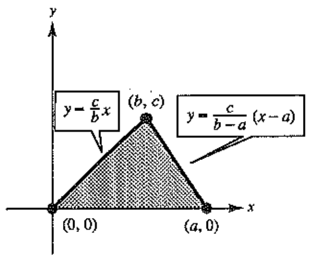


62. $A = \int_0^c \left[\left(\frac{b-a}{c}y + a\right) - \frac{b}{c}y\right] dy$

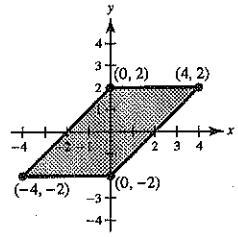
$= \int_0^c \left(-\frac{a}{c}y + a\right) dy$

$= \left[-\frac{a}{2c}y^2 + ay\right]_0^c$

$= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2}(\text{base})(\text{height})\right)$



63.



Left boundary line: $y = x + 2 \Leftrightarrow x = y - 2$

Right boundary line: $y = x - 2 \Leftrightarrow x = y + 2$

$A = \int_{-2}^{2} [(y + 2) - (y - 2)]\, dy$

$= \int_{-2}^{2} 4\, dy = 4y\Big]_{-2}^{2} = 8 - (-8) = 16$

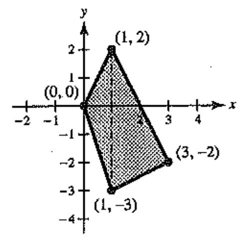
64. $A = \int_0^1 [2x - (-3x)]\, dx + \int_1^3 \left[(-2x + 4) - \left(\frac{1}{2}x - \frac{7}{2}\right)\right] dx$

$= \int_0^1 5x\, dx + \int_1^3 \left(-\frac{5}{2}x + \frac{15}{2}\right) dx$

$= \frac{5x^2}{2}\Big]_0^1 + \left[-\frac{5x^2}{4} + \frac{15}{2}x\right]_1^3$

$= \frac{5}{2} + \left[-\frac{45}{4} + \frac{45}{2} + \frac{5}{4} - \frac{15}{2}\right]$

$= \frac{15}{2}$



65. Answers will vary. If you let $\Delta x = 6$ and $n = 10$, $b - a = 10(6) = 60$.

(a) Area $\approx \frac{60}{2(10)}[0 + 2(14) + 2(14) + 2(12) + 2(12) + 2(15) + 2(20) + 2(23) + 2(25) + 2(26) + 0]$

$= 3[322] = 966$ sq ft

(b) Area $\approx \frac{60}{3(10)}[0 + 4(14) + 2(14) + 4(12) + 2(12) + 4(15) + 2(20) + 4(23) + 2(25) + 4(26) + 0]$

$= 2[502] = 1004$ sq ft

66. $\Delta x = 4, n = 8, b - a = (8)(4) = 32$

(a) Area $\approx \frac{32}{2(8)}[0 + 2(11) + 2(13.5) + 2(14.2) + 2(14) + 2(14.2) + 2(15) + 2(13.5) + 0]$

$= 2[190.8] = 381.6$ sq mi

(b) Area $\approx \frac{32}{3(8)}[0 + 4(11) + 2(13.5) + 4(14.2) + 2(14) + 4(14.2) + 2(15) + 4(13.5) + 0]$

$= \frac{4}{3}[296.6] \approx 395.5$ sq mi

67. $f(x) = x^3$

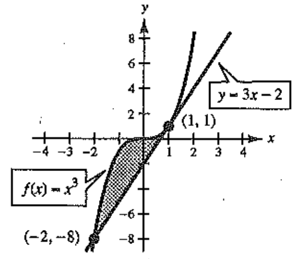
$f'(x) = 3x^2$

At $(1, 1)$, $f'(1) = 3$.

Tangent line: $y - 1 = 3(x - 1)$ or $y = 3x - 2$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^{1} [x^3 - (3x - 2)]\,dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \frac{27}{4}$$



68. $y = x^3 - 2x, \quad (-1, 1)$

$y' = 3x^2 - 2$

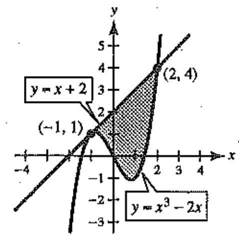
$y'(-1) = 3 - 2 = 1$

Tangent line: $y - 1 = 1(x + 1) \Rightarrow y = x + 2$

Intersection points: $(-1, 1)$ and $(2, 4)$

$$A = \int_{-1}^{2} [(x + 2) - (x^3 - 2x)]\,dx = \int_{-1}^{2} (-x^3 + 3x + 2)\,dx$$

$$= \left[-\frac{x^4}{4} + \frac{3x^2}{2} + 2x\right]_{-1}^{2} = \left[(-4 + 6 + 4) - \left(-\frac{1}{4} + \frac{3}{2} - 2\right)\right] = \frac{27}{4}$$



69. $f(x) = \dfrac{1}{x^2 + 1}$

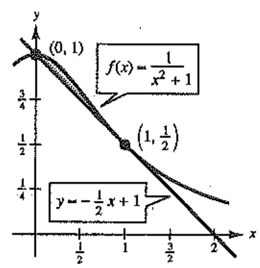
$f'(x) = -\dfrac{2x}{(x^2 + 1)^2}$

At $\left(1, \frac{1}{2}\right)$, $f'(1) = -\frac{1}{2}$.

Tangent line: $y - \frac{1}{2} = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x + 1$

The tangent line intersects $f(x) = \dfrac{1}{x^2 + 1}$ at $x = 0$.

$$A = \int_0^1 \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1\right)\right] dx = \left[\arctan x + \frac{x^2}{4} - x\right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$



70. $y = \dfrac{2}{1 + 4x^2}, \quad \left(\frac{1}{2}, 1\right)$

$y' = \dfrac{-16x}{(1 + 4x^2)^2}$

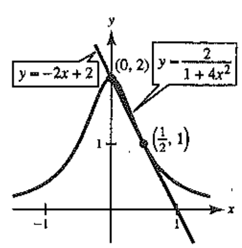
$y'\left(\frac{1}{2}\right) = \dfrac{-8}{2^2} = -2$

Tangent line: $y - 1 = -2\left(x - \frac{1}{2}\right)$

$y = -2x + 2$

Intersection points: $\left(\frac{1}{2}, 1\right)$, $(0, 2)$

$$A = \int_0^{1/2} \left[\frac{2}{1 + 4x^2} - (-2x + 2)\right] dx = \left[\arctan(2x) + x^2 - 2x\right]_0^{1/2} = \arctan(1) + \frac{1}{4} - 1 = \frac{\pi}{4} - \frac{3}{4} \approx 0.0354$$



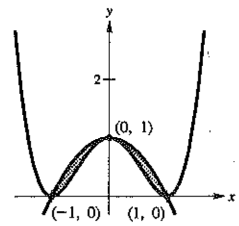
71. $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$A = \int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\,dx$$

$$= \int_{-1}^{1} (x^2 - x^4)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$$

You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.



72. $x^3 \ge x$ on $[-1, 0]$, $x^3 \le x$ on $[0, 1]$

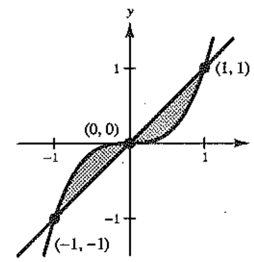
Both functions symmetric to origin.

$$\int_{-1}^{0} (x^3 - x)\,dx = -\int_{0}^{1} (x^3 - x)\,dx$$

Thus, $\int_{-1}^{1} (x^3 - x)\,dx = 0$.

$$A = 2\int_{0}^{1} (x - x^3)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2}$$



73. Offer 2 is better because the accumulated salary (area under the curve) is larger.

74. Proposal 2 is better since the cumulative deficit (the area under the curve) is less.

75.

$$A = \int_{-3}^{3} (9 - x^2)\,dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b]\,dx = 18$$

$$\int_{0}^{\sqrt{9-b}} [(9 - b) - x^2]\,dx = 9$$

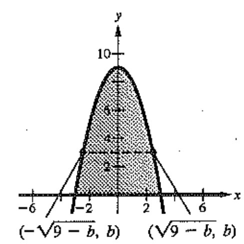
$$\left[(9 - b)x - \frac{x^3}{3}\right]_{0}^{\sqrt{9-b}} = 9$$

$$\frac{2}{3}(9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



76. $A = 2\int_{0}^{9} (9 - x)\,dx = 2\left[9x - \frac{x^2}{2}\right]_{0}^{9} = 81$

$$2\int_{0}^{9-b} [(9 - x) - b]\,dx = \frac{81}{2}$$

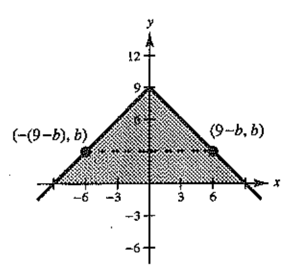
$$2\int_{0}^{9-b} [(9 - b) - x]\,dx = \frac{81}{2}$$

$$2\left[(9 - b)x - \frac{x^2}{2}\right]_{0}^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



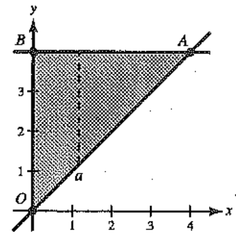
77. Area of triangle OAB is $\frac{1}{2}(4)(4) = 8$.

$$4 = \int_0^a (4 - x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^a = 4a - \frac{a^2}{2}$$

$$a^2 - 8a + 8 = 0$$

$$a = 4 \pm 2\sqrt{2}$$

Since $0 < a < 4$, select $a = 4 - 2\sqrt{2} \approx 1.172$.



78. Total area $= \int_{-2}^{2} (4 - y^2)\,dy = 2\int_0^2 (4 - y^2)\,dy$

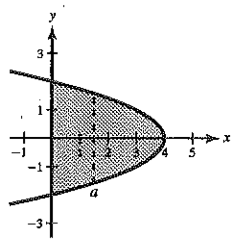
$$= 2\left[4y - \frac{y^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

$$\frac{16}{3} = 2\int_a^4 \sqrt{4 - x}\,dx = -\frac{4}{3}(4 - x)^{3/2}\Big]_a^4 = \frac{4}{3}(4 - a)^{3/2}$$

$$4 = (4 - a)^{3/2}$$

$$4^{2/3} = 4 - a$$

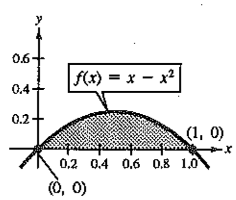
$$a = 4 - 4^{2/3} \approx 1.48$$



79. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\,\Delta x$

where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

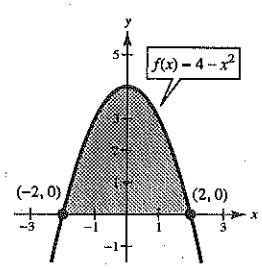
$$\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$



80. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\,\Delta x$

where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ is the same as

$$\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$



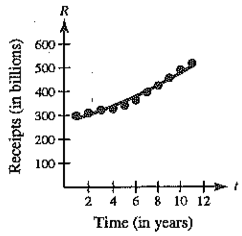
81. $\int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)]\,dt = \int_0^5 0.13t\,dt = \left[\frac{0.13t^2}{2}\right]_0^5 = \$1.625$ billion

82. $\int_0^5 [(7.21 + 0.26t + 0.02t^2) - (7.21 + 0.1t + 0.01t^2)]\,dt = \int_0^5 (0.01t^2 + 0.16t)\,dt$

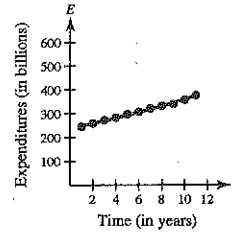
$$= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2}\right]_0^5$$

$$= \frac{29}{12} \text{ billion} \approx \$2.417 \text{ billion}$$

83. (a) $y_1 = (270.3151)(1.0586)^t = 270.3151e^{0.05695t}$



(b) $y_2 = (239.9704)(1.0416)^t = 239.9704e^{0.04074t}$



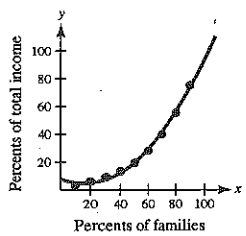
(c) Surplus $= \int_{12}^{17} (y_1 - y_2)\, dt \approx 926.4$ billion dollars

(Answers will vary.)

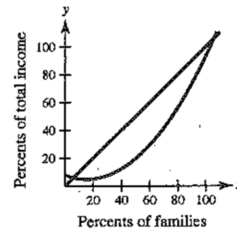
(d) No, $y_1 > y_2$ forever because $1.0586 > 1.0416$. No, these models are not accurate for the future. According to news, $E > R$ eventually.

84. (a) $y_1 = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) Income inequality $= \int_0^{100} [x - y_1]\, dx \approx 2006.7$

85. 5%: $P_1 = 893{,}000e^{(0.05)t}$

$3\frac{1}{2}$%: $P_2 = 893{,}000e^{(0.035)t}$

Difference in profits over 5 years: $\int_0^5 [893{,}000e^{0.05t} - 893{,}000e^{0.035t}]\, dt = 893{,}000\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5$

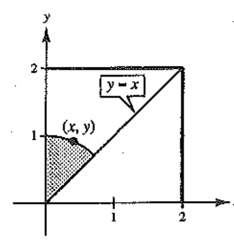
$\approx 893{,}000[(25.6805 - 34.0356) - (20 - 28.5714)]$

$\approx 893{,}000(0.2163) \approx \$193{,}156$

Note: Using a graphing utility, you obtain $193,183.

86. The total area is 8 times the area of the shaded region to the right. A point (x, y) is on the upper boundary of the region if

$$\sqrt{x^2 + y^2} = 2 - y$$
$$x^2 + y^2 = 4 - 4y + y^2$$
$$x^2 = 4 - 4y$$
$$4y = 4 - x^2$$
$$y = 1 - \frac{x^2}{4}.$$



We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$
$$x^2 + 4x - 4 = 0$$
$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \implies x = -2 + 2\sqrt{2}$$

Total area $= 8\int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx = 8\left[x - \frac{x^3}{12} - \frac{x^2}{2}\right]_0^{-2+2\sqrt{2}} = \frac{16}{3}(4\sqrt{2} - 5) \approx 8(0.4379) = 3.503$

87. The curves intersect at the point where the slope of y_2 equals that of y_1, 1.

$$y_2 = 0.08x^2 + k \Rightarrow y'_2 = 0.16x = 1 \Rightarrow x = \frac{1}{0.16} = 6.25$$

(a) The value of k is given by

$$y_1 = y_2$$
$$6.25 = (0.08)(6.25)^2 + k$$
$$k = 3.125.$$

(b)
$$\text{Area} = 2\int_0^{6.25} (y_2 - y_1)\,dx$$
$$= 2\int_0^{6.25} (0.08x^2 + 3.125 - x)\,dx$$
$$= 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25}$$
$$= 2(6.510417) \approx 13.02083$$

88. (a)
$$A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5-x}\right)dx + \int_5^{5.5} (1 - 0)\,dx\right]$$
$$= 2\left(\left[x + \frac{2}{9}(5-x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right)$$
$$= 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\,V \approx 5000(12.062) = 60{,}310$ pounds

89. (a) $A \approx 6.031 - 2\left[\pi\left(\frac{1}{16}\right)^2\right] - 2\left[\pi\left(\frac{1}{8}\right)^2\right] \approx 5.908$

(b) $V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$

(c) $5000V \approx 5000(11.816) = 59{,}082$ pounds

90. True

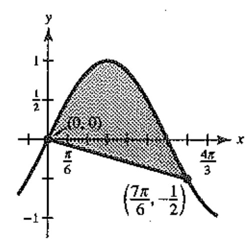
91. True

92. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)]\,dx = \int_0^2 [x - (2x - x^2)]\,dx = \frac{2}{3} \neq 0.$$

93. Line: $y = \dfrac{-3}{7\pi}x$

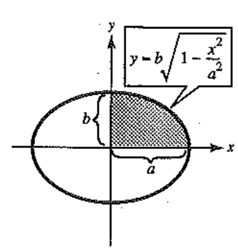
$$A = \int_0^{7\pi/6} \left[\sin x + \frac{3x}{7\pi}\right]dx$$
$$= \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6}$$
$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1$$
$$\approx 2.7823$$



94.
$$A = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$\int_0^a \sqrt{a^2 - x^2}\,dx$ is the area of $\frac{1}{4}$ of a circle $= \dfrac{\pi a^2}{4}$.

Hence, $A = \dfrac{4b}{a}\left(\dfrac{\pi a^2}{4}\right) = \pi ab.$

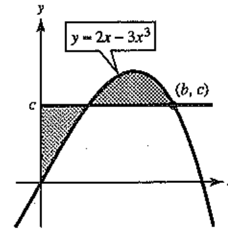


95. We want to find c such that:

$$\int_0^b [(2x - 3x^3) - c]\,dx = 0$$

$$\left[x^2 - \frac{3}{4}x^4 - cx\right]_0^b = 0$$

$$b^2 - \frac{3}{4}b^4 - cb = 0$$



But, $c = 2b - 3b^3$ because (b, c) is on the graph.

$$b^2 - \frac{3}{4}b^4 - (2b - 3b^3)b = 0$$

$$4 - 3b^2 - 8 + 12b^2 = 0$$

$$9b^2 = 4$$

$$b = \frac{2}{3}$$

$$c = \frac{4}{9}$$

Section 7.2 Volume: The Disk Method

1. $V = \pi\int_0^1 (-x + 1)^2\,dx = \pi\int_0^1 (x^2 - 2x + 1)\,dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}$

2. $V = \pi\int_0^2 (4 - x^2)^2\,dx = \pi\int_0^2 (x^4 - 8x^2 + 16)\,dx = \pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2 = \frac{256\pi}{15}$

3. $V = \pi\int_1^4 (\sqrt{x})^2\,dx = \pi\int_1^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

4. $V = \pi\int_0^3 (\sqrt{9 - x^2})^2\,dx = \pi\int_0^3 (9 - x^2)\,dx$

$= \pi\left[9x - \frac{x^3}{3}\right]_0^3 = 18\pi$

5. $V = \pi\int_0^1 [(x^2)^2 - (x^3)^2]\,dx = \pi\int_0^1 (x^4 - x^6)\,dx = \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{2\pi}{35}$

6. $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$V = \pi\int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4}\right)^2 - (2)^2\right]dx$

$= 2\pi\int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12\right]dx$

$= 2\pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x\right]_0^{2\sqrt{2}}$

$= 2\pi\left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2}\right]$

$= \frac{448\sqrt{2}}{15}\pi \approx 132.69$

7. $y = x^2 \implies x = \sqrt{y}$

$V = \pi\int_0^4 (\sqrt{y})^2\,dy = \pi\int_0^4 y\,dy$

$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$

8. $y = \sqrt{16 - x^2} \implies x = \sqrt{16 - y^2}$

$V = \pi\int_0^4 (\sqrt{16 - y^2})^2\,dy = \pi\int_0^4 (16 - y^2)\,dy$

$= \pi\left[16y - \frac{y^3}{3}\right]_0^4 = \frac{128\pi}{3}$

9. $y = x^{2/3} \implies x = y^{3/2}$

$V = \pi\int_0^1 (y^{3/2})^2\,dy = \pi\int_0^1 y^3\,dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$

10. $V = \pi \int_1^4 (-y^2 + 4y)^2 \, dy = \pi \int_1^4 (y^4 - 8y^3 + 16y^2) \, dy$

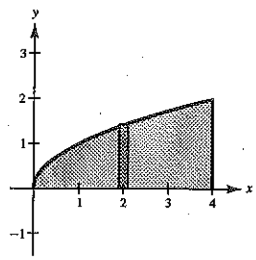
$= \pi \left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3} \right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5}$

11. $y = \sqrt{x}, \; y = 0, \; x = 4$

(a) $R(x) = \sqrt{x}, \; r(x) = 0$

$V = \pi \int_0^4 (\sqrt{x})^2 \, dx$

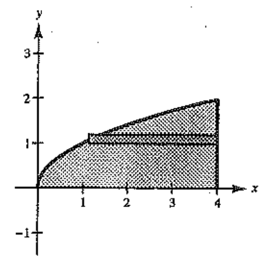
$= \pi \int_0^4 x \, dx = \left[\frac{\pi}{2} x^2 \right]_0^4 = 8\pi$



(b) $R(y) = 4, \; r(y) = y^2$

$V = \pi \int_0^2 (16 - y^4) \, dy$

$= \pi \left[16y - \frac{1}{5} y^5 \right]_0^2 = \frac{128\pi}{5}$

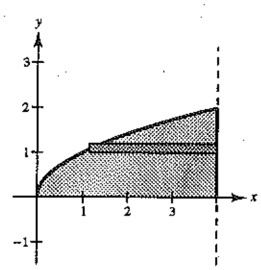


(c) $R(y) = 4 - y^2, \; r(y) = 0$

$V = \pi \int_0^2 (4 - y^2)^2 \, dy$

$= \pi \int_0^2 (16 - 8y^2 + y^4) \, dy$

$= \pi \left[16y - \frac{8}{3} y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{256\pi}{15}$

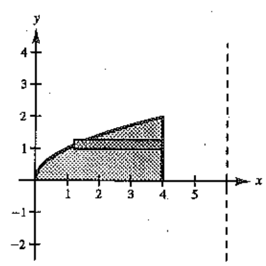


(d) $R(y) = 6 - y^2, \; r(y) = 2$

$V = \pi \int_0^2 [(6 - y^2)^2 - 4] \, dy$

$= \pi \int_0^2 (32 - 12y^2 + y^4) \, dy$

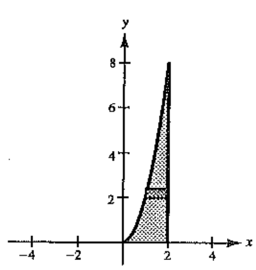
$= \pi \left[32y - 4y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{192\pi}{5}$



12. $y = 2x^2, \; y = 0, \; x = 2$

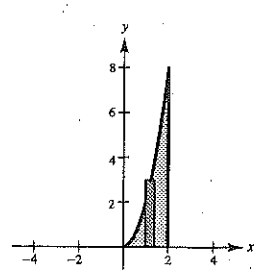
(a) $R(y) = 2, \; r(y) = \sqrt{y/2}$

$V = \pi \int_0^8 \left(4 - \frac{y}{2} \right) dy = \pi \left[4y - \frac{y^2}{4} \right]_0^8 = 16\pi$



(b) $R(x) = 2x^2, \; r(x) = 0$

$V = \pi \int_0^2 4x^4 \, dx = \pi \left[\frac{4x^5}{5} \right]_0^2 = \frac{128\pi}{5}$



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12. —CONTINUED—

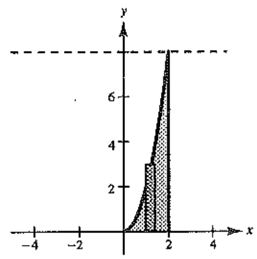
(c) $R(x) = 8,\ r(x) = 8 - 2x^2$

$$V = \pi\int_0^2 [64 - (64 - 32x^2 + 4x^4)]\,dx$$

$$= \pi\int_0^2 (32x^2 - 4x^4)\,dx = 4\pi\int_0^2 (8x^2 - x^4)\,dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$



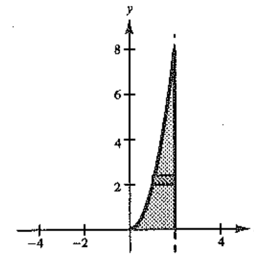
(d) $R(y) = 2 - \sqrt{y/2},\ r(y) = 0$

$$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8$$

$$= \frac{16\pi}{3}$$



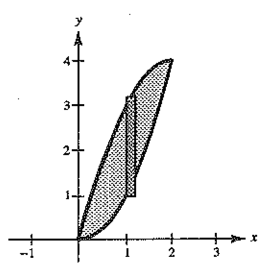
13. $y = x^2,\ y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2,\ r(x) = x^2$

$$V = \pi\int_0^2 [(4x - x^2)^2 - x^4]\,dx$$

$$= \pi\int_0^2 (16x^2 - 8x^3)\,dx$$

$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$

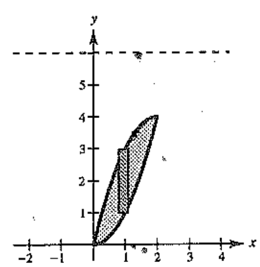


(b) $R(x) = 6 - x^2,\ r(x) = 6 - (4x - x^2)$

$$V = \pi\int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2]\,dx$$

$$= 8\pi\int_0^2 (x^3 - 5x^2 + 6x)\,dx$$

$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$



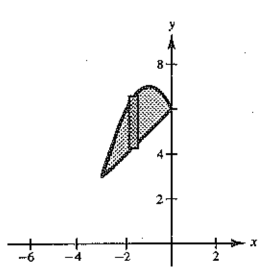
14. $y = 6 - 2x - x^2,\ y = x + 6$ intersect at $(-3, 3)$ and $(0, 6)$.

(a) $R(x) = 6 - 2x - x^2,\ r(x) = x + 6$

$$V = \pi\int_{-3}^0 [(6 - 2x - x^2)^2 - (x + 6)^2]\,dx$$

$$= \pi\int_{-3}^0 (x^4 + 4x^3 - 9x^2 - 36x)\,dx$$

$$= \pi\left[\frac{1}{5}x^5 + x^4 - 3x^3 - 18x^2\right]_{-3}^0 = \frac{243\pi}{5}$$

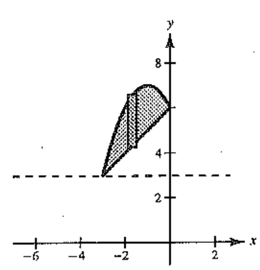


(b) $R(x) = (6 - 2x - x^2) - 3,\ r(x) = (x + 6) - 3$

$$V = \pi\int_{-3}^0 [(3 - 2x - x^2)^2 - (x + 3)^2]\,dx$$

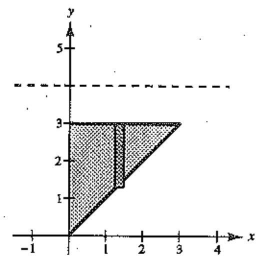
$$= \pi\int_{-3}^0 (x^4 + 4x^3 - 3x^2 - 18x)\,dx$$

$$= \pi\left[\frac{1}{5}x^5 + x^4 - x^3 - 9x^2\right]_{-3}^0 = \frac{108\pi}{5}$$



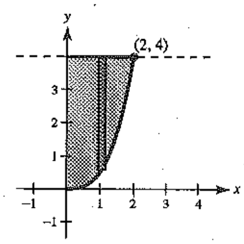
15. $R(x) = 4 - x,\ r(x) = 1$

$$V = \pi\int_0^3 [(4-x)^2 - (1)^2]\,dx$$
$$= \pi\int_0^3 (x^2 - 8x + 15)\,dx$$
$$= \pi\left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3$$
$$= 18\pi$$



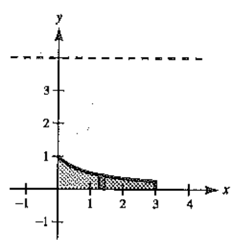
16. $R(x) = 4 - \frac{x^3}{2},\ r(x) = 0$

$$V = \pi\int_0^1 \left(4 - \frac{x^3}{2}\right)^2 dx$$
$$= \pi\int_0^2 \left[16 - 4x^3 + \frac{x^6}{4}\right] dx$$
$$= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_0^2$$
$$= \pi\left[32 - 16 + \frac{128}{28}\right]$$
$$= \frac{144}{7}\pi$$



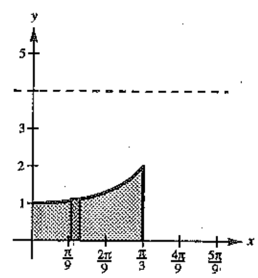
17. $R(x) = 4,\ r(x) = 4 - \frac{1}{1+x}$

$$V = \pi\int_0^3 \left[4^2 - \left(4 - \frac{1}{1+x}\right)^2\right] dx$$
$$= \pi\int_0^3 \left[\frac{8}{1+x} - \frac{1}{(1+x)^2}\right] dx$$
$$= \pi\left[8\ln(1+x) + \frac{1}{1+x}\right]_0^3$$
$$= \pi\left[8\ln 4 + \frac{1}{4} - 1\right]$$
$$= \left(8\ln 4 - \frac{3}{4}\right)\pi$$
$$\approx 32.485$$



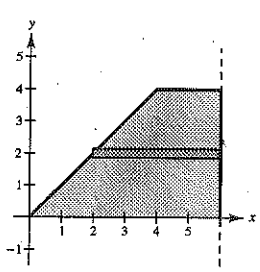
18. $R(x) = 4,\ r(x) = 4 - \sec x$

$$V = \pi\int_0^{\pi/3} [(4)^2 - (4 - \sec x)^2]\,dx$$
$$= \pi\int_0^{\pi/3} (8\sec x - \sec^2 x)\,dx$$
$$= \pi\left[8\ln|\sec x + \tan x| - \tan x\right]_0^{\pi/3}$$
$$= \pi\left[\left(8\ln|2+\sqrt{3}| - \sqrt{3}\right) - \left(8\ln|1+0| - 0\right)\right]$$
$$= \pi\left[8\ln\left(2+\sqrt{3}\right) - \sqrt{3}\right] \approx 27.66$$



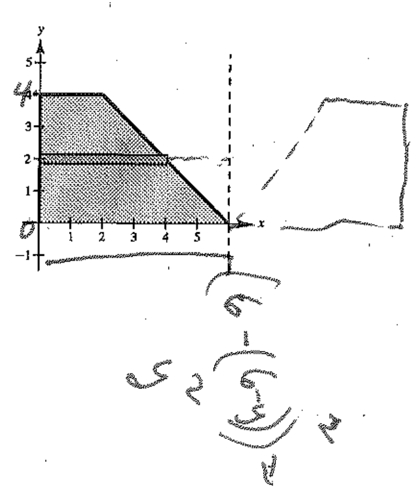
19. $R(y) = 6 - y,\ r(y) = 0$

$$V = \pi\int_0^4 (6-y)^2\,dy$$
$$= \pi\int_0^4 (y^2 - 12y + 36)\,dy$$
$$= \pi\left[\frac{y^3}{3} - 6y^2 + 36y\right]_0^4$$
$$= \frac{208\pi}{3}$$



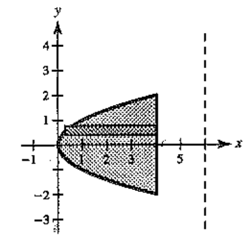
20. $R(y) = 6,\ r(y) = 6 - (6 - y) = y$

$$V = \pi\int_0^4 [(6)^2 - (y)^2]\,dy$$
$$= \pi\left[36y - \frac{y^3}{3}\right]_0^4 = \frac{368\pi}{3}$$



21. $R(y) = 6 - y^2$, $r(y) = 2$

$$V = \pi\int_{-2}^{2} [(6 - y^2)^2 - (2)^2]\,dy$$
$$= 2\pi\int_0^2 (y^4 - 12y^2 + 32)\,dy$$
$$= 2\pi\left[\frac{y^5}{5} - 4y^3 + 32y\right]_0^2$$
$$= \frac{384\pi}{5}$$

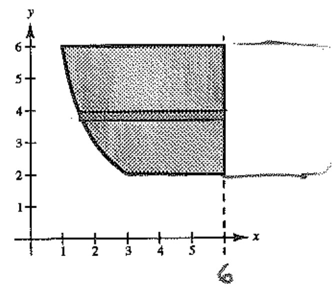


$(6-y^2)^2 - 2^2$

22. $R(y) = 6 - \dfrac{6}{y}$, $r(y) = 0$

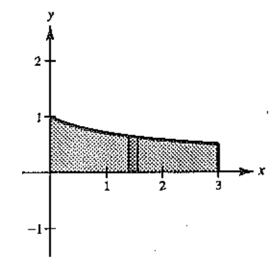
$$V = \pi\int_2^6 \left(6 - \frac{6}{y}\right)^2 dy \quad -0^2$$
$$= 36\pi\int_2^6 \left(1 - \frac{2}{y} + \frac{1}{y^2}\right) dy$$
$$= 36\pi\left[y - 2\ln|y| - \frac{1}{y}\right]_2^6$$
$$= 36\pi\left[\left(\frac{35}{6} - 2\ln 6\right) - \left(\frac{3}{2} - 2\ln 2\right)\right]$$
$$= 36\pi\left(\frac{13}{3} + 2\ln\frac{1}{3}\right)$$
$$= 12\pi(13 - 6\ln 3)$$
$$\approx 241.59$$

just $R(x)^2$ solid



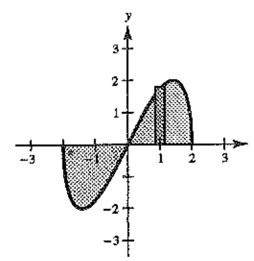
23. $R(x) = \dfrac{1}{\sqrt{x+1}}$, $r(x) = 0$

$$V = \pi\int_0^3 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx$$
$$= \pi\int_0^3 \frac{1}{x+1}\,dx$$
$$= \Big[\pi\ln|x+1|\Big]_0^3$$
$$= \pi\ln 4 \approx 4.355$$



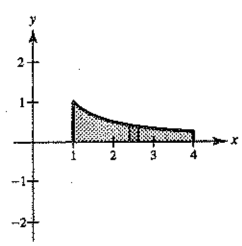
24. $R(x) = x\sqrt{4 - x^2}$, $r(x) = 0$

$$V = 2\pi\int_0^2 \left[x\sqrt{4 - x^2}\right]^2 dx$$
$$= 2\pi\int_0^2 (4x^2 - x^4)\,dx$$
$$= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= \frac{128\pi}{15}$$



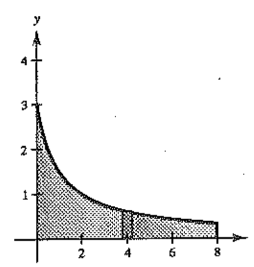
25. $R(x) = \dfrac{1}{x}$, $r(x) = 0$

$$V = \pi\int_1^4 \left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left[-\frac{1}{x}\right]_1^4$$
$$= \frac{3\pi}{4}$$



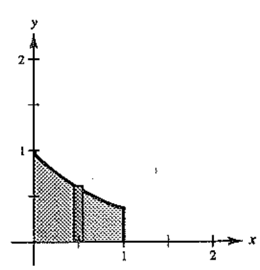
26. $R(x) = \dfrac{3}{x+1}$, $r(x) = 0$

$$V = \pi\int_0^8 \left(\frac{3}{x+1}\right)^2 dx$$
$$= 9\pi\int_0^8 (x+1)^{-2}\,dx$$
$$= 9\pi\left[-\frac{1}{x+1}\right]_0^8 = 8\pi$$



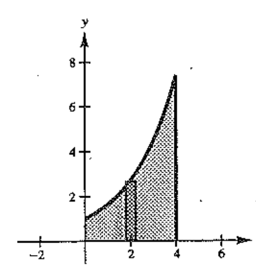
27. $R(x) = e^{-x}$, $r(x) = 0$

$$V = \pi\int_0^1 (e^{-x})^2\,dx$$
$$= \pi\int_0^1 e^{-2x}\,dx$$
$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$
$$= \frac{\pi}{2}(1 - e^{-2}) \approx 1.358$$



28. $R(x) = e^{x/2}$, $r(x) = 0$

$$V = \pi\int_0^4 (e^{x/2})^2\,dx$$
$$= \pi\int_0^4 e^x\,dx$$
$$= \Big[\pi e^x\Big]_0^4$$
$$= \pi(e^4 - 1) \approx 168.38$$



29. $x^2 + 1 = -x^2 + 2x + 5$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

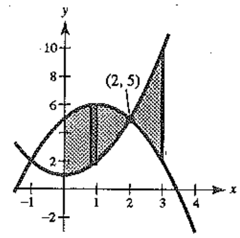
The curves intersect at $(-1, 2)$ and $(2, 5)$.

$$V = \pi\int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2)]\,dx + \pi\int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2]\,dx$$

$$= \pi\int_0^2 (-4x^3 - 8x^2 + 20x + 24)\,dx + \pi\int_2^3 (4x^3 + 8x^2 - 20x - 24)\,dx$$

$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$

$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

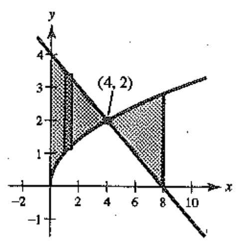


30. $$V = \pi\int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi\int_4^8 \left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx$$

$$= \pi\int_0^4 \left(\frac{x^2}{4} - 5x + 16\right) dx + \pi\int_4^8 \left(-\frac{x^2}{4} + 5x - 16\right) dx$$

$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8$$

$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$



31. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

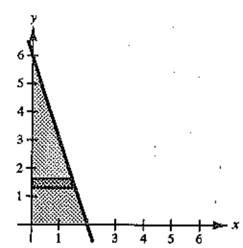
$$V = \pi\int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy$$

$$= \frac{\pi}{9}\int_0^6 [36 - 12y + y^2]\,dy$$

$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$

$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$

$= 8\pi = \frac{1}{3}\pi r^2 h$, Volume of cone



32. $y = 9 - x^2,\ y = 0,\ x = 2,\ x = 3$

$x = \sqrt{9 - y}$

$$V = \pi\int_0^5 \left(\sqrt{9 - y}\right)^2 - 2]^2\,dy$$

$$= \pi\int_0^5 (5 - y)\,dy$$

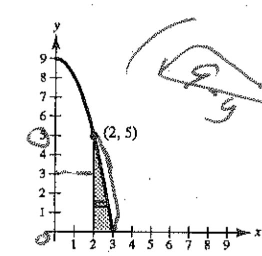
$$= \pi\left[5y - \frac{y^2}{2}\right]_0^5$$

$$= \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2}$$

9 - y - 4 = 5 - y

misprint ?

$(9-y-2)^2$ $(7-y)^2$



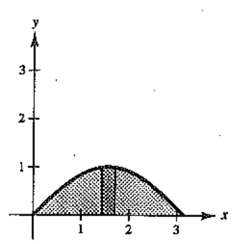
33. $$V = \pi\int_0^\pi (\sin x)^2\,dx$$

$$= \pi\int_0^\pi \frac{1 - \cos 2x}{2}\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$

$$= \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

Numerical approximation: 4.9348



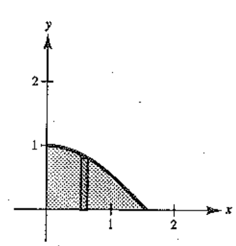
34. $$V = \pi\int_0^{\pi/2} [\cos x]^2\,dx$$

$$= \pi\int_0^{\pi/2} \frac{1 + \cos 2x}{2}\,dx$$

$$= \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$

Numerical approximation: 2.4674



35. $V = \pi\int_1^2 (e^{x-1})^2\,dx$

$= \pi\int_1^2 e^{2x-2}\,dx$

$= \frac{\pi}{2}e^{2x-2}\Big]_1^2$

$= \frac{\pi}{2}(e^2 - 1)$

Numerical approximation: 10.0359

36. $V = \pi\int_{-1}^2 [e^{x/2} + e^{-x/2}]^2\,dx$

$= \pi\int_{-1}^2 [e^x + e^{-x} + 2]\,dx$

$= \pi\Big[e^x - e^{-x} + 2x\Big]_{-1}^2$

$= \pi[(e^2 - e^{-2} + 4) - (e^{-1} - e - 2)]$

$= \pi[e^2 + e + 6 - e^{-2} - e^{-1}]$

Numerical approximation: 49.0218

37. $V = \pi\int_0^2 [e^{-x^2}]^2\,dx \approx 1.9686$

38. $V = \pi\int_1^3 [\ln x]^2\,dx \approx 3.2332$

39. $V = \pi\int_0^5 [2\arctan(0.2x)]^2\,dx$

≈ 15.4115

40. $x^2 = \sqrt{2x}$

$x^4 = 2x$

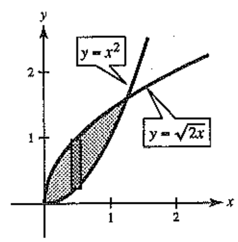
$x^3 = 2$

$x = 2^{1/3} \approx 1.2599$

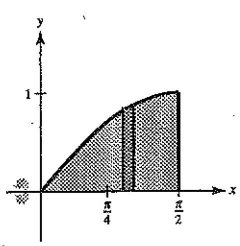
$V = \pi\int_0^{2^{1/3}} \left[(\sqrt{2x})^2 - (x^2)^2\right]dx$

$= \pi\int_0^{2^{1/3}} (2x - x^4)\,dx$

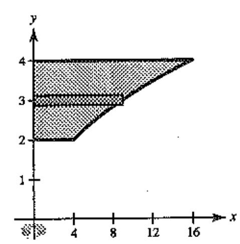
$= \frac{3\cdot 2^{2/3}\pi}{5} \approx 2.9922$



41. $\pi\int_0^{\pi/2} \sin^2 x\,dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi/2$ about the x-axis.

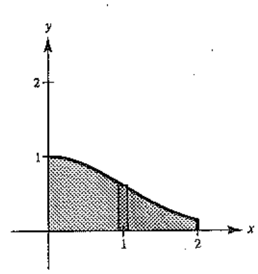


42. $\pi\int_2^4 y^4\,dy$ represents the volume of the solid generated by revolving the region bounded by $x = y^2$, $x = 0$, $y = 2$, $y = 4$ about the y-axis.



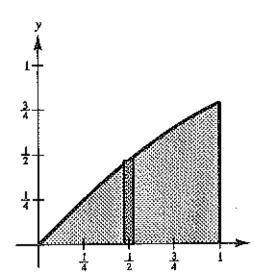
43. $A \approx 3$

Matches (a)

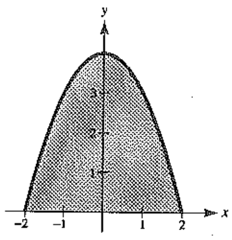


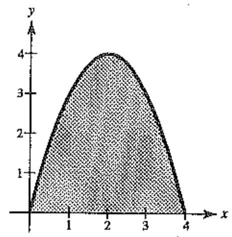
44. $A \approx \frac{3}{4}$

Matches (b)



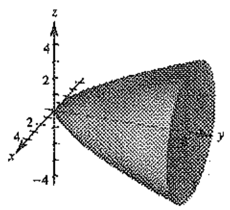
45.



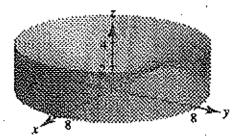


The volumes are the same because the solid has been translated horizontally. $(4x - x^2 = 4 - (x - 2)^2)$

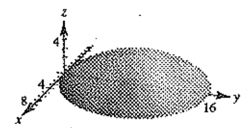
46. (a)



(b)



(c)



$a < c < b$

47. $R(x) = \frac{1}{2}x,\ r(x) = 0$

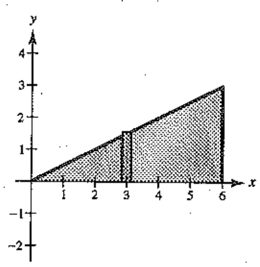
$$V = \pi\int_0^6 \frac{1}{4}x^2\,dx$$

$$= \left[\frac{\pi}{12}x^3\right]_0^6 = 18\pi$$

Note: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$

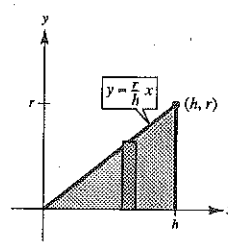


48. $R(x) = \frac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx$$

$$= \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h$$

$$= \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$



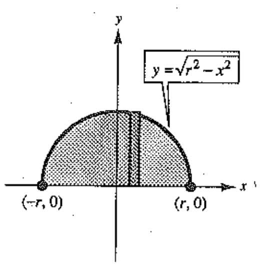
49. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$V = \pi\int_{-r}^{r} (r^2 - x^2)\,dx$$

$$= 2\pi\int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi\left[r^2x - \frac{1}{3}x^3\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$$



50. $x = \sqrt{r^2 - y^2},\ R(y) = \sqrt{r^2 - y^2},\ r(y) = 0$

$$V = \pi\int_h^r \left(\sqrt{r^2 - y^2}\right)^2 dy$$

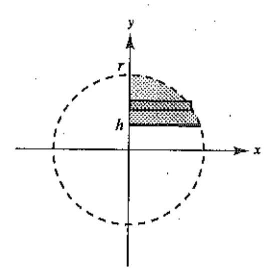
$$= \pi\int_h^r (r^2 - y^2)\,dy$$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_h^r$$

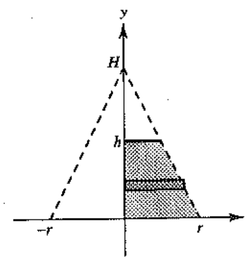
$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2h - \frac{h^3}{3}\right)\right]$$

$$= \pi\left(\frac{2r^3}{3} - r^2h + \frac{h^3}{3}\right)$$

$$= \frac{\pi}{3}(2r^3 - 3r^2h + h^3)$$



51. $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right),\ R(y) = r\left(1 - \frac{y}{H}\right),\ r(y) = 0$



$$V = \pi\int_0^h \left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right) dy$$
$$= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h$$
$$= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right)$$
$$= \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$$

52. (a) $V = \pi\int_0^4 (\sqrt{x})^2\, dx = \pi\int_0^4 x\, dx = \left[\frac{\pi x^2}{2}\right]_0^4 = 8\pi$

Let $0 < c < 4$ and set

$$\pi\int_0^c x\, dx = \left[\frac{\pi x^2}{2}\right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$
$$c^2 = 8$$
$$c = \sqrt{8} = 2\sqrt{2}$$

Thus, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

(b) Set $\pi\int_0^c x\, dx = \frac{8\pi}{3}$ (one third of the volume). Then

$$\frac{\pi c^2}{2} = \frac{8\pi}{3},\ c^2 = \frac{16}{3},\ c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$$

To find the other value, set

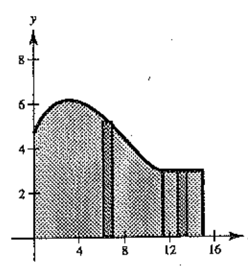
$\pi\int_0^d x\, dx = \frac{16\pi}{3}$ (two thirds of the volume).

Then $\frac{\pi d^2}{2} = \frac{16\pi}{3},\ d^2 = \frac{32}{3},\ d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}.$

The x-values that divide the solid into three parts of equal volume are $x = (4\sqrt{3})/3$ and $x = (4\sqrt{6})/3$.

53. $V = \pi\int_0^2 \left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\, dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}$

54. $y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$

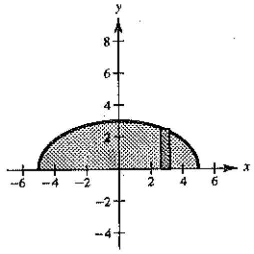


$$V = \pi\int_0^{11.5} \left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\, dx$$
$$= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2 x\right]_{11.5}^{15}$$

≈ 1031.9016 cubic centimeters

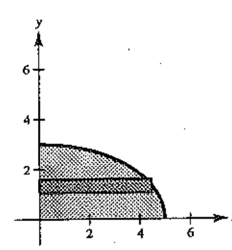
55. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2},\ r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^5 (25 - x^2)\, dx$$
$$= \frac{18\pi}{25}\int_0^5 (25 - x^2)\, dx$$
$$= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5$$
$$= 60\pi$$



(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2},\ r(y) = 0,\ x \ge 0$

$$V = \frac{25\pi}{9}\int_0^3 (9 - y^2)\, dy$$
$$= \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3$$
$$= 50\pi$$



56. (a) First find where $y = b$ intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$

$$x^2 = 16 - 4b = 4(4 - b)$$

$$x = 2\sqrt{4 - b}$$

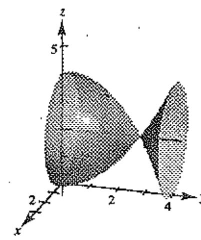
$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^4 \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$

$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$

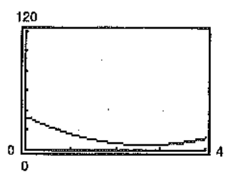
$$= \pi\int_0^4 \left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right] dx$$

$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$

$$= \pi\left[\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right] = \pi\left[4b^2 - \frac{64}{3}b + \frac{512}{15}\right]$$



(b) Graph of $V(b) = \pi\left[4b^2 - \frac{64}{3}b + \frac{512}{15}\right]$

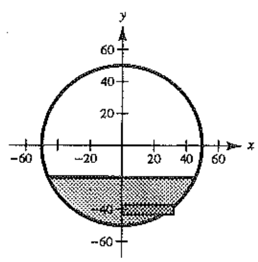


Minimum volume is 17.87 for $b = 2.67$.

(c) $V'(b) = \pi\left[8b - \frac{64}{3}\right] = 0 \Rightarrow b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \frac{8}{3}$ is a relative minimum.

57. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500{,}000\pi}{3}\ \text{ft}^3$



Volume of water in the tank:

$$\pi\int_{-50}^{y_0} \left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0} (2500 - y^2)\, dy$$

$$= \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0}$$

$$= \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

When the tank is one-fourth of its capacity:

$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$

$$y_0^3 - 7500y_0 - 125{,}000 = 0$$

$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

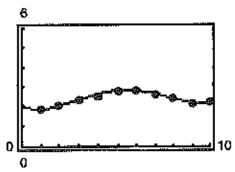
58. (a) $V = \int_0^{10} \pi[f(x)]^2\,dx$

Simpson's Rule: $b - a = 10 - 0 = 10,\ n = 10$

$$V \approx \frac{\pi}{3}[(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2]$$

$$\approx \frac{\pi}{3}[178.405] \approx 186.83 \text{ cm}^3$$

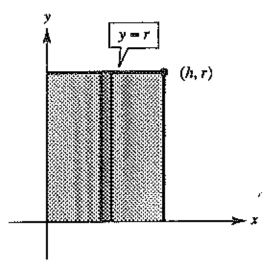
(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$



(c) $V \approx \int_0^{10} \pi f(x)^2\,dx \approx 186.35 \text{ cm}^3$

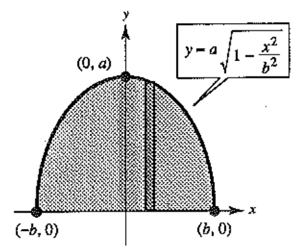
59. (a) $\pi\int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



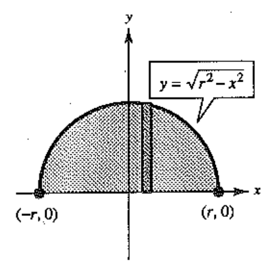
(b) $\pi\int_{-b}^{b}\left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



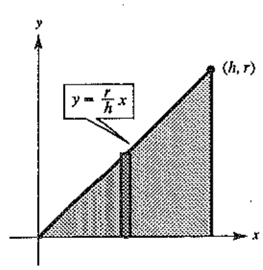
(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



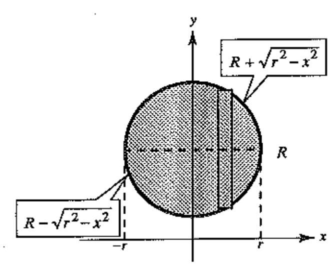
(d) $\pi\int_0^h \left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.



(e) $\pi\int_{-r}^{r}\left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.

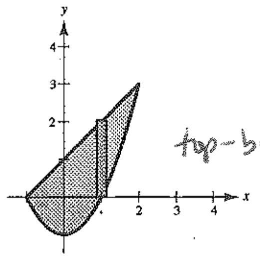


60. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2.$$

Thus, the volumes are the same.

61.



top-bottom $V = \int$ area of section.

Base of cross section $= (x + 1) - (x^2 - 1) = 2 + x - x^2$

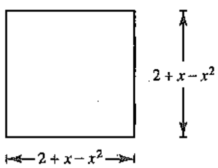
base · ht

(a) $A(x) = b^2 = (2 + x - x^2)^2$

$= 4 + 4x - 3x^2 - 2x^3 + x^4$

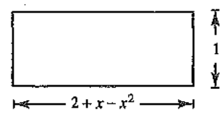
$$V = \int_{-1}^{2} (4 + 4x - 3x^2 - 2x^3 + x^4)\, dx$$

$$= \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

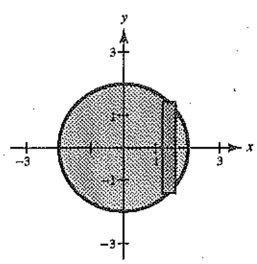


(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



62.

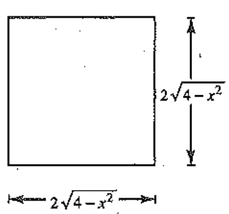


Base of cross section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$$V = \int_{-2}^{2} 4(4 - x^2)\, dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{128}{3}$$



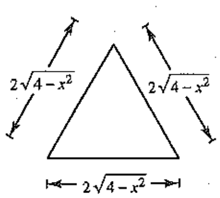
(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right)$

$= \sqrt{3}(4 - x^2)$

$$V = \sqrt{3}\int_{-2}^{2} (4 - x^2)\, dx$$

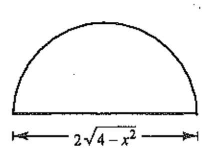
$$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{32\sqrt{3}}{3}$$



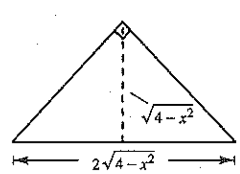
(c) $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}(4 - x^2)$

$$V = \frac{\pi}{2}\int_{-2}^{2} (4 - x^2)\, dx = \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16\pi}{3}$$

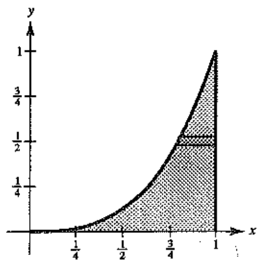


(d) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$

$$V = \int_{-2}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$



63.



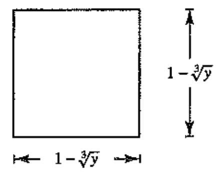
Base of cross section $= 1 - \sqrt[3]{y}$

(a) $A(y) = b^2 = \left(1 - \sqrt[3]{y}\right)^2$

$$V = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy$$

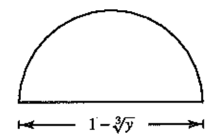
$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3})\,dy$$

$$= \left[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3}\right]_0^1 = \frac{1}{10}$$



(b) $A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1 - \sqrt[3]{y}}{2}\right)^2 = \frac{1}{8}\pi\left(1 - \sqrt[3]{y}\right)^2$

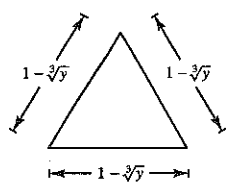
$$V = \frac{1}{8}\pi \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{8}\left(\frac{1}{10}\right) = \frac{\pi}{80}$$



(c) $A(y) = \frac{1}{2}bh = \frac{1}{2}\left(1 - \sqrt[3]{y}\right)\left(\frac{\sqrt{3}}{2}\right)\left(1 - \sqrt[3]{y}\right)$

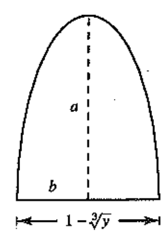
$$= \frac{\sqrt{3}}{4}\left(1 - \sqrt[3]{y}\right)^2$$

$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\sqrt{3}}{4}\left(\frac{1}{10}\right) = \frac{\sqrt{3}}{40}$$



(d) $A(y) = \frac{1}{2}\pi ab = \frac{\pi}{2}(2)\left(1 - \sqrt[3]{y}\right)\frac{1 - \sqrt[3]{y}}{2} = \frac{\pi}{2}\left(1 - \sqrt[3]{y}\right)^2$

$$V = \frac{\pi}{2}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{2}\left(\frac{1}{10}\right) = \frac{\pi}{20}$$



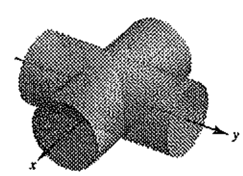
64. The cross sections are squares. By symmetry, we can set up an integral for an eighth of the volume and multiply by 8.

$A(y) = b^2 = \left(\sqrt{r^2 - y^2}\right)^2$

$$V = 8\int_0^r (r^2 - y^2)\,dy$$

$$= 8\left[r^2 y - \frac{1}{3}y^3\right]_0^r$$

$$= \frac{16}{3}r^3$$



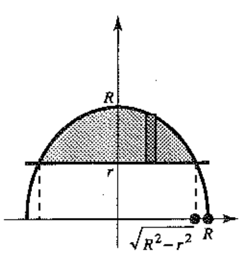
65. $V = \pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$

$$= 2\pi \int_0^{\sqrt{R^2 - r^2}} (R^2 - r^2 - x^2)\,dx$$

$$= 2\pi\left[(R^2 - r^2)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2 - r^2}}$$

$$= 2\pi\left[(R^2 - r^2)^{3/2} - \frac{(R^2 - r^2)^{3/2}}{3}\right]$$

$$= \frac{4}{3}\pi(R^2 - r^2)^{3/2}$$



66. $\frac{4}{3}\pi(25 - r^2)^{3/2} = \frac{1}{2}\left(\frac{4}{3}\right)\pi(125)$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2}\right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. $V = \pi\int_0^1 y^2\,dy = \pi\frac{y^3}{3}\Big]_0^1 = \frac{\pi}{3}$

68. $V = \pi\int_0^1 [1^2 - (1 - y)^2]\,dy$

$$= \pi\int_0^1 [2y - y^2]\,dy$$

$$= \pi\left[y^2 - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[1 - \frac{1}{3}\right] = \frac{2}{3}\pi$$

69. $V = \pi\int_0^1 (x^2 - x^4)\,dx$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{2\pi}{15}$$

70. $V = \pi\int_0^1 [(1 - x^2)^2 - (1 - x)^2]\,dx$

$$= \pi\int_0^1 [1 - 2x^2 + x^4 - 1 + 2x - x^2]\,dx$$

$$= \pi\int_0^1 [2x - 3x^2 + x^4]\,dx$$

$$= \pi\left[x^2 - x^3 + \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\frac{1}{5}\right] = \frac{\pi}{5}$$

71. $V = \pi\int_0^1 (1 - y)\,dy$

$$= \pi\left[y - \frac{y^2}{2}\right]_0^1$$

$$= \pi\left[1 - \frac{1}{2}\right]$$

$$= \frac{\pi}{2}$$

72. $V = \pi\int_0^1 \left(1 - \sqrt{y}\right)^2 dy$

$$= \pi\int_0^1 \left(1 - 2\sqrt{y} + y\right) dy$$

$$= \pi\left[y - \frac{4}{3}y^{3/2} + \frac{y^2}{2}\right]_0^1$$

$$= \pi\left[1 - \frac{4}{3} + \frac{1}{2}\right]$$

$$= \frac{\pi}{6}$$

73. $V = \pi\int_0^1 (y - y^2)\,dy$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{\pi}{6}$$

74. $V = \pi\int_0^1 \left[(1-y)^2 - \left(1-\sqrt{y}\right)^2\right] dy$

$= \pi\int_0^1 \left[1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right] dy$

$= \pi\int_0^1 \left[2\sqrt{y} - 3y + y^2\right] dy$

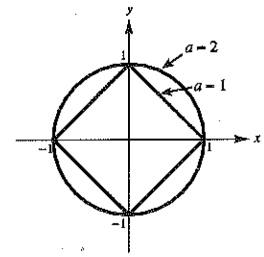
$= \pi\left[\frac{4}{3}y^{3/2} - \frac{3y^2}{2} + \frac{y^3}{3}\right]_0^1$

$= \pi\left[\frac{4}{3} - \frac{3}{2} + \frac{1}{3}\right]$

$= \frac{\pi}{6}$

75. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.

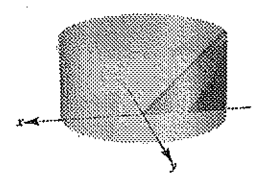


(b) $|y| = (1 - |x|^a)^{1/a}$

$A = 2\int_{-1}^{1} (1 - |x|^a)^{1/a}\,dx = 4\int_0^1 (1 - x^a)^{1/a}\,dx$

To approximate the volume of the solid, form n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.

76. (a) Since the cross sections are isosceles right triangles:



$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2-y^2}\right)\left(\sqrt{r^2-y^2}\right) = \frac{1}{2}(r^2 - y^2)$

$V = \frac{1}{2}\int_{-r}^{r} (r^2 - y^2)\,dy = \int_0^r (r^2 - y^2)\,dy = \left[r^2y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3$

(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2-y^2}\left(\sqrt{r^2-y^2}\tan\theta\right) = \frac{\tan\theta}{2}(r^2 - y^2)$

$V = \frac{\tan\theta}{2}\int_{-r}^{r} (r^2 - y^2)\,dy = \tan\theta\int_0^r (r^2 - y^2)\,dy = \tan\theta\left[r^2y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3\tan\theta$

As $\theta \to 90°$, $V \to \infty$.

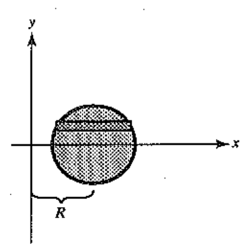
77. (a) $(x - R)^2 + y^2 = r^2$

$x = R \pm \sqrt{r^2 - y^2}$

$V = 2\pi\int_0^r \left(\left[R + \sqrt{r^2-y^2}\right]^2 - \left[R - \sqrt{r^2-y^2}\right]^2\right) dy$

$= 2\pi\int_0^r 4R\sqrt{r^2-y^2}\,dy$

$= 8\pi R\int_0^r \sqrt{r^2-y^2}\,dy$



(b) $\int_0^r \sqrt{r^2-y^2}\,dy$ is one-quarter of the area of a circle of radius r, $\frac{1}{4}\pi r^2$.

$V = 8\pi R\left(\frac{1}{4}\pi r^2\right) = 2\pi^2 r^2 R$

Section 7.3 Volume: The Shell Method

1. $p(x) = x, \; h(x) = x$

$$V = 2\pi\int_0^2 x(x)\,dx$$

$$= \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

2. $p(x) = x, \; h(x) = 1 - x$

$$V = 2\pi\int_0^1 x(1 - x)\,dx$$

$$= 2\pi\int_0^1 (x - x^2)\,dx$$

$$= 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

3. $p(x) = x, \; h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 x\sqrt{x}\,dx$$

$$= 2\pi\int_0^4 x^{3/2}\,dx$$

$$= \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

4. $p(x) = x, \; h(x) = 8 - (x^2 + 4) = 4 - x^2$

$$V = 2\pi\int_0^2 x(4 - x^2)\,dx$$

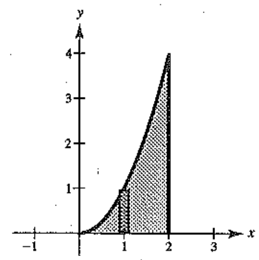
$$= 2\pi\int_0^2 (4x - x^3)\,dx$$

$$= 2\pi\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8\pi$$

5. $p(x) = x, \; h(x) = x^2$

$$V = 2\pi\int_0^2 x^3\,dx$$

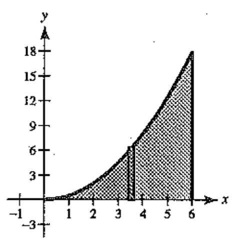
$$= \left[\frac{\pi}{2}x^4\right]_0^2 = 8\pi$$



6. $p(x) = x, \; h(x) = \frac{1}{2}x^2$

$$V = 2\pi\int_0^6 \frac{1}{2}x^3\,dx$$

$$= \left[\pi\frac{x^4}{4}\right]_0^6 = 324\pi$$

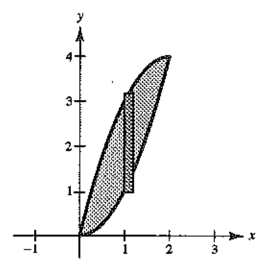


7. $p(x) = x, \; h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 x(4x - 2x^2)\,dx$$

$$= 4\pi\int_0^2 (2x^2 - x^3)\,dx$$

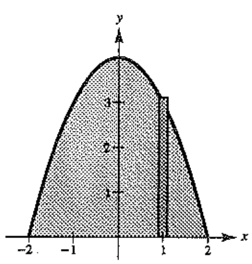
$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$



8. $p(x) = x, \; h(x) = 4 - x^2$

$$V = 2\pi\int_0^2 (4x - x^3)\,dx$$

$$= 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 8\pi$$



9. $p(x) = x$

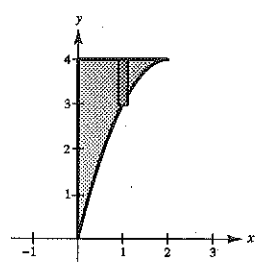
$$h(x) = 4 - (4x - x^2)$$

$$= x^2 - 4x + 4$$

$$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2$$

$$= \frac{8\pi}{3}$$

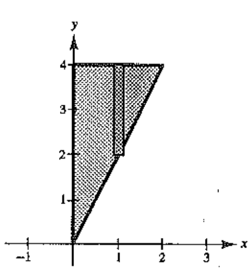


10. $p(x) = x, \; h(x) = 4 - 2x$

$$V = 2\pi\int_0^2 x(4 - 2x)\,dx$$

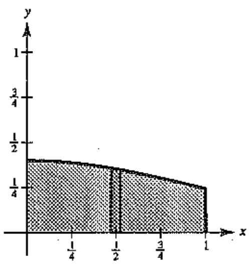
$$= 2\pi\int_0^2 (4x - 2x^2)\,dx$$

$$= 2\pi\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{16\pi}{3}$$



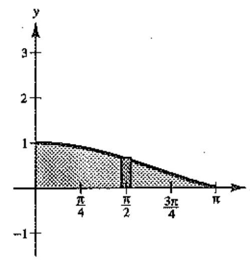
11. $p(x) = x,\ h(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$$V = 2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx$$
$$= \sqrt{2\pi}\int_0^1 e^{-x^2/2}x\,dx$$
$$= \left[-\sqrt{2\pi}e^{-x^2/2}\right]_0^1$$
$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$
$$\approx 0.986$$



12. $p(x) = x,\ h(x) = \dfrac{\sin x}{x}$

$$V = 2\pi\int_0^{\pi} x\left[\frac{\sin x}{x}\right]dx$$
$$= 2\pi\int_0^{\pi}\sin x\,dx$$
$$= \Big[-2\pi\cos x\Big]_0^{\pi} = 4\pi$$



13. $p(y) = y,\ h(y) = 2 - y$

$$V = 2\pi\int_0^2 y(2 - y)\,dy$$
$$= 2\pi\int_0^2 (2y - y^2)\,dy$$
$$= 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}$$

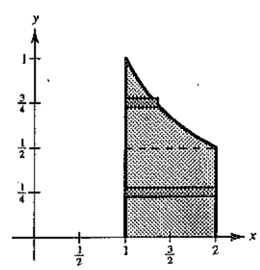
14. $p(y) = -y,\quad (p(y) \ge 0 \text{ on } [-2, 0])$

$$h(y) = 4 - (2 - y) = 2 + y$$
$$V = 2\pi\int_{-2}^{0} (-y)(2 + y)\,dy$$
$$= 2\pi\int_{-2}^{0} (-2y - y^2)\,dy$$
$$= 2\pi\left[-y^2 - \frac{y^3}{3}\right]_{-2}^{0} = \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \le y < \dfrac{1}{2}$.

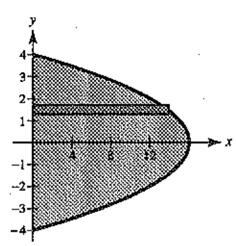
$p(y) = y$ and $h(y) = \dfrac{1}{y} - 1$ if $\dfrac{1}{2} \le y \le 1$.

$$V = 2\pi\int_0^{1/2} y\,dy + 2\pi\int_{1/2}^{1}(1 - y)\,dy$$
$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



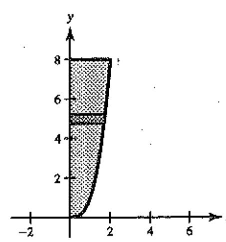
16. $p(y) = y,\ h(y) = 16 - y^2$

$$V = 2\pi\int_0^4 y(16 - y^2)\,dy$$
$$= 2\pi\int_0^4 (16y - y^3)\,dy$$
$$= 2\pi\left[8y^2 - \frac{y^4}{4}\right]_0^4$$
$$= 2\pi[128 - 64] = 128\pi$$



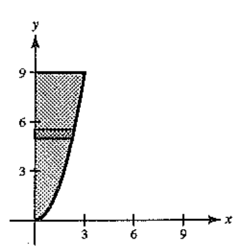
17. $p(y) = y,\ h(y) = \sqrt[3]{y}$

$$V = 2\pi\int_0^8 y\sqrt[3]{y}\,dy$$
$$= 2\pi\int_0^8 y^{4/3}\,dy$$
$$= \left[2\pi\left(\frac{3}{7}\right)y^{7/3}\right]_0^8$$
$$= \frac{6\pi}{7}(2^7) = \frac{768\pi}{7}$$



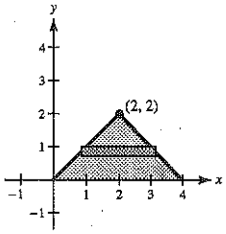
18. $p(y) = y,\ h(y) = \sqrt{y}$

$$V = 2\pi\int_0^9 y\sqrt{y}\,dy = 2\pi\int_0^9 y^{3/2}\,dy$$
$$= 2\pi\left(\frac{2}{5}\right)y^{5/2}\Big]_0^9$$
$$= \frac{4\pi}{5}(9)^{5/2} = \frac{4\pi}{5}(243) = \frac{972}{5}\pi$$



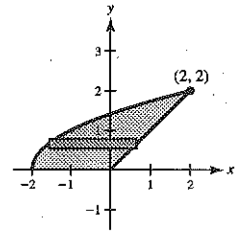
19. $p(y) = y,\ h(y) = (4 - y) - (y) = 4 - 2y$

$$V = 2\pi\int_0^2 y(4 - 2y)\,dy$$
$$= 2\pi\int_0^2 (4y - 2y^2)\,dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3\right]_0^2$$
$$= 2\pi\left[8 - \frac{16}{3}\right] = \frac{16\pi}{3}$$



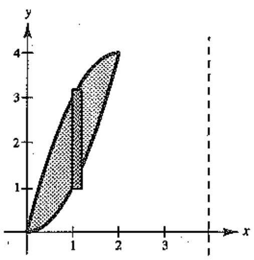
20. $p(y) = y,\ h(y) = y - (y^2 - 2) = 2 + y - y^2$

$$V = 2\pi\int_0^2 y(2 + y - y^2)\,dy$$
$$= 2\pi\int_0^2 (2y + y^2 - y^3)\,dy$$
$$= 2\pi\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$
$$= 2\pi\left[4 + \frac{8}{3} - 4\right] = \frac{16\pi}{3}$$



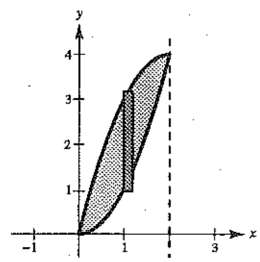
21. $p(x) = 4 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 (4 - x)(4x - 2x^2)\,dx$$
$$= 2\pi(2)\int_0^2 (x^3 - 6x^2 + 8x)\,dx$$
$$= 4\pi\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 16\pi$$



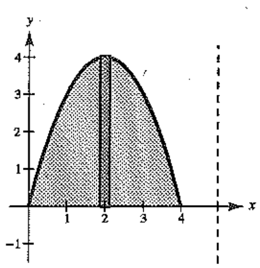
22. $p(x) = 2 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 (2 - x)(4x - 2x^2)\,dx$$
$$= 2\pi\int_0^2 (8x - 8x^2 + 2x^3)\,dx$$
$$= 2\pi\left[4x^2 - \frac{8}{3}x^3 + \frac{1}{2}x^4\right]_0^2 = \frac{16\pi}{3}$$



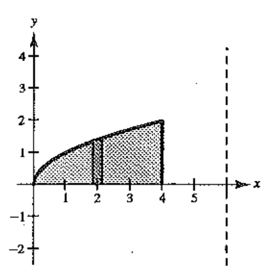
23. $p(x) = 5 - x,\ h(x) = 4x - x^2$

$$V = 2\pi\int_0^4 (5 - x)(4x - x^2)\,dx$$
$$= 2\pi\int_0^4 (x^3 - 9x^2 + 20x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 64\pi$$



24. $p(x) = 6 - x,\ h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 (6 - x)\sqrt{x}\,dx$$
$$= 2\pi\int_0^4 (6x^{1/2} - x^{3/2})\,dx$$
$$= 2\pi\left[4x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 = \frac{192\pi}{5}$$



25. The shell method would be easier: $V = 2\pi\int_0^4 [4 - (y - 2)^2]y\,dy$ shells

Using the disk method: $V = \pi\int_0^4 \left[\left(2 + \sqrt{4 - x}\right)^2 - \left(2 - \sqrt{4 - x}\right)^2\right] dx$ $\left[\textbf{Note: } V = \frac{128\pi}{3}\right]$

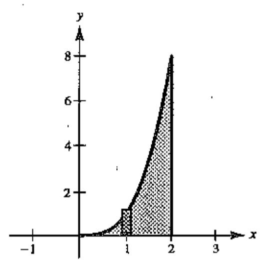
26. The shell method is easier: $V = 2\pi \int_0^{\ln 4} x(4 - e^x)\,dx$

Using the disk method, $x = \ln(4 - y)$ and $V = \pi \int_0^3 (\ln(4 - y))^2\,dy$. [**Note:** $V = \pi[8(\ln 2)^2 - 8\ln 2 + 3]$]

27. (a) **Disk**

$R(x) = x^3,\ r(x) = 0$

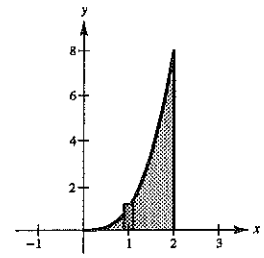
$$V = \pi \int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$



(b) **Shell**

$p(x) = x,\ h(x) = x^3$

$$V = 2\pi \int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$



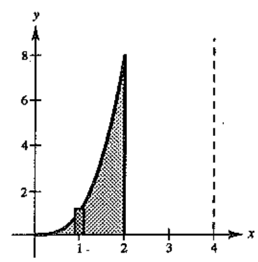
(c) **Shell**

$p(x) = 4 - x,\ h(x) = x^3$

$$V = 2\pi \int_0^2 (4 - x)x^3\,dx$$

$$= 2\pi \int_0^2 (4x^3 - x^4)\,dx$$

$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$



28. (a) **Disk**

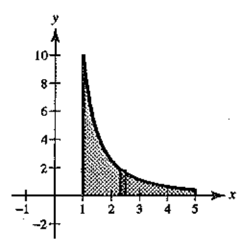
$R(x) = \frac{10}{x^2},\ r(x) = 0$

$$V = \pi \int_1^5 \left(\frac{10}{x^2}\right)^2 dx$$

$$= 100\pi \int_1^5 x^{-4}\,dx$$

$$= 100\pi\left[\frac{x^{-3}}{-3}\right]_1^5$$

$$= -\frac{100\pi}{3}\left[\frac{1}{125} - 1\right] = \frac{496}{15}\pi$$



(b) **Shell**

$R(x) = x,\ r(x) = 0$

$$V = 2\pi \int_1^5 x\left(\frac{10}{x^2}\right) dx$$

$$= 20\pi \int_1^5 \frac{1}{x}\,dx$$

$$= 20\pi\Big[\ln|x|\Big]_1^5 = 20\pi \ln 5$$

(c) **Disk**

$R(x) = 10,\ r(x) = 10 - \frac{10}{x^2}$

$$V = \pi \int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2}\right)^2\right] dx$$

$$= \pi\left[\frac{100}{3x^3} - \frac{200}{x}\right]_1^5 = \frac{1904}{15}\pi$$

29. (a) **Shell**

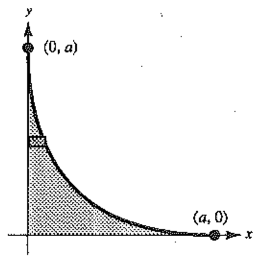
$p(y) = y, \ h(y) = (a^{1/2} - y^{1/2})^2$

$$V = 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y)\, dy$$

$$= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2)\, dy$$

$$= 2\pi \left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi \left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right] = \frac{\pi a^3}{15}$$



(b) Same as part (a) by symmetry

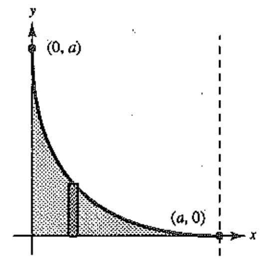
(c) **Shell**

$p(x) = a - x, \ h(x) = (a^{1/2} - x^{1/2})^2$

$$V = 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2\, dx$$

$$= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2)\, dx$$

$$= 2\pi \left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a = \frac{4\pi a^3}{15}$$



30. (a) **Disk**

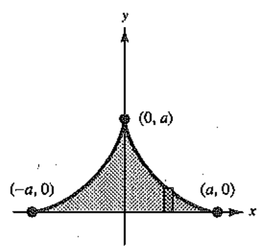
$R(x) = (a^{2/3} - x^{2/3})^{3/2}, \ r(x) = 0$

$$V = \pi \int_{-a}^a (a^{2/3} - x^{2/3})^3\, dx$$

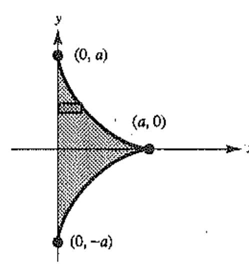
$$= 2\pi \int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)\, dx$$

$$= 2\pi \left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3\right]_0^a$$

$$= 2\pi \left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3\right) = \frac{32\pi a^3}{105}$$



(b) Same as part (a) by symmetry

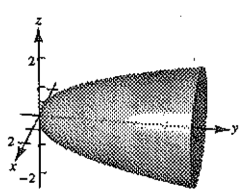


31. Answers will vary.

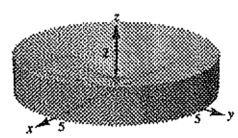
(a) The rectangles would be vertical.

(b) The rectangles would be horizontal.

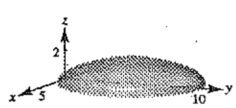
32. (a)



(b)



(c)



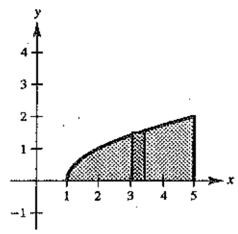
$a < c < b$

33. $\pi \int_1^5 (x - 1)\, dx = \pi \int_1^5 (\sqrt{x - 1})^2\, dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x-axis by using the disk method.

$2\pi \int_0^2 y[5 - (y^2 + 1)]\, dy$

represents this same volume by using the shell method.



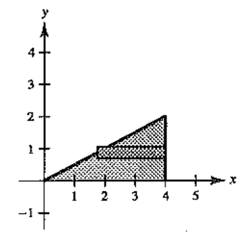
Disk method

34. $2\pi \int_0^4 x\left(\frac{x}{2}\right) dx$

represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the y-axis by using the shell method.

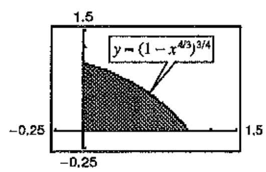
$\pi \int_0^2 [16 - (2y)^2]\, dy = \pi \int_0^2 [(4)^2 - (2y)^2]\, dy$

represents this same volume by using the disk method.



Disk method

35. (a)

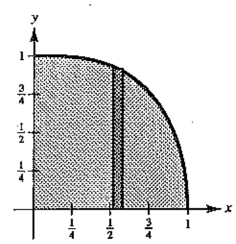


(b) $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$

$y = (1 - x^{4/3})^{3/4}$

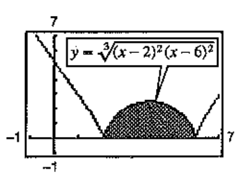
$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4}\, dx \approx 1.5056$

36. (a)



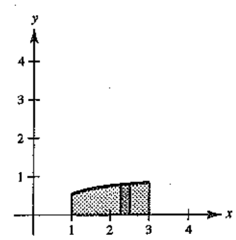
(b) $V = 2\pi \int_0^1 x\sqrt{1 - x^3}\, dx \approx 2.3222$

37. (a)



(b) $V = 2\pi \int_2^6 x\sqrt[3]{(x - 2)^2(x - 6)^2}\, dx \approx 187.249$

38. (a)

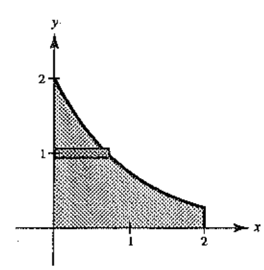


(b) $V = 2\pi \int_1^3 \frac{2x}{1 + e^{1/x}}\, dx \approx 19.0162$

39. $y = 2e^{-x},\ y = 0,\ x = 0,\ x = 2$

Volume ≈ 7.5

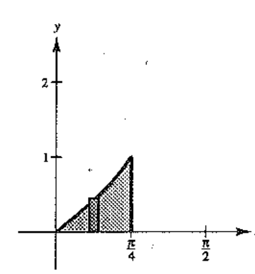
Matches (d)



40. $y = \tan x,\ y = 0,\ x = 0,\ x = \frac{\pi}{4}$

Volume ≈ 1

Matches (e)



41. $p(x) = x,\ h(x) = 2 - \frac{1}{2}x^2$

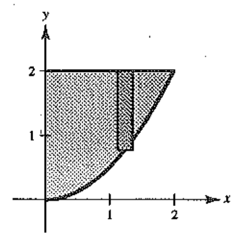
$$V = 2\pi\int_0^2 x\left(2 - \frac{1}{2}x^2\right)dx = 2\pi\int_0^2\left(2x - \frac{1}{2}x^3\right)dx = 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \quad \text{(total volume)}$$

Now find x_0 such that:

$$\pi = 2\pi\int_0^{x_0}\left(2x - \frac{1}{2}x^3\right)dx$$

$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$



$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \qquad \text{(Quadratic Formula)}$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$, since the other root is too large.

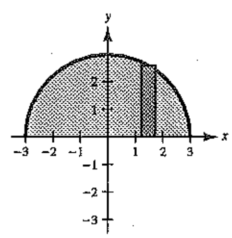
Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

42. Total volume of the hemisphere is $\frac{1}{2}\left(\frac{4}{3}\right)\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method, $p(x) = x$, $h(x) = \sqrt{9 - x^2}$. Find x_0 such that:

$$6\pi = 2\pi\int_0^{x_0} x\sqrt{9 - x^2}\,dx$$

$$6 = -\int_0^{x_0}(9 - x^2)^{1/2}(-2x)\,dx$$

$$= \left[-\frac{2}{3}(9 - x^2)^{3/2}\right]_0^{x_0} = 18 - \frac{2}{3}(9 - x_0^2)^{3/2}$$



$$(9 - x_0^2)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460$$

Diameter: $2\sqrt{9 - 18^{2/3}} \approx 2.920$

43. $$V = 4\pi\int_{-1}^{1}(2 - x)\sqrt{1 - x^2}\,dx$$

$$= 8\pi\int_{-1}^{1}\sqrt{1 - x^2}\,dx - 4\pi\int_{-1}^{1}x\sqrt{1 - x^2}\,dx$$

$$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi\int_{-1}^{1}x(1 - x^2)^{1/2}(-2)\,dx$$

$$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_{-1}^{1} = 4\pi^2$$

44. $$V = 4\pi\int_{-r}^{r}(R - x)\sqrt{r^2 - x^2}\,dx$$

$$= 4\pi R\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx - 4\pi\int_{-r}^{r}x\sqrt{r^2 - x^2}\,dx$$

$$= 4\pi R\left(\frac{\pi r^2}{2}\right) + \left[2\pi\left(\frac{2}{3}\right)(r^2 - x^2)^{3/2}\right]_{-r}^{r}$$

$$= 2\pi^2 r^2 R$$

45. (a) $\frac{d}{dx}[\sin x - x\cos x + C] = \cos x + x\sin x - \cos x = x\sin x$

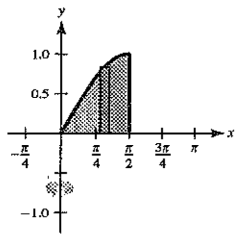
Hence, $\int x\sin x\,dx = \sin x - x\cos x + C$.

—CONTINUED—

45. —CONTINUED—

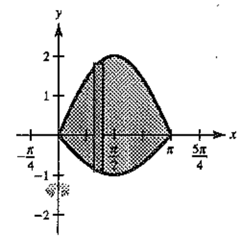
(b) (i) $p(x) = x, h(x) = \sin x$

$$V = 2\pi \int_0^{\pi/2} x \sin x \, dx$$
$$= 2\pi \Big[\sin x - x \cos x \Big]_0^{\pi/2}$$
$$= 2\pi[(1 - 0) - 0] = 2\pi$$



(ii) $p(x) = x, h(x) = 2 \sin x - (-\sin x) = 3 \sin x$

$$V = 2\pi \int_0^{\pi} x(3 \sin x) \, dx$$
$$= 6\pi \int_0^{\pi} x \sin x \, dx$$
$$= 6\pi \Big[\sin x - x \cos x \Big]_0^{\pi}$$
$$= 6\pi[\pi] = 6\pi^2$$

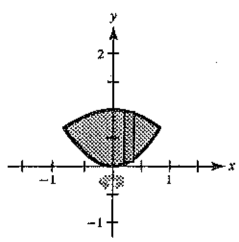


46. (a) $\frac{d}{dx}[\cos x + x \sin x + C] = -\sin x + \sin x + x \cos x = x \cos x$

Hence, $\int x \cos x \, dx = \cos x + x \sin x + C$.

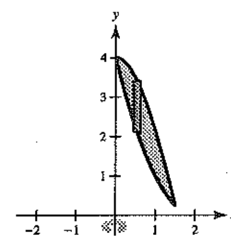
(b) (i) $x^2 = \cos x \implies x \approx \pm 0.8241$

$$V \approx 2(2\pi) \int_0^{0.8241} x[\cos x - x^2] \, dx$$
$$= 4\pi \left[\cos x + x \sin x - \frac{x^4}{4} \right]_0^{0.8241}$$
$$\approx 2.1205$$



(ii) $4 \cos x = (x - 2)^2 \implies x = 0, 1.5110$

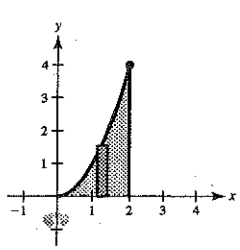
$$V \approx 2\pi \int_0^{1.511} x[4 \cos x - (x - 2)^2] \, dx$$
$$= 2\pi \int_0^{1.511} \left[4 \cos x + 4x \sin x - \frac{(x - 2)^3}{3} \right]_0^{1.511}$$
$$= 6.2993$$



47. $2\pi \int_0^2 x^3 \, dx = 2\pi \int_0^2 x(x^2) \, dx$

(a) Plane region bounded by $y = x^2, y = 0, x = 0, x = 2$

(b) Revolved about the y-axis

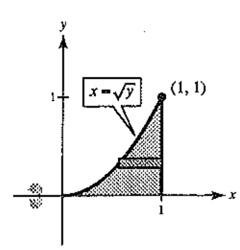


Other answers possible

48. $2\pi \int_0^1 (y - y^{3/2}) \, dy = 2\pi \int_0^1 y(1 - \sqrt{y}) \, dy$

(a) Plane region bounded by $x = \sqrt{y}, x = 1, y = 0$

(b) Revolved about the x-axis

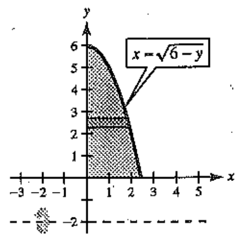


Other answers possible

49. $2\pi\int_0^6 (y+2)\sqrt{6-y}\,dy$

(a) Plane region bounded by $x = \sqrt{6-y}, x = 0, y = 0$

(b) Revolved around line $y = -2$

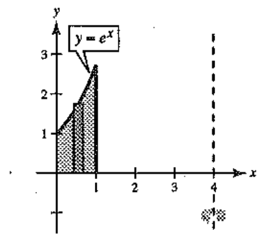


Other answers possible

50. $2\pi\int_0^1 (4-x)e^x\,dx$

(a) Plane region bounded by $y = e^x, y = 0, x = 0, x = 1$

(b) Revolved about the line $x = 4$



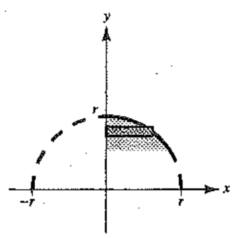
51. Disk Method

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

$$V = \pi\int_{r-h}^{r} (r^2 - y^2)\,dy$$

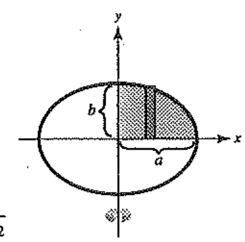
$$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$$



52. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$



$$p(x) = x,\ h(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$V = 2(2\pi)\int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

$$= \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - x^2}\,x\,dx$$

$$= \frac{4\pi b}{a}\left(\frac{-(a^2 - x^2)^{3/2}}{3}\right)\Bigg]_0^a$$

$$= \frac{4\pi b}{3a}a^3 = \frac{4}{3}\pi a^2 b$$

Note: If $a = b$, then volume is that of a sphere.

53. (a) Area region $= \displaystyle\int_0^b [ab^n - ax^n]\,dx$

$$= \left[ab^n x - a\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= ab^{n+1} - a\frac{b^{n+1}}{n+1}$$

$$= ab^{n+1}\left(1 - \frac{1}{n+1}\right) = ab^{n+1}\left(\frac{n}{n+1}\right)$$

$$R_1(n) = \frac{ab^{n+1}[n/(n+1)]}{(ab^n)b} = \frac{n}{n+1}$$

(b) $\displaystyle\lim_{n\to\infty} R_1(n) = \lim_{n\to\infty}\frac{n}{n+1} = 1$

$\displaystyle\lim_{n\to\infty}(ab^n)b = \infty$

(c) **Disk Method:**

$$V = 2\pi\int_0^b x(ab^n - ax^n)\,dx$$

$$= 2\pi a\int_0^b (xb^n - x^{n+1})\,dx$$

$$= 2\pi a\left[\frac{b^n}{2}x^2 - \frac{x^{n+2}}{n+2}\right]_0^b$$

$$= 2\pi a\left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\frac{n}{n+2}\right)$$

$$R_2(n) = \frac{\pi ab^{n+2}[n/(n+2)]}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2}\right)$$

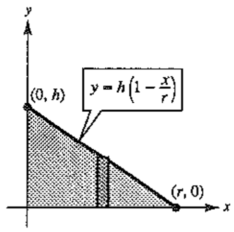
(d) $\displaystyle\lim_{n\to\infty} R_2(n) = \lim_{n\to\infty}\left(\frac{n}{n+2}\right) = 1$

$\displaystyle\lim_{n\to\infty}(\pi b^2)(ab^n) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = 1$.

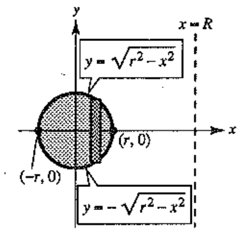
54. (a) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h.



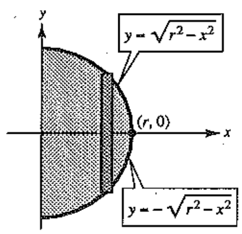
(b) $2\pi \int_{-r}^{r} (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



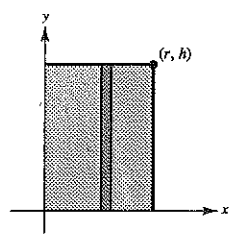
(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx$ (iii)

is the volume of a sphere with radius r.



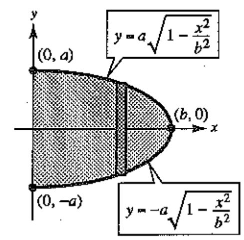
(d) $2\pi \int_0^r hx\, dx$ (i)

is the volume of a right circular cylinder with a radius of r and a height of h.



(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)}\, dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



55. (a) $V = 2\pi \int_0^4 xf(x)\, dx$

$$= \frac{2\pi(40)}{3(4)}[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0]$$

$$= \frac{20\pi}{3}[5800] \approx 121{,}475 \text{ cubic feet}$$

(b) Top line: $y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \implies y = -\frac{1}{2}x + 50$

Bottom line: $y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \implies y = -2x + 80$

$$V = 2\pi \int_0^{20} x\left(-\frac{1}{2}x + 50\right) dx + 2\pi \int_{20}^{40} x(-2x + 80)\, dx$$

$$= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x\right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x)\, dx$$

$$= 2\pi\left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40}$$

$$= 2\pi\left[\frac{26{,}000}{3}\right] + 2\pi\left[\frac{32{,}000}{3}\right]$$

$$\approx 121{,}475 \text{ cubic feet}$$

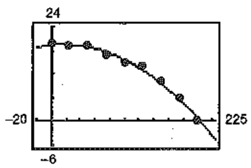
(Note that Simpson's Rule is exact for this problem.)

56. (a) $V = 2\pi\int_0^{200} xf(x)\,dx$

$\approx \frac{2\pi(200)}{3(8)}[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$

$\approx 1{,}366{,}593$ cubic feet

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi\int_0^{200} xd(x)\,dx \approx 2\pi(213{,}800)$

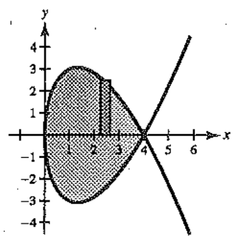
$= 1{,}343{,}345$ cubic feet

(d) Number gallons $\approx V(7.48) = 10{,}048{,}221$ gallons

57. $y^2 = x(4-x)^2, \quad 0 \le x \le 4$

$y_1 = \sqrt{x(4-x)^2} = (4-x)\sqrt{x}$

$y_2 = -\sqrt{x(4-x)^2} = -(4-x)\sqrt{x}$



(a) $V = \pi\int_0^4 x(4-x)^2\,dx$

$= \pi\int_0^4 (x^3 - 8x^2 + 16x)\,dx$

$= \pi\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^4 = \frac{64\pi}{3}$

(b) $V = 4\pi\int_0^4 x(4-x)\sqrt{x}\,dx$

$= 4\pi\int_0^4 (4x^{3/2} - x^{5/2})\,dx$

$= 4\pi\left[\frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{2048\pi}{35}$

(c) $V = 4\pi\int_0^4 (4-x)(4-x)\sqrt{x}\,dx$

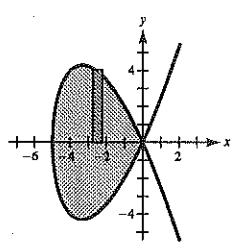
$= 4\pi\int_0^4 \left(16\sqrt{x} - 8x^{3/2} + x^{5/2}\right)dx$

$= 4\pi\left[\frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{8192\pi}{105}$

58. $y^2 = x^2(x+5), \quad -5 \le x \le 0$

$y_1 = \sqrt{x^2(x+5)} = x\sqrt{x+5}$

$y_2 = -\sqrt{x^2(x+5)} = -x\sqrt{x+5}$



(a) $V = \pi\int_{-5}^0 x^2(x+5)\,dx$

$= \pi\left[\frac{x^4}{4} + \frac{5x^3}{3}\right]_{-5}^0 = \frac{625\pi}{12}$

(b) $V = 4\pi\int_{-5}^0 x\left(x\sqrt{x+5}\right)dx$

Let $u = x + 5$, $du = dx$.

$V = 4\pi\int_0^5 (u-5)^2\sqrt{u}\,du$

$= 4\pi\int_0^5 (u^{5/2} - 10u^{3/2} + 25u^{1/2})\,du$

$= 4\pi\left[\frac{2}{7}u^{7/2} - 4u^{5/2} + \frac{50}{3}u^{3/2}\right]_0^5 = \frac{1600\sqrt{5}\,\pi}{21}$

(c) $V = 4\pi\int_{-5}^0 (-5-x)x\sqrt{x+5}\,dx$

Let $u = x + 5$, $du = dx$.

$V = 4\pi\int_0^5 (-u)(u-5)\sqrt{u}\,du$

$= 4\pi\int_0^5 (-u^{5/2} + 5u^{3/2})\,du$

$= 4\pi\left[-\frac{2}{7}u^{7/2} + 2u^{5/2}\right]_0^5 = \frac{400\sqrt{5}\,\pi}{7}$

59. $V_1 = \pi\int_{1/4}^{c} \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_{1/4}^{c} = \pi\left[-\frac{1}{c}+4\right] = \frac{4c-1}{c}\pi$

$V_2 = 2\pi\int_{1/4}^{c} x\left(\frac{1}{x}\right)dx = 2\pi x\Big]_{1/4}^{c} = 2\pi\left(c-\frac{1}{4}\right)$

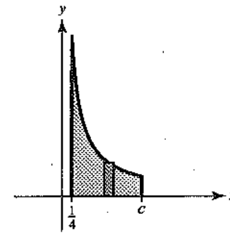
$V_1 = V_2 \Rightarrow \frac{4c-1}{c}\pi = 2\pi\left(c-\frac{1}{4}\right)$

$4c-1 = 2c\left(c-\frac{1}{4}\right)$

$4c^2-9c+2=0$

$(4c-1)(c-2)=0$

$c=2 \quad \left(c=\frac{1}{4}\text{ yields no volume.}\right)$



Section 7.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2+(12-0)^2}$

$= 13$

(b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$s = \int_0^5 \sqrt{1+\left(\frac{12}{5}\right)^2}\,dx$

$= \left[\frac{13}{5}x\right]_0^5 = 13$

2. $(1, 2), (7, 10)$

(a) $d = \sqrt{(7-1)^2+(10-2)^2}$

$= 10$

(b) $y = \frac{4}{3}x+\frac{2}{3}$

$y' = \frac{4}{3}$

$s = \int_1^7 \sqrt{1+\left(\frac{4}{3}\right)^2}\,dx$

$= \left[\frac{5}{3}x\right]_1^7 = 10$

3. $y = \frac{2}{3}x^{3/2}+1$

$y' = x^{1/2}, \quad 0 \le x \le 1$

$s = \int_0^1 \sqrt{1+x}\,dx$

$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$

$= \frac{2}{3}\left(\sqrt{8}-1\right) \approx 1.219$

4. $y = 2x^{3/2}+3$

$y' = 3x^{1/2}, \quad 0 \le x \le 9$

$s = \int_0^9 \sqrt{1+9x}\,dx$

$= \left[\frac{2}{27}(1+9x)^{3/2}\right]_0^9$

$= \frac{2}{27}(82^{3/2}-1) \approx 54.929$

5. $y = \frac{3}{2}x^{2/3}$

$y' = \frac{1}{x^{1/3}}, \quad 1 \le x \le 8$

$s = \int_1^8 \sqrt{1+\left(\frac{1}{x^{1/3}}\right)^2}\,dx$

$= \int_1^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx$

$= \frac{3}{2}\int_1^8 \sqrt{x^{2/3}+1}\left(\frac{2}{3x^{1/3}}\right)dx$

$= \frac{3}{2}\left[\frac{2}{3}(x^{2/3}+1)^{3/2}\right]_1^8$

$= 5\sqrt{5}-2\sqrt{2} \approx 8.352$

6. $y = \frac{x^4}{8}+\frac{1}{4x^2}$

$y' = \frac{1}{2}x^3-\frac{1}{2x^3}, \quad 1 \le x \le 2$

$1+(y')^2 = \left(\frac{1}{2}x^3+\frac{1}{2x^3}\right)^2, \quad [1, 2]$

$s = \int_a^b \sqrt{1+(y')^2}\,dx$

$= \int_1^2 \left(\frac{1}{2}x^3+\frac{1}{2x^3}\right)dx$

$= \left[\frac{1}{8}x^4-\frac{1}{4x^2}\right]_1^2$

$= \frac{33}{16} \approx 2.063$

7.
$$y = \frac{x^5}{10} + \frac{1}{6x^3}$$
$$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$$
$$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, \quad 1 \le x \le 2$$
$$s = \int_a^b \sqrt{1 + (y')^2}\,dx$$
$$= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2}\,dx$$
$$= \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)dx$$
$$= \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.2458$$

8.
$$y = \frac{3}{2}x^{2/3} + 4$$
$$y' = x^{-1/3}, \quad 1 \le x \le 27$$
$$s = \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx$$
$$= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\,dx$$
$$= \frac{3}{2}\int_1^{27} \sqrt{x^{2/3} + 1}\left(\frac{2}{3x^{1/3}}\right)dx$$
$$= \left[\frac{3}{2} \cdot \frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^{27}$$
$$= 10^{3/2} - 2^{3/2} \approx 28.794$$

9.
$$y = \ln(\sin x), \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$
$$y' = \frac{1}{\sin x}\cos x = \cot x$$
$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$
$$s = \int_{\pi/4}^{3\pi/4} \csc x\,dx$$
$$= \left[\ln\left|\csc x - \cot x\right|\right]_{\pi/4}^{3\pi/4}$$
$$= \ln\left(\sqrt{2} + 1\right) - \ln\left(\sqrt{2} - 1\right) \approx 1.763$$

10.
$$y = \ln(\cos x), \quad 0 \le x \le \frac{\pi}{3}$$
$$y' = \frac{-\sin x}{\cos x} = -\tan x$$
$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$
$$s = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx$$
$$= \int_0^{\pi/3} \sec x\,dx$$
$$= \ln\left|\sec x + \tan x\right|\Big]_0^{\pi/3}$$
$$= \ln\left(2 + \sqrt{3}\right) \approx 1.3170$$

11.
$$y = \frac{1}{2}(e^x + e^{-x})$$
$$y' = \frac{1}{2}(e^x - e^{-x}), \quad [0, 2]$$
$$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, \quad [0, 2]$$
$$s = \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}\,dx$$
$$= \frac{1}{2}\int_0^2 (e^x + e^{-x})\,dx$$
$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

12.
$$y = \ln\left(\frac{e^x + 1}{e^x - 1}\right) = \ln(e^x + 1) - \ln(e^x - 1)$$
$$\frac{dy}{dx} = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{-2e^x}{e^{2x} - 1} = \frac{2e^x}{1 - e^{2x}}$$
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4e^{2x}}{1 - 2e^{2x} + e^{4x}}$$
$$= \frac{1 + 2e^{2x} + e^{4x}}{(1 - e^{2x})^2} = \left(\frac{1 + e^{2x}}{1 - e^{2x}}\right)^2$$
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{\ln 2}^{\ln 3} \frac{1 + e^{2x}}{e^{2x} - 1}\,dx$$
$$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int_{\ln 2}^{\ln 3} \coth x\,dx$$
$$= \ln\left(\sinh(x)\right)\Big]_{\ln 2}^{\ln 3} = \ln\left(\frac{4}{3}\right) - \ln\left(\frac{3}{4}\right)$$
$$= \ln\left(\frac{4/3}{3/4}\right) = \ln\frac{16}{9} - 2\ln\left(\frac{4}{3}\right) \approx 0.57536$$

13. $x = \frac{1}{3}(y^2 + 2)^{3/2}, \quad 0 \le y \le 4$

$$\frac{dx}{dy} = y(y^2 + 2)^{1/2}$$

$$s = \int_0^4 \sqrt{1 + y^2(y^2 + 2)}\, dy$$

$$= \int_0^4 \sqrt{y^4 + 2y^2 + 1}\, dy$$

$$= \int_0^4 (y^2 + 1)\, dy$$

$$= \left[\frac{y^3}{3} + y\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

14. $x = \frac{1}{3}\sqrt{y}(y - 3), \quad 1 \le y \le 4$

$$x = \frac{1}{3}(y^{3/2} - 3y^{1/2})$$

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y + \frac{1}{4}y^{-1} - \frac{1}{2}$$

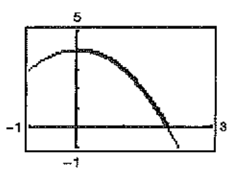
$$= \frac{1}{4}(y + 2 + y^{-1}) = \frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2$$

$$s = \int_1^4 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy$$

$$= \left[\frac{1}{2}\left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right)\right]_1^4$$

$$= \frac{1}{2}\left(\frac{16}{3} + 4\right) - \frac{1}{2}\left(\frac{2}{3} + 2\right) = \frac{10}{3}$$

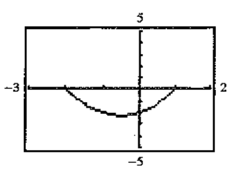
15. (a) $y = 4 - x^2, \quad 0 \le x \le 2$



(b)
$$y' = -2x$$
$$1 + (y')^2 = 1 + 4x^2$$
$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx$$

(c) $L \approx 4.647$

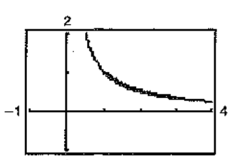
16. (a) $y = x^2 + x - 2, \quad -2 \le x \le 1$



(b)
$$y' = 2x + 1$$
$$1 + (y')^2 = 1 + 4x^2 + 4x + 1$$
$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\, dx$$

(c) $L \approx 5.653$

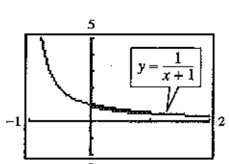
17. (a) $y = \frac{1}{x}, \quad 1 \le x \le 3$



(b)
$$y' = -\frac{1}{x^2}$$
$$1 + (y')^2 = 1 + \frac{1}{x^4}$$
$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

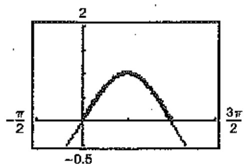
18. (a) $y = \frac{1}{1 + x}, \quad 0 \le x \le 1$



(b)
$$y' = -\frac{1}{(1 + x)^2}$$
$$1 + (y')^2 = 1 + \frac{1}{(1 + x)^4}$$
$$L = \int_0^1 \sqrt{1 + \frac{1}{(1 + x)^4}}\, dx$$

(c) $L \approx 1.132$

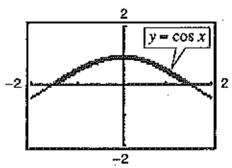
19. (a) $y = \sin x, \quad 0 \le x \le \pi$



(b) $$y' = \cos x$$
$$1 + (y')^2 = 1 + \cos^2 x$$
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

20. (a) $y = \cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



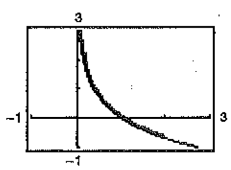
(b) $$y' = -\sin x$$
$$1 + (y')^2 = 1 + \sin^2 x$$
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

21. (a) $x = e^{-y}, \quad 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $$y' = -\frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

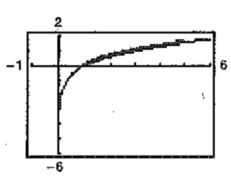
Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y}, \quad 0 \le y \le 2$

(b) $$\frac{dx}{dy} = -e^{-y}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$
$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

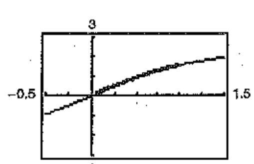
22. (a) $y = \ln x, \quad 1 \le x \le 5$



(b) $$y' = \frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 4.367$

23. (a) $y = 2 \arctan x, \quad 0 \le x \le 1$

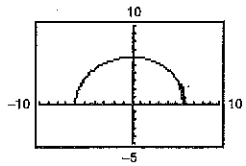


(b) $y' = \dfrac{2}{1 + x^2}$
$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\, dx$$

(c) $L \approx 1.871$

24. (a) $x = \sqrt{36 - y^2},\quad 0 \le y \le 3$

$y = \sqrt{36 - x^2},\quad 3\sqrt{3} \le x \le 6$



(b) $\dfrac{dx}{dy} = \dfrac{1}{2}(36 - y^2)^{-1/2}(-2y)$

$$= \frac{-y}{\sqrt{36 - y^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\, dy$$

$$= \int_0^3 \frac{6}{\sqrt{36 - y^2}}\, dy$$

(c) $L \approx 3.142 \quad (\pi!)$

Alternatively, you can convert to a function of x.

$$y = \sqrt{36 - x^2}$$

$$y' = \frac{dy}{dx} = -\frac{x}{\sqrt{36 - x^2}}$$

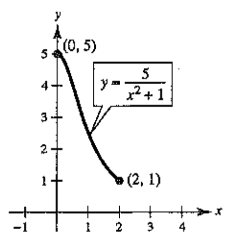
$$L = \int_{3\sqrt{3}}^{6} \sqrt{1 + \frac{x^2}{36 - x^2}}\, dx = \int_{3\sqrt{3}}^{6} \frac{6}{\sqrt{36 - x^2}}\, dx$$

Although this integral is undefined at $x = 0$, a graphing utility still gives $L \approx 3.142$.

25. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$

$s \approx 5$

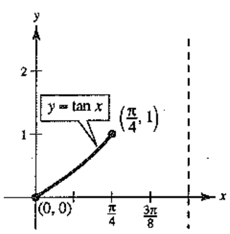
Matches (b)



26. $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$

$s \approx 1$

Matches (e)



27. $y = x^3, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$

(b) $d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2}$

≈ 64.525

(c) $s = \displaystyle\int_0^4 \sqrt{1 + (3x^2)^2}\, dx = \int_0^4 \sqrt{1 + 9x^4}\, dx \approx 64.666$ (Simpson's Rule, $n = 10$)

(d) 64.672

28. $f(x) = (x^2 - 4)^2, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

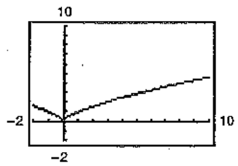
(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2}$

≈ 160.151

(c) $s = \displaystyle\int_0^4 \sqrt{1 + [4x(x^2 - 4)]^2}\, dx \approx 159.087$

(d) 160.287

29. (a) $f(x) = x^{2/3}$



(b) No, $f'(0)$ is not defined.

(c) $$f'(x) = \frac{2}{3}x^{-1/3}$$

$$1 + f'(x)^2 = 1 + \frac{4}{9x^{2/3}} = \frac{9x^{2/3}+4}{9x^{2/3}}$$

Divide $[-1, 8]$ into two intervals.

$$[-1, 0]\text{: } s_1 = \int_{-1}^{0}\sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\,dx$$
$$= \frac{-1}{3}\int_{-1}^{0}\sqrt{9x^{2/3}+4}\,\frac{1}{x^{1/3}}\,dx, \quad (x0)$$
$$= -\frac{1}{18}\int_{-1}^{0}(9x^{2/3}+4)^{1/2}\left(\frac{6}{x^{1/3}}\right)dx$$
$$= -\frac{1}{27}(9x^{2/3}+4)^{3/2}\Big]_{-1}^{0}$$
$$= -\frac{1}{27}(4^{3/2} - 13^{3/2})$$
$$= -\frac{1}{27}(8 - 13^{3/2}) \approx 1.4397$$

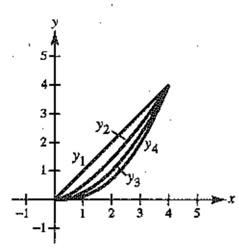
$$[0, 8]\text{: } s_2 = \int_{0}^{8}\sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\,dx$$
$$= \frac{1}{3}\int_{0}^{8}\sqrt{9x^{2/3}+4}\,\frac{1}{x^{1/3}}\,dx, \quad (x \ge 0)$$
$$= \frac{1}{27}(9x^{2/3}+4)^{3/2}\Big]_{0}^{8}$$
$$= \frac{1}{27}(40^{3/2} - 4^{3/2})$$
$$= \frac{1}{27}(40^{3/2} - 8) \approx 9.0734$$

$$s_1 + s_2 = \frac{1}{27}[40^{3/2} - 8 - 8 + 13^{3/2}]$$
$$= \frac{1}{27}[40^{3/2} + 13^{3/2} - 16] \approx 10.5131$$

30. $$x^{2/3} + y^{2/3} = 4$$
$$y^{2/3} = 4 - x^{2/3}$$
$$y = (4 - x^{2/3})^{3/2}, \quad 0 \le x \le 8$$
$$y' = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$
$$\frac{1}{4}s = \int_0^8 \sqrt{\frac{4}{x^{2/3}}}\,dx$$
$$= 2\int_0^8 x^{-2/3}\,dx = 6x^{1/3}\Big]_0^8 = 12$$

Total length: $s = 4(12) = 48$

31. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1,\ L_1 = \int_0^4 \sqrt{2}\,dx \approx 5.657$

$y_2' = \frac{3}{4}x^{1/2},\ L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\,dx \approx 5.759$

$y_3' = \frac{1}{2}x,\ L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\,dx \approx 5.916$

$y_4' = \frac{5}{16}x^{3/2},\ L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\,dx \approx 6.063$

32. Let $y = \ln x$, $1 \le x \le e$, $y' = \dfrac{1}{x}$ and $L_1 = \displaystyle\int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx$.

Equivalently, $x = e^y$, $0 \le y \le 1$, $\dfrac{dx}{dy} = e^y$, and $L_2 = \displaystyle\int_0^1 \sqrt{1 + e^{2y}}\, dy = \int_0^1 \sqrt{1 + e^{2x}}\, dx$.

Numerically, both integrals yield $L = 2.0035$.

33. $y = \dfrac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0$, $y = \frac{2}{3}$. Thus, the fleeing object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3}\left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right] = \left(\frac{1}{2}\right)\frac{x-1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x+1}{2x^{1/2}}\, dx = \frac{1}{2}\int_0^1 (x^{1/2} + x^{-1/2})\, dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

34.

$$y = 31 - 10(e^{x/20} + e^{-x/20})$$

$$y' = -\frac{1}{2}(e^{x/20} - e^{-x/20})$$

$$1 + (y')^2 = 1 + \frac{1}{4}(e^{x/10} - 2 + e^{-x/10})$$

$$= \left[\frac{1}{2}(e^{x/20} + e^{-x/20})\right]^2$$

$$s = \int_{-20}^{20} \sqrt{\left[\frac{1}{2}(e^{x/20} + e^{-x/20})\right]^2}\, dx$$

$$= \frac{1}{2}\int_{-20}^{20} (e^{x/20} + e^{-x/20})\, dx$$

$$= \Big[10(e^{x/20} - e^{-x/20})\Big]_{-20}^{20}$$

$$= 20\left(e - \frac{1}{e}\right) \approx 47 \text{ ft}$$

Thus, there are $100(47) = 4700$ square feet of roofing on the barn.

35.

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

$$y' = \sinh\frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\, dx = 2\int_0^{20} \cosh\frac{x}{20}\, dx$$

$$= 2(20)\sinh\frac{x}{20}\Big]_0^{20} = 40\sinh(1) \approx 47.008 \text{ m}$$

36. $y = 693.8597 - 68.7672\cosh 0.0100333x$

$$y' = -0.6899619478\sinh 0.0100333x$$

$$s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478\sinh 0.0100333x)^2}\, dx$$

$$\approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

37.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\, dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= \left[3\arcsin\frac{x}{3}\right]_0^2 = 3\left(\arcsin\frac{2}{3} - \arcsin 0\right)$$

$$= 3\arcsin\frac{2}{3} \approx 2.1892$$

38.

$$y = \sqrt{25 - x^2}$$

$$y' = \frac{-x}{\sqrt{25 - x^2}}$$

$$1 + (y')^2 = \frac{25}{25 - x^2}$$

$$s = \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}}\, dx = \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}}\, dx$$

$$= \left[5\arcsin\frac{x}{5}\right]_{-3}^4 = 5\left[\arcsin\frac{4}{5} - \arcsin\left(-\frac{3}{5}\right)\right]$$

$$\approx 7.8540$$

$$\frac{1}{4}[2\pi(5)] \approx 7.8540 = s$$

39. $y = \dfrac{x^3}{3}$

$y' = x^2, \quad [0, 3]$

$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\, dx$$
$$= \frac{\pi}{6}\int_0^3 (1 + x^4)^{1/2}(4x^3)\, dx$$
$$= \left[\frac{\pi}{9}(1 + x^4)^{3/2}\right]_0^3$$
$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

40. $y = 2\sqrt{x}$

$y' = \dfrac{1}{\sqrt{x}}, \quad [4, 9]$

$$S = 2\pi \int_4^9 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$$
$$= 4\pi \int_4^9 \sqrt{x + 1}\, dx$$
$$= \frac{8}{3}\pi(x + 1)^{3/2}\Big]_4^9$$
$$= \frac{8\pi}{3}(10^{3/2} - 5^{3/2}) \approx 171.258$$

41. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$

$y' = \dfrac{x^2}{2} - \dfrac{1}{2x^2}$

$1 + (y')^2 = \left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right)^2, \quad [1, 2]$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$
$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$
$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

42. $y = \dfrac{x}{2}$

$y' = \dfrac{1}{2}$

$1 + (y')^2 = \dfrac{5}{4}, \quad [0, 6]$

$$S = 2\pi \int_0^6 \frac{x}{2}\sqrt{\frac{5}{4}}\, dx$$
$$= \left[\frac{2\pi\sqrt{5}}{8}x^2\right]_0^6 = 9\sqrt{5}\pi$$

43. $y = \sqrt[3]{x} + 2$

$y' = \dfrac{1}{3x^{2/3}}, \quad [1, 8]$

$$S = 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx$$
$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\, dx$$
$$= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\, dx$$
$$= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8$$
$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

44. $y = 9 - x^2, \quad [0, 3]$

$y' = -2x$

$$S = 2\pi \int_0^3 x\sqrt{1 + 4x^2}\, dx$$
$$= \frac{\pi}{4}\int_0^3 (1 + 4x^2)^{1/2}(8x)\, dx$$
$$= \left[\frac{\pi}{6}(1 + 4x^2)^{3/2}\right]_0^3$$
$$= \frac{\pi}{6}(37^{3/2} - 1) \approx 117.319$$

45. $y = \sin x$

$y' = \cos x, \quad [0, \pi]$

$$S = 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx$$
$$\approx 14.4236$$

46. $y = \ln x$

$y' = \dfrac{1}{x}$

$1 + (y')^2 = \dfrac{x^2 + 1}{x^2}, \quad [1, e]$

$$S = 2\pi \int_1^e x\sqrt{\frac{x^2 + 1}{x^2}}\, dx = 2\pi \int_1^e \sqrt{x^2 + 1}\, dx$$
$$\approx 22.943$$

47. A rectifiable curve is one that has a finite arc length.

48. The precalculus formula is the distance formula between two points. The representative element is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta xi.$$

49. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

50. The surface of revolution given by f_1 will be larger. $r(x)$ is larger for f_1.

51.

$$y = \frac{hx}{r}$$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi \int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\,dx$$

$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

52.

$$y = \sqrt{r^2 - x^2}$$

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$

$$S = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\,dx$$

$$= 2\pi \int_{-r}^{r} r\,dx = \Big[2\pi r x\Big]_{-r}^{r} = 4\pi r^2$$

53.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}}\,dx$$

$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\,dx$$

$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$

$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 54.

54. From Exercise 53 we have:

$$S = 2\pi \int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\,dx$$

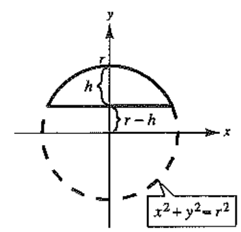
$$= -r\pi \int_0^a \frac{-2x\,dx}{\sqrt{r^2 - x^2}}$$

$$= \left[-2r\pi\sqrt{r^2 - x^2}\right]_0^a$$

$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$

$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$

$= 2\pi rh$ (where h is the height of the zone)



55.

$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2})\,dx$$

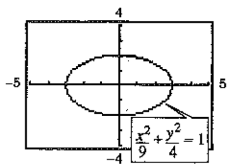
$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in.}^2$$

Amount of glass needed: $V = \dfrac{\pi}{27}\left(\dfrac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in.}^3$

56. (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \frac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \frac{x^2}{9}}$



(b) $y = 2\sqrt{1 - \frac{x^2}{9}}, \quad 0 \le x \le 3$

$$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(\frac{-2x}{9}\right)$$

$$= \frac{-2x}{9\sqrt{1 - (x^2/9)}} = \frac{-2x}{3\sqrt{9 - x^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\, dx$$

(c) You cannot evaluate this definite integral, since the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

57. (a) We approximate the volume by summing six disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

$$V \approx \sum_{i=1}^{6} \pi r_i^2(3) = \sum_{i=1}^{6} \pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6} C_i^2$$

$$= \frac{3}{4\pi}\left[\left(\frac{50 + 65.5}{2}\right)^2 + \left(\frac{65.5 + 70}{2}\right)^2 + \left(\frac{70 + 66}{2}\right)^2 + \left(\frac{66 + 58}{2}\right)^2 + \left(\frac{58 + 51}{2}\right)^2 + \left(\frac{51 + 48}{2}\right)^2\right]$$

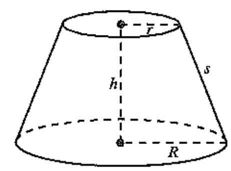
$$= \frac{3}{4\pi}[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2]$$

$$= \frac{3}{4\pi}[21813.625] = 5207.62 \text{ cubic inches}$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum:

$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$

$$= \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}.$$



Adding the six frustums together:

$$S \approx \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5 + 70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2} +$$

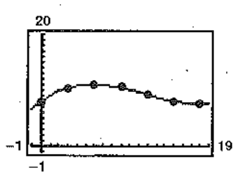
$$\left(\frac{70 + 66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66 + 58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2} +$$

$$\left(\frac{58 + 51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51 + 48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$

$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37$$

$$= 1168.64$$

(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



(d) $V = \int_0^{18} \pi r^2\, dy \approx 5275.9$ cubic inches

$$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\, dy$$

≈ 1179.5 square inches

58. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

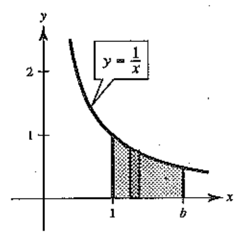
(b) Area $= \int_0^{400} f(x)\,dx \approx 131{,}734.5$ square feet ≈ 3.0 acres (1 acre = 43,560 square feet)

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2}\,dx \approx 794.9$ feet

(Answers will vary.)

59. (a) $V = \pi \int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$



(b) $S = 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$

$= 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$

$= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Since

$$\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \Big[\ln x\Big]_1^b = \ln b$$

and $\lim_{b\to\infty} \ln b \to \infty$. Thus,

$$\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx = \infty.$$

60. (a) Area of circle with radius L: $A = \pi L^2$

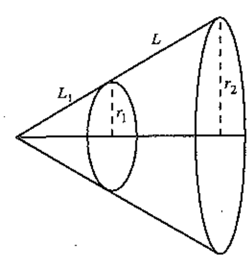
Area of sector with central angle θ (in radians):

$$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2}L^2\theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL.$$

(c) The lateral surface area of the frustum is the difference of the large cone and the small one.



$S = \pi r_2(L + L_1) - \pi r_1 L_1$

$= \pi r_2 L + \pi L_1(r_2 - r_1)$

By similar triangles, $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1)$. Hence,

$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$

$= \pi L(r_1 + r_2)$.

61. Individual project

62. Essay

63. $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$

$$y = (4 - x^{2/3})^{3/2}, \quad 0 \le x \le 8$$

$$y' = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$

$$S = 2\pi\int_0^8 (4 - x^{2/3})^{3/2}\sqrt{\frac{4}{x^{2/3}}}\,dx$$

$$= 4\pi\int_0^8 \frac{(4 - x^{2/3})^{3/2}}{x^{1/3}}\,dx$$

$$= \left[-\frac{12\pi}{5}(4 - x^{2/3})^{5/2}\right]_0^8 = \frac{192\pi}{5}$$

[Surface area of portion above the x-axis]

64. $y^2 = \frac{1}{12}x(4 - x)^2, \quad 0 \le x \le 4$

$$y = \frac{(4 - x)\sqrt{x}}{\sqrt{12}}$$

$$y' = \frac{(4 - 3x)\sqrt{3}}{12\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{(4 - 3x)^2}{48x}$$

$$= \frac{48x + 16 - 24x + 9x^2}{48x} = \frac{(4 + 3x)^2}{48x}, \quad x \ne 0$$

$$S = 2\pi\int_0^4 \frac{(4 - x)\sqrt{x}}{\sqrt{12}} \cdot \frac{(4 + 3x)}{\sqrt{48x}}\,dx$$

$$= 2\pi\int_0^4 \frac{(4 - x)(4 + 3x)}{24}\,dx$$

$$= \frac{\pi}{12}\int_0^4 (16 + 8x - 3x^2)\,dx$$

$$= \frac{\pi}{12}\left[16x + 4x^2 - x^3\right]_0^4$$

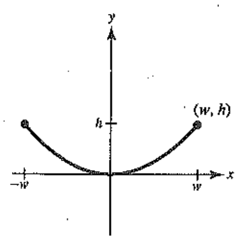
$$= \frac{\pi}{12}[64 + 64 - 64] = \frac{16\pi}{3}$$

65. $y = kx^2, y' = 2kx$

$$1 + (y')^2 = 1 + 4k^2x^2$$

$$h = kw^2 \implies k = \frac{h}{w^2} \implies 1 + (y') = 1 + \frac{4h^2}{w^4}x^2$$

By symmetry, $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx.$



66. $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx$

$$= 2\int_0^{700} \sqrt{1 + \frac{4(155)^2x^2}{700^4}}\,dx$$

≈ 1444.5 meters

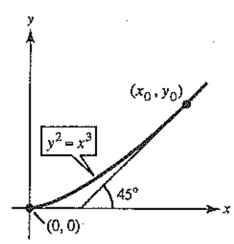
67. Let (x_0, y_0) be the point on the graph of $y^2 = x^3$ where the tangent line makes an angle of 45° with the x-axis.

$$y = x^{3/2}$$

$$y' = \tfrac{3}{2}x^{1/2} = 1$$

$$x_0 = \tfrac{4}{9}$$

$$L = \int_0^{4/9} \sqrt{1 + \tfrac{9}{4}x}\,dx = \tfrac{8}{27}\left(2\sqrt{2} - 1\right)$$



Section 7.5 Work

1. $W = Fd = (100)(10)$
$= 1000 \text{ ft} \cdot \text{lb}$

2. $W = Fd = (2800)(4)$
$= 11{,}200 \text{ ft} \cdot \text{lb}$

3. $W = Fd = (112)(4)$
$= 448$ joules (newton-meters)

4. $W = Fd = [9(2000)]\left[\frac{1}{2}(5280)\right] = 47{,}520{,}000 \text{ ft} \cdot \text{lb}$

5. Work equals force times distance, $W = FD$.

6. $W = \int_a^b F(x)\,dx$ is the work done by a force F moving an object along a straight line from $x = a$ to $x = b$.

7. Since the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

8. (a) $W = \int_0^9 6\,dx = 54 \text{ ft} \cdot \text{lbs}$

(b) $W = \int_0^7 20\,dx + \int_7^9 (-10x + 90)\,dx = 140 + 20$
$= 160 \text{ ft} \cdot \text{lbs}$

(c) $W = \int_0^9 \frac{1}{27}x^2\,dx = \left[\frac{x^3}{81}\right]_0^9 = 9 \text{ ft} \cdot \text{lbs}$

(d) $W = \int_0^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3}(27) = 18 \text{ ft} \cdot \text{lbs}$

9. $F(x) = kx$
$5 = k(4)$
$k = \frac{5}{4}$
$W = \int_0^7 \frac{5}{4}x\,dx = \left[\frac{5}{8}x^2\right]_0^7$
$= \frac{245}{8} \text{ in.} \cdot \text{lb}$
$= 30.625 \text{ in.} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$

10. $W = \int_5^9 \frac{5}{4}x\,dx = \left[\frac{5}{8}x^2\right]_5^9$
$= 35 \text{ in.} \cdot \text{lb}$

11. $F(x) = kx$
$250 = k(30) \implies k = \frac{25}{3}$
$W = \int_{20}^{50} F(x)\,dx$
$= \int_{20}^{50} \frac{25}{3}x\,dx = \left[\frac{25x^2}{6}\right]_{20}^{50}$
$= 8750 \text{ n} \cdot \text{cm}$
$= 87.5$ joules or Nm

12. $F(x) = kx$
$800 = k(70) \implies k = \frac{80}{7}$
$W = \int_0^{70} F(x)\,dx$
$= \int_0^{70} \frac{80}{7}x\,dx = \left[\frac{40x^2}{7}\right]_0^{70}$
$= 28{,}000 \text{ n} \cdot \text{cm} = 280 \text{ Nm}$

13. $F(x) = kx$
$20 = k(9)$
$k = \frac{20}{9}$
$W = \int_0^{12} \frac{20}{9}x\,dx = \left[\frac{10}{9}x^2\right]_0^{12} = 160 \text{ in.} \cdot \text{lb} = \frac{40}{3} \text{ ft} \cdot \text{lb}$

14. $F(x) = kx$
$15 = k(1) = k$
$W = 2\int_0^4 15x\,dx = \left[15x^2\right]_0^4 = 240 \text{ ft} \cdot \text{lb}$

15. $W = 18 = \int_0^{1/3} kx\,dx = \left[\frac{kx^2}{2}\right]_0^{1/3} = \frac{k}{18} \implies k = 324$
$W = \int_{1/3}^{7/12} 324x\,dx = \left[162x^2\right]_{1/3}^{7/12} = 37.125 \text{ ft} \cdot \text{lbs}$
[**Note:** 4 inches $= \frac{1}{3}$ foot]

16. $W = 7.5 = \int_0^{1/6} kx\,dx = \left[\frac{kx^2}{2}\right]_0^{1/6} = \frac{k}{72} \implies k = 540$
$W = \int_{1/6}^{5/24} 540x\,dx = \left[270x^2\right]_{1/6}^{5/24} = 4.21875 \text{ ft} \cdot \text{lbs}$

17. Assume that Earth has a radius of 4000 miles.

$$F(x) = \frac{k}{x^2}$$

$$5 = \frac{k}{(4000)^2}$$

$$k = 80{,}000{,}000$$

$$F(x) = \frac{80{,}000{,}000}{x^2}$$

(a) $W = \int_{4000}^{4100} \frac{80{,}000{,}000}{x^2}\,dx = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4100}$

$\approx 487.8 \text{ mi} \cdot \text{tons} \approx 5.15 \times 10^9 \text{ ft} \cdot \text{lb}$

(b) $W = \int_{4000}^{4300} \frac{80{,}000{,}000}{x^2}\,dx$

$\approx 1395.3 \text{ mi} \cdot \text{ton} \approx 1.47 \times 10^{10} \text{ ft} \cdot \text{ton}$

18. $W = \int_{4000}^{h} \frac{80{,}000{,}000}{x^2}\,dx = \left[-\frac{80{,}000{,}000}{x}\right]_{4000}^{h} = \frac{-80{,}000{,}000}{h} + 20{,}000$

$\lim_{h\to\infty} W = 20{,}000 \text{ mi/ton} \approx 2.1 \times 10^{11} \text{ ft} \cdot \text{lb}$

19. Assume that Earth has a radius of 4000 miles.

$$F(x) = \frac{k}{x^2}$$

$$10 = \frac{k}{(4000)^2}$$

$$k = 160{,}000{,}000$$

$$F(x) = \frac{160{,}000{,}000}{x^2}$$

(a) $W = \int_{4000}^{15{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333 \text{ mi} \cdot \text{ton}$

$\approx 2.93 \times 10^4 \text{ mi} \cdot \text{ton}$

$\approx 3.10 \times 10^{11} \text{ ft} \cdot \text{lb}$

(b) $W = \int_{4000}^{26{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154 \text{ mi} \cdot \text{ton}$

$\approx 3.38 \times 10^4 \text{ mi} \cdot \text{ton}$

$\approx 3.57 \times 10^{11} \text{ ft} \cdot \text{lb}$

20. Weight on surface of moon: $\frac{1}{6}(12) = 2$ tons

Weight varies inversely as the square of distance from the center of the moon. Therefore:

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2}\,dx = \left[\frac{-2.42 \times 10^6}{x}\right]_{1100}^{1150} = 2.42 \times 10^6\left(\frac{1}{1100} - \frac{1}{1150}\right)$$

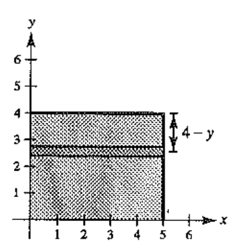
$\approx 95.652 \text{ mi} \cdot \text{ton} \approx 1.01 \times 10^9 \text{ ft} \cdot \text{lb}$

21. Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \int_{2}^{4} 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_{2}^{4} = 2496 \text{ ft} \cdot \text{lb}$

(b) $W = \int_{0}^{4} 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_{0}^{4} = 9984 \text{ ft} \cdot \text{lb}$



22. The bottom half had to be pumped a greater distance than the top half.

23. Volume of disk: $\pi(2)^2\,\Delta y = 4\pi\,\Delta y$

Weight of disk of water: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5-y)(9800)4\pi\,dy = 39{,}200\pi\int_0^4 (5-y)\,dy$$

$$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$$

$$= 39{,}200\pi(12) = 470{,}400\pi \text{ newton–meters}$$

24. Volume of disk: $4\pi\,\Delta y$

Weight of disk: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: y

$$W = \int_{10}^{12} y(9800)(4\pi)\,dy = 39{,}200\pi\left[\frac{y^2}{2}\right]_{10}^{12}$$

$$= 39{,}200\pi(22)$$

$$= 862{,}400\pi \text{ newton–meters}$$

25. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\Delta y$

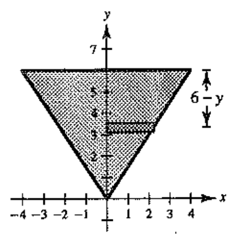
Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9}\int_0^6 (6-y)y^2\,dy$$

$$= \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6$$

$$= 2995.2\pi \text{ ft} \cdot \text{lb}$$



26. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\Delta y$

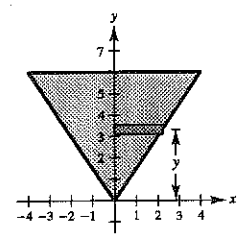
Distance: y

(a) $W = \frac{4}{9}(62.4)\pi\int_0^2 y^3\,dy$

$$= \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$$

(b) $W = \frac{4}{9}(62.4)\pi\int_4^6 y^3\,dy$

$$= \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_4^6 \approx 7210.7\pi \text{ ft} \cdot \text{lb}$$



27. Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2\Delta y$

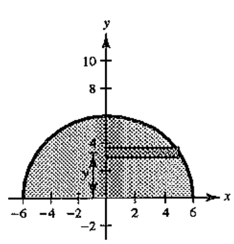
Weight of disk: $62.4\pi(36 - y^2)\,\Delta y$

Distance: y

$$W = 62.4\pi\int_0^6 y(36 - y^2)\,dy$$

$$= 62.4\pi\int_0^6 (36y - y^3)\,dy = 62.4\pi\left[18y^2 - \frac{1}{4}y^4\right]_0^6$$

$$= 20{,}217.6\pi \text{ ft} \cdot \text{lb}$$



28. Volume of each layer: $\frac{y+3}{3}(3)\,\Delta y = (y+3)\,\Delta y$

Weight of each layer: $53.1(y+3)\,\Delta y$

Distance: $6 - y$

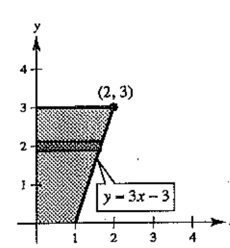
$$W = \int_0^3 53.1(6-y)(y+3)\,dy$$

$$= 53.1\int_0^3 (18 + 3y - y^2)\,dy$$

$$= 53.1\left[18y + \frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$$

$$= 53.1\left[\frac{117}{2}\right]$$

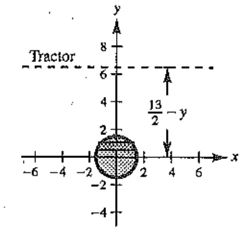
$$= 3106.35 \text{ ft} \cdot \text{lb}$$



29. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$

Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

Distance: $\dfrac{13}{2} - y$



$$W = \int_{-1.5}^{1.5} 42(8)\sqrt{\frac{9}{4} - y^2}\left(\frac{13}{2} - y\right) dy = 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,y\,dy\right]$$

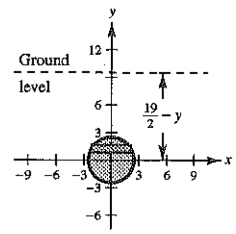
The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336\left(\frac{13}{2}\right)\pi\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) = 2457\pi \text{ ft} \cdot \text{lb}.$$

30. Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2}\,\Delta y$

Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2}\,\Delta y$

Distance: $\dfrac{19}{2} - y$



$$W = \int_{-2.5}^{2.5} 42(24)\sqrt{\frac{25}{4} - y^2}\left(\frac{19}{2} - y\right) dy$$

$$= 1008\left[\frac{19}{2}\int_{-2.5}^{2.5}\sqrt{\frac{25}{4} - y^2}\,dy + \int_{-2.5}^{2.5}\sqrt{\frac{25}{4} - y^2}(-y)\,dy\right]$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. Thus, the work is

$$W = 1008\left(\frac{19}{2}\right)\pi\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right) = 29{,}925\pi \text{ ft} \cdot \text{lb} \approx 94{,}012.16 \text{ ft} \cdot \text{lb}.$$

31. Weight of section of chain: $3\,\Delta y$

Distance: $15 - y$

$$W = 3\int_0^{15} (15 - y)\,dy$$

$$= \left[-\frac{3}{2}(15 - y)^2\right]_0^{15}$$

$$= 337.5 \text{ ft} \cdot \text{lb}$$

32. The lower 10 feet of chain are raised 5 feet with a constant force.

$W_1 = 3(10)5 = 150$ ft · lb

The top 5 feet will be raised with variable force.

Weight of section: $3\,\Delta y$

Distance: $5 - y$

$$W_2 = 3\int_0^5 (5 - y)\,dy = \left[-\frac{3}{2}(5 - y)^2\right]_0^5 = \frac{75}{2} \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 150 + \frac{75}{2} = \frac{375}{2} \text{ ft} \cdot \text{lb}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$W_1 = 3(5)(10) = 150$ ft · lb

The top 10 feet of chain are raised with a variable force.

Weight per section: $3\,\Delta y$

Distance: $10 - y$

$$W_2 = 3\int_0^{10} (10 - y)\,dy = \left[-\frac{3}{2}(10 - y)^2\right]_0^{10} = 150 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

34. The work required to lift the chain is 337.5 ft · lb (from Exercise 31). The work required to lift the 500-pound load is $W = (500)(15) = 7500$. The work required to lift the chain with a 100-pound load attached is

$$W = 337.5 + 7500 = 7837.5 \text{ ft} \cdot \text{lbs}.$$

35. Weight of section of chain: $3\ \Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5}(15 - 2y)\,dy = \left[-\frac{3}{4}(15 - 2y)^2\right]_0^{7.5}$$

$$= \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

36. $W = 3\int_0^{6}(12 - 2y)\,dy = \left[-\frac{3}{4}(12 - 2y)^2\right]_0^{6}$

$$= \frac{3}{4}(12)^2 = 108 \text{ ft} \cdot \text{lb}$$

37. Work to pull up the ball: $W_1 = 500(15) = 7500$ ft · lb

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1\ \Delta y$

Distance: $15 - x$

$$W_2 = \int_0^{15}(15 - x)\,dx = \left[-\frac{1}{2}(15 - x)^2\right]_0^{15}$$

$$= 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375$$

$$= 7987.5 \text{ ft} \cdot \text{lb}$$

38. Work to pull up the ball: $W_1 = 500(40) = 20{,}000$ ft · lb

Work to pull up the cable: force is variable

Weight per section: $1\ \Delta y$

Distance: $40 - x$

$$W_2 = \int_0^{40}(40 - x)\,dx = \left[-\frac{1}{2}(40 - x)^2\right]_0^{40}$$

$$= 800 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 20{,}000 + 800 = 20{,}800 \text{ ft} \cdot \text{lb}$$

39. $p = \dfrac{k}{V}$

$$1000 = \frac{k}{2}$$

$$k = 2000$$

$$W = \int_2^3 \frac{2000}{V}\,dV$$

$$= \Big[2000 \ln|V|\Big]_2^3$$

$$= 2000 \ln\left(\frac{3}{2}\right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

40. $p = \dfrac{k}{V}$

$$2500 = \frac{k}{1} \implies k = 2500$$

$$W = \int_1^3 \frac{2500}{V}\,dV$$

$$= \Big[2500 \ln V\Big]_1^3$$

$$= 2500 \ln 3$$

$$\approx 2746.53 \text{ ft} \cdot \text{lb}$$

41. $F(x) = \dfrac{k}{(2 - x)^2}$

$$W = \int_{-2}^{1} \frac{k}{(2 - x)^2}\,dx$$

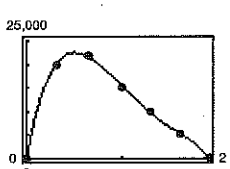
$$= \left[\frac{k}{2 - x}\right]_{-2}^{1} = k\left(1 - \frac{1}{4}\right)$$

$$= \frac{3k}{4} \text{ (units of work)}$$

42. (a) $W = FD = (8000\pi)(2) = 16{,}000\pi$ ft · lbs

(b) $W \approx \dfrac{2 - 0}{3(6)}[0 + 4(20{,}000) + 2(22{,}000) + 4(15{,}000) + 2(10{,}000) + 4(5000) + 0] \approx 24{,}88.889$ ft · lb

(c) $F(x) = -16{,}261.36x^4 + 85{,}295.45x^3 - 157{,}738.64x^2 + 104{,}386.36x - 32.4675$



(d) $F(x) = 0$ when $x \approx 0.524$ feet. $F(x)$ is a maximum when $x \approx 0.524$ feet.

(e) $W = \displaystyle\int_0^2 F(x)\,dx \approx 25{,}180.5$ ft · lbs

43. $W = \int_0^5 1000[1.8 - \ln(x+1)]\,dx \approx 3249.44 \text{ ft} \cdot \text{lb}$

44. $W = \int_0^4 \left(\dfrac{e^{x^2} - 1}{100}\right) dx \approx 11{,}494 \text{ ft} \cdot \text{lb}$

45. $W = \int_0^5 100x\sqrt{125 - x^3}\,dx \approx 10{,}330.3 \text{ ft} \cdot \text{lb}$

46. $W = \int_0^2 1000 \sinh x\,dx \approx 2762.2 \text{ ft} \cdot \text{lb}$

Section 7.6 Moments, Centers of Mass, and Centroids

1. $\bar{x} = \dfrac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\dfrac{6}{7}$

2. $\bar{x} = \dfrac{7(-3) + 4(-2) + 3(5) + 8(6)}{7 + 4 + 3 + 8} = \dfrac{17}{11}$

3. $\bar{x} = \dfrac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

4. $\bar{x} = \dfrac{12(-6) + 1(-4) + 6(-2) + 3(0) + 11(8)}{12 + 1 + 6 + 3 + 11} = 0$

5. (a) $\bar{x} = \dfrac{(7+5) + (8+5) + (12+5) + (15+5) + (18+5)}{5} = 17 = 12 + 5$

(b) $\bar{x} = \dfrac{12(-6-3) + 1(-4-3) + 6(-2-3) + 3(0-3) + 11(8-3)}{12 + 1 + 6 + 3 + 11} = \dfrac{-99}{33} = -3$

6. The center of mass is translated k units as well.

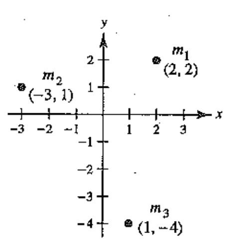
7.
$50x = 75(L - x) = 75(10 - x)$
$50x = 750 - 75x$
$125x = 750$
$x = 6$ feet

8.
$200x = 550(5 - x)$ (Person on left)
$200x = 2750 - 550x$
$750x = 2750$
$x = 3\frac{2}{3}$ feet

9.
$\bar{x} = \dfrac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \dfrac{10}{9}$

$\bar{y} = \dfrac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\dfrac{1}{9}$

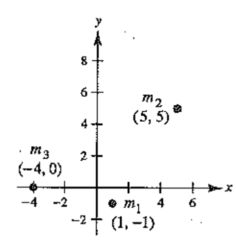
$(\bar{x}, \bar{y}) = \left(\dfrac{10}{9}, -\dfrac{1}{9}\right)$



10.
$\bar{x} = \dfrac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$

$\bar{y} = \dfrac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$

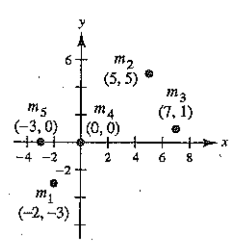
$(\bar{x}, \bar{y}) = (0, 0)$



11.
$\bar{x} = \dfrac{3(-2) + 4(5) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = \dfrac{5}{8}$

$\bar{y} = \dfrac{3(-3) + 4(5) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = \dfrac{13}{16}$

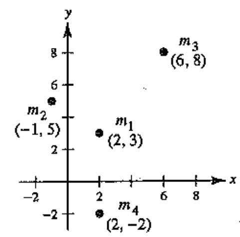
$(\bar{x}, \bar{y}) = \left(\dfrac{5}{8}, \dfrac{13}{16}\right)$



12. $\bar{x} = \dfrac{12(2) + 6(-1) + (15/2)(6) + 15(2)}{12 + 6 + (15/2) + 15} = \dfrac{93}{40.5} = \dfrac{62}{27}$

$\bar{y} = \dfrac{12(3) + 6(5) + (15/2)(8) + 15(-2)}{12 + 6 + (15/2) + 15} = \dfrac{96}{40.5} = \dfrac{64}{27}$

$(\bar{x}, \bar{y}) = \left(\dfrac{62}{27}, \dfrac{64}{27}\right)$



13. $m = \rho \displaystyle\int_0^4 \sqrt{x}\, dx = \left[\frac{2\rho}{3} x^{3/2}\right]_0^4 = \frac{16\rho}{3}$

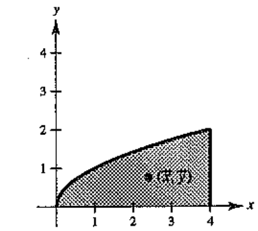
$M_x = \rho \displaystyle\int_0^4 \frac{\sqrt{x}}{2}(\sqrt{x})\, dx = \left[\rho \frac{x^2}{4}\right]_0^4 = 4\rho$

$\bar{y} = \dfrac{M_x}{m} = 4\rho\left(\dfrac{3}{16\rho}\right) = \dfrac{3}{4}$

$M_y = \rho \displaystyle\int_0^4 x\sqrt{x}\, dx = \left[\rho \frac{2}{5} x^{5/2}\right]_0^4 = \frac{64\rho}{5}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{64\rho}{5}\left(\dfrac{3}{16\rho}\right) = \dfrac{12}{5}$

$(\bar{x}, \bar{y}) = \left(\dfrac{12}{5}, \dfrac{3}{4}\right)$



14. $m = \rho \displaystyle\int_0^2 \frac{1}{2}x^2\, dx = \left[\rho \frac{x^3}{6}\right]_0^2 = \frac{4}{3}\rho$

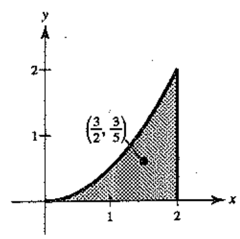
$M_x = \rho \displaystyle\int_0^2 \frac{1}{2}\left(\frac{1}{2}x^2\right)\left(\frac{1}{2}x^2\right) dx = \frac{\rho}{8}\int_0^2 x^4\, dx = \left[\frac{\rho}{40}x^5\right]_0^2 = \frac{32}{40}\rho = \frac{4}{5}\rho$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{(4/5)\rho}{(4/3)\rho} = \dfrac{3}{5}$

$M_y = \rho \displaystyle\int_0^2 x\left(\frac{1}{2}x^2\right) dx = \frac{1}{2}\rho \int_0^2 x^3\, dx = \left[\frac{\rho}{8}x^4\right]_0^2 = 2\rho$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{2\rho}{(4/3)\rho} = \dfrac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{3}{2}, \dfrac{3}{5}\right)$



15. $m = \rho \displaystyle\int_0^1 (x^2 - x^3)\, dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$

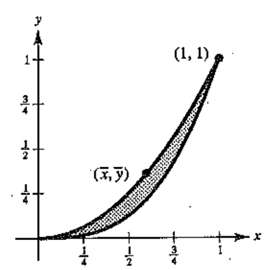
$M_x = \rho \displaystyle\int_0^1 \frac{(x^2 + x^3)}{2}(x^2 - x^3)\, dx = \frac{\rho}{2}\int_0^1 (x^4 - x^6)\, dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{\rho}{35}\left(\dfrac{12}{\rho}\right) = \dfrac{12}{35}$

$M_y = \rho \displaystyle\int_0^1 x(x^2 - x^3)\, dx = \rho \int_0^1 (x^3 - x^4)\, dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{\rho}{20}\left(\dfrac{12}{\rho}\right) = \dfrac{3}{5}$

$(\bar{x}, \bar{y}) = \left(\dfrac{3}{5}, \dfrac{12}{35}\right)$



16. $m = \rho \int_0^1 \left(\sqrt{x} - x\right) dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{\rho}{6}$

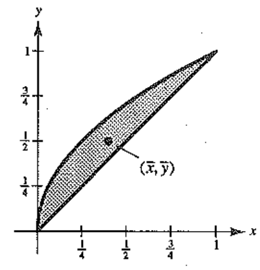
$M_x = \rho \int_0^1 \frac{\left(\sqrt{x} + x\right)}{2}\left(\sqrt{x} - x\right) dx = \frac{\rho}{2}\int_0^1 (x - x^2)\,dx = \frac{\rho}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\rho}{12}$

$\bar{y} = \frac{M_x}{m} = \frac{\rho}{12}\left(\frac{6}{\rho}\right) = \frac{1}{2}$

$M_y = \rho \int_0^1 x\left(\sqrt{x} - x\right) dx = \rho \int_0^1 (x^{3/2} - x^2)\,dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{x^3}{3}\right]_0^1 = \frac{\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{\rho}{15}\left(\frac{6}{\rho}\right) = \frac{2}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, \frac{1}{2}\right)$



17. $m = \rho \int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\,dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$

$M_x = \rho \int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2}\right][(-x^2 + 4x + 2) - (x + 2)]\,dx$

$= \frac{\rho}{2}\int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x)\,dx = \frac{\rho}{2}\int_0^3 (x^4 - 8x^3 + 11x^2 + 12x)\,dx$

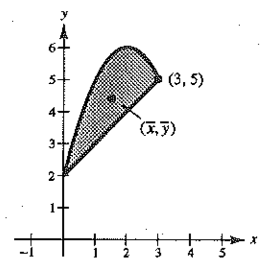
$= \frac{\rho}{2}\left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2\right]_0^3 = \frac{99\rho}{5}$

$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{22}{5}$

$M_y = \rho \int_0^3 x[(-x^2 + 4x - 2) - (x + 2)]\,dx = \rho \int_0^3 (-x^3 + 3x^2)\,dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$

$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5}\right)$



18. $m = \rho\int_0^9 \left[\left(\sqrt{x} + 1\right) - \left(\frac{1}{3}x + 1\right)\right] dx = \rho\int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right) dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{6}\right]_0^9 = \rho\left(18 - \frac{27}{2}\right) = \frac{9}{2}\rho$

$M_x = \rho\int_0^9 \frac{\sqrt{x} + 1 + (1/3)x + 1}{2}\left(\sqrt{x} + 1 - \frac{1}{3}x - 1\right) dx = \frac{\rho}{2}\int_0^9 \left(\sqrt{x} + \frac{1}{3}x + 2\right)\left(\sqrt{x} - \frac{1}{3}x\right) dx$

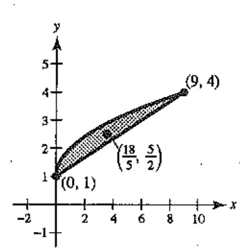
$= \frac{\rho}{2}\int_0^9 \left(x - \frac{1}{3}x^{3/2} + \frac{1}{3}x^{3/2} - \frac{1}{9}x^2 + 2\sqrt{x} - \frac{2}{3}x\right) dx = \frac{\rho}{2}\int_0^9 \left(\frac{1}{3}x - \frac{1}{9}x^2 + 2\sqrt{x}\right) dx$

$= \frac{\rho}{2}\left[\frac{x^2}{6} - \frac{x^3}{27} + \frac{4}{3}x^{3/2}\right]_0^9 = \frac{\rho}{2}\left[\frac{27}{2} - 27 + 36\right] = \frac{45}{4}\rho$

$M_y = \rho\int_0^9 x\left[\sqrt{x} + 1 - \frac{1}{3}x - 1\right] dx = \rho\int_0^9 \left(x^{3/2} - \frac{1}{3}x^2\right) dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{1}{9}x^3\right]_0^9 = \rho\left[\frac{486}{5} - 81\right] = \frac{81}{5}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{(81/5)\rho}{(9/2)\rho} = \frac{18}{5};\ \bar{y} = \frac{M_x}{m} = \frac{(45/4)\rho}{(9/2)\rho} = \frac{5}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{18}{5}, \frac{5}{2}\right)$



19. $m = \rho \int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$

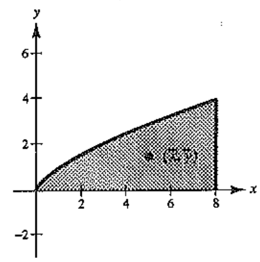
$M_x = \rho \int_0^8 \frac{x^{2/3}}{2}(x^{2/3})\,dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$

$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$

$M_y = \rho \int_0^8 x(x^{2/3})\,dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$

$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$

$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$



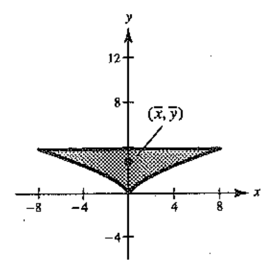
20. $m = 2\rho \int_0^8 (4 - x^{2/3})\,dx = 2\rho\left[4x - \frac{3}{5}x^{5/3}\right]_0^8 = \frac{128\rho}{5}$

By symmetry, M_y and $\bar{x} = 0$.

$M_x = 2\rho \int_0^8 \left(\frac{4 + x^{2/3}}{2}\right)(4 - x^{2/3})\,dx = \rho\left[16x - \frac{3}{7}x^{7/3}\right]_0^8 = \frac{512\rho}{7}$

$\bar{y} = \frac{512\rho}{7}\left(\frac{5}{128\rho}\right) = \frac{20}{7}$

$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7}\right)$



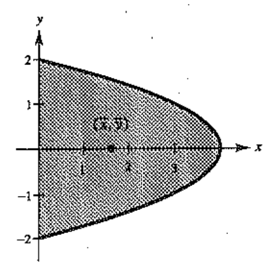
21. $m = 2\rho \int_0^2 (4 - y^2)\,dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$

$M_y = 2\rho \int_0^2 \left(\frac{4 - y^2}{2}\right)(4 - y^2)\,dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$

By symmetry, M_x and $\bar{y} = 0$.

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$



22. $m = \rho \int_0^2 (2y - y^2)\,dy = \rho\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{4\rho}{3}$

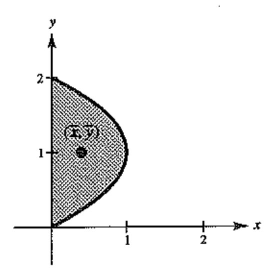
$M_y = \rho \int_0^2 \left(\frac{2y - y^2}{2}\right)(2y - y^2)\,dy = \frac{\rho}{2}\left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right]_0^2 = \frac{8\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{8\rho}{15}\left(\frac{3}{4\rho}\right) = \frac{2}{5}$

$M_x = \rho \int_0^2 y(2y - y^2)\,dy = \rho\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \frac{4\rho}{3}$

$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{3}\left(\frac{3}{4\rho}\right) = 1$

$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, 1\right)$



23. $m = \rho \int_0^3 [(2y - y^2) - (-y)]\,dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$

$M_y = \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2}[(2y - y^2) - (-y)]\,dy = \frac{\rho}{2}\int_0^3 (y - y^2)(3y - y^2)\,dy$

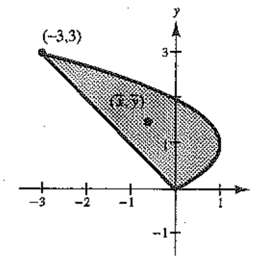
$= \frac{\rho}{2}\int_0^3 (y^4 - 4y^3 + 3y^2)\,dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$

$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$

$M_x = \rho \int_0^3 y[(2y - y^2) - (-y)]\,dy = \rho \int_0^3 (3y^2 - y^3)\,dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$



24. $m = \rho \int_{-1}^2 [(y + 2) - y^2]\,dy = \rho\left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2 = \frac{9\rho}{2}$

$M_y = \rho \int_{-1}^2 \frac{[(y + 2) + y^2]}{2}[(y + 2) - y^2]\,dy$

$= \frac{\rho}{2}\int_{-1}^2 [(y + 2)^2 - y^4]\,dy = \frac{\rho}{2}\left[\frac{(y + 2)^3}{3} - \frac{y^5}{5}\right]_{-1}^2 = \frac{36\rho}{5}$

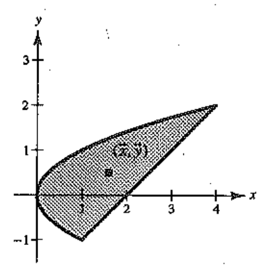
$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{8}{5}$

$M_x = \rho \int_{-1}^2 y[(y + 2) - y^2]\,dy$

$= \rho \int_{-1}^2 (2y + y^2 - y^3)\,dy = \rho\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^2 = \frac{9\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{1}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$



25. $A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

$M_x = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$

$M_y = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}$

26. $A = \int_1^4 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^4 = \ln 4$

$M_x = \frac{1}{2}\int_1^4 \frac{1}{x^2}\,dx = \left[\frac{1}{2}\left(-\frac{1}{x}\right)\right]_1^4 = \left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{3}{8}$

$M_y = \int_1^4 x\left(\frac{1}{x}\right)dx = \left[x\right]_1^4 = 3$

27. $A = \int_0^3 (2x + 4)\,dx = \left[x^2 + 4x\right]_0^3 = 9 + 12 = 21$

$M_x = \frac{1}{2}\int_0^3 (2x + 4)^2\,dx = \int_0^3 (2x^2 + 8x + 8)\,dx$

$= \left[\frac{2x^3}{3} + 4x^2 + 8x\right]_0^3 = 18 + 36 + 24 = 78$

$M_y = \int_0^3 (2x^2 + 4x)\,dx = \left[\frac{2x^3}{3} + 2x^2\right]_0^3 = 18 + 18 = 36$

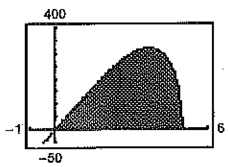
28. $A = \int_{-2}^{2} -(x^2 - 4)\,dx = 2\int_0^2 (4 - x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$

$M_x = \frac{1}{2}\int_{-2}^{2} (x^2 - 4)(4 - x^2)\,dx = -\frac{1}{2}\int_{-2}^{2} (x^4 - 8x^2 + 16)\,dx$

$= -\frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{2} = -\left[\frac{32}{5} - \frac{64}{3} + 32\right] = -\frac{256}{15}$

$M_y = 0$ by symmetry.

29. $m = \rho\int_0^5 10x\sqrt{125 - x^3}\,dx \approx 1033.0\rho$



$M_x = \rho\int_0^5 \left(\frac{10x\sqrt{125 - x^3}}{2}\right)\left(10x\sqrt{125 - x^3}\right)dx$

$= 50\rho\int_0^5 x^2(125 - x^3)\,dx = \frac{3{,}124{,}375\rho}{24} \approx 130{,}208\rho$

$M_y = \rho\int_0^5 10x^2\sqrt{125 - x^3}\,dx = -\frac{10\rho}{3}\int_0^5 \sqrt{125 - x^3}(-3x^2)\,dx = \frac{12{,}500\sqrt{5}\rho}{9} \approx 3105.6\rho$

$\bar{x} = \frac{M_y}{m} \approx 3.0$

$\bar{y} = \frac{M_x}{m} \approx 126.0$

Therefore, the centroid is (3.0, 126.0).

30. $m = \rho\int_0^4 xe^{-x/2}\,dx \approx 2.3760\rho$

$M_x = \rho\int_0^4 \left(\frac{xe^{-x/2}}{2}\right)(xe^{-x/2})\,dx$

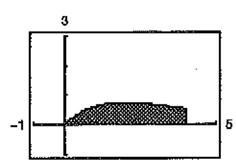
$= \frac{\rho}{2}\int_0^4 x^2e^{-x}\,dx \approx 0.7619\rho$

$M_y = \rho\int_0^4 x^2e^{-x/2}\,dx \approx 5.1732\rho$

$\bar{x} = \frac{M_y}{m} \approx 2.2$

$\bar{y} = \frac{M_x}{m} \approx 0.3$

Therefore, the centroid is (2.2, 0.3).



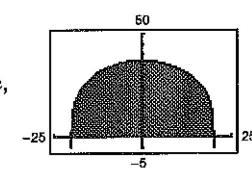
31. $m = \rho\int_{-20}^{20} 5\sqrt[3]{400 - x^2}\,dx \approx 1239.76\rho$

$M_x = \rho\int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2}\left(5\sqrt[3]{400 - x^2}\right)dx$

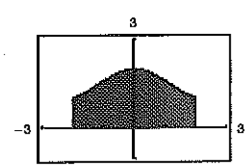
$= \frac{25\rho}{2}\int_{-20}^{20} (400 - x^2)^{2/3}\,dx \approx 20064.27$

$\bar{y} = \frac{M_x}{m} \approx 16.18$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 16.2).



32. $m = \rho\int_{-2}^{2} \frac{8}{x^2 + 4}\,dx \approx 6.2832\rho$



$M_x = \rho\int_{-2}^{2} \frac{1}{2}\left(\frac{8}{x^2 + 4}\right)\left(\frac{8}{x^2 + 4}\right)dx = 32\rho\int_{-2}^{2} \frac{1}{(x^2 + 4)^2}\,dx \approx 5.14149\rho$

$\bar{y} = \frac{M_x}{m} \approx 0.8$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 0.8).

33. $A = \frac{1}{2}(2a)c = ac$

$\frac{1}{A} = \frac{1}{ac}$

$$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b-a}{c}y + a\right)^2 - \left(\frac{b+a}{c}y - a\right)^2\right] dy$$

$$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right] dy$$

$$= \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$$

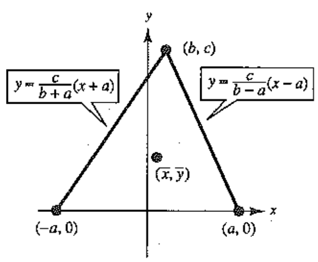
$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b-a}{c}y + a\right) - \left(\frac{b+a}{c}y - a\right)\right] dy$$

$$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right) dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right) dy$$

$$= \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

From elementary geometry, $(b/3, c/3)$ is the point of intersection of the medians.



34. $A = bh = ac$

$\frac{1}{A} = \frac{1}{ac}$

$$\bar{x} = \frac{1}{ac}\frac{1}{2}\int_0^c \left[\left(\frac{b}{c}y + a\right)^2 - \left(\frac{b}{c}y\right)^2\right] dy$$

$$= \frac{1}{2ac}\int_0^c \left(\frac{2ab}{c}y + a^2\right) dy$$

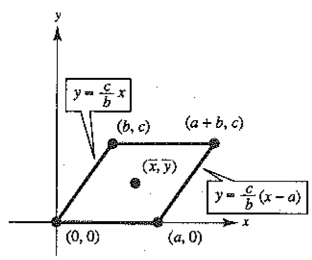
$$= \frac{1}{2ac}\left[\frac{ab}{c}y^2 + a^2y\right]_0^c$$

$$= \frac{1}{2ac}[abc + a^2c] = \frac{1}{2}(b + a).$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b}{c}y + a\right) - \left(\frac{b}{c}y\right)\right] dy = \left[\frac{1}{c}\frac{y^2}{2}\right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b+a}{2}, \frac{c}{2}\right)$$

This is the point of intersection of the diagonals.



35. $A = \frac{c}{2}(a + b)$

$\frac{1}{A} = \frac{2}{c(a + b)}$

$$\bar{x} = \frac{2}{c(a+b)}\int_0^c x\left(\frac{b-a}{c}x + a\right) dx = \frac{2}{c(a+b)}\int_0^c \left(\frac{b-a}{c}x^2 + ax\right) dx = \frac{2}{c(a+b)}\left[\frac{b-a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$$

$$= \frac{2}{c(a+b)}\left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a+b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}$$

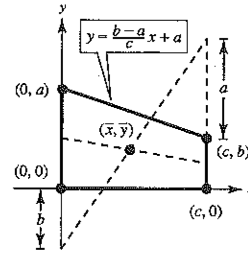
$$\bar{y} = \frac{2}{c(a+b)}\frac{1}{2}\int_0^c \left(\frac{b-a}{c}x + a\right)^2 dx = \frac{1}{c(a+b)}\int_0^c \left[\left(\frac{b-a}{c}\right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2\right] dx$$

$$= \frac{1}{c(a+b)}\left[\left(\frac{b-a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b-a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a+b)}\left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c\right]$$

$$= \frac{1}{3c(a+b)}[(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$$

$$= \frac{1}{3(a+b)}[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a+b)}$$

Thus, $(\bar{x}, \bar{y}) = \left(\frac{(a+2b)c}{3(a+b)}, \frac{a^2 + ab + b^2}{3(a+b)}\right)$.



The one line passes through $(0, a/2)$ and $(c, b/2)$. It's equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$. The other line passes through $(0, -b)$ and $(c, a + b)$. It's equation is $y = \frac{a+2b}{c}x - b$. $(\bar{x}, \bar{y})$ is the point of intersection of these two lines.

36. $\bar{x} = 0$ by symmetry.

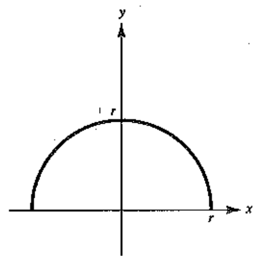
$$A = \frac{1}{2}\pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\bar{y} = \frac{2}{\pi r^2}\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{1}{\pi r^2}\left[\frac{4r^3}{3}\right] = \frac{4r}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$$



37. $\bar{x} = 0$ by symmetry.

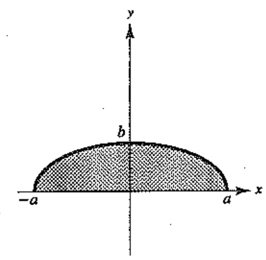
$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

$$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left[\frac{4a^3}{3}\right] = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$$



38. $$A = \int_0^1 [1 - (2x - x^2)]\,dx = \frac{1}{3}$$

$$\frac{1}{A} = 3$$

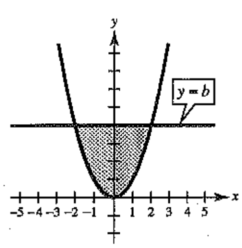
$$\bar{x} = 3\int_0^1 x[1 - (2x - x^2)]\,dx = 3\int_0^1 [x - 2x^2 + x^3]\,dx = 3\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\bar{y} = 3\int_0^1 \frac{[1 + (2x - x^2)]}{2}[1 - (2x - x^2)]\,dx \qquad = \frac{3}{2}\int_0^1 [1 - (2x - x^2)^2]\,dx$$

$$= \frac{3}{2}\int_0^1 [1 - 4x^2 + 4x^3 - x^4]\,dx = \frac{3}{2}\left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5}\right]_0^1 = \frac{7}{10}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{7}{10}\right)$$

39. (a)



(b) $\bar{x} = 0$ by symmetry.

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\,dx = 0$ because $bx - x^3$ is odd.

(d) $\bar{y} > \frac{b}{2}$ since there is more area above $y = \frac{b}{2}$ than below.

(e) $$M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2}\,dx$$

$$= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2}\,dx = \frac{1}{2}\left[b^2 x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2)\,dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b$$

40. (a) $M_y = 0$ by symmetry.

$$M_y = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} x(b - x^{2n})\,dx = 0$$

because $bx - x^{2n+1}$ is an odd function.

(c) $$M_x = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{(b + x^{2n})(b - x^{2n})}{2}\,dx = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{1}{2}(b^2 - x^{4n})\,dx$$

$$= \frac{1}{2}\left(b^2x - \frac{x^{4n+1}}{4n+1}\right)\Bigg]_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}}$$

$$= b^2b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1}b^{(4n+1)/2n}$$

$$A = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} (b - x^{2n})\,dx = 2\left[bx - \frac{x^{2n+1}}{2n+1}\right]_0^{\sqrt[2n]{b}}$$

$$= 2\left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1}\right] = \frac{4n}{2n+1}b^{(2n+1)/2n}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4n\,b^{(4n+1)/2n}/(4n+1)}{4n\,b^{(24n+1)/2n}/(2n+1)} = \frac{2n+1}{4n+1}b$$

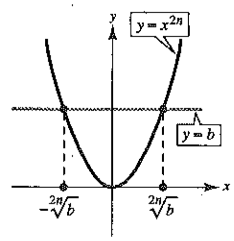
(b) $\bar{y} > \dfrac{b}{2}$ because there is more area above $y = \dfrac{b}{2}$ than below.

(d)

n	1	2	3	4
$\bar{y}$	$\frac{3}{5}b$	$\frac{5}{9}b$	$\frac{7}{13}b$	$\frac{9}{17}b$

(e) $$\lim_{n\to\infty} \bar{y} = \lim_{n\to\infty} \frac{2n+1}{4n+1}b = \frac{1}{2}b$$

(f) As $n \to \infty$, the figure gets narrower.



41. (a) $\bar{x} = 0$ by symmetry.

$$A = 2\int_0^{40} f(x)\,dx = \frac{2(40)}{3(4)}[30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2}\,dx = \frac{40}{3(4)}[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3}(7216) = \frac{72{,}160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72{,}160/3}{5560/3} = \frac{72{,}160}{5560} \approx 12.98$$

$(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$ (Use nine data points.)

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{23{,}697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

42. Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

x	0	0.5	1.0	1.5	2.0
f	2.0	1.93	1.73	1.32	0
d	0.50	0.48	0.43	0.33	0

—CONTINUED—

42. —CONTINUED—

(a) $\text{Area} = 2\int_0^2 [f(x) - d(x)]\,dx$

$\approx 2\dfrac{2}{3(4)}[1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0]$

$= \dfrac{1}{3}[13.86] = 4.62$

$M_x = \displaystyle\int_{-2}^{2} \frac{f(x) + d(x)}{2}(f(x) - d(x))\,dx$

$= \displaystyle\int_0^2 [f(x)^2 - d(x)^2]\,dx$

$= \dfrac{2}{3(4)}[3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0]$

$= \dfrac{1}{6}[29.878] = 4.9797$

$\bar{y} = \dfrac{M_x}{A} = \dfrac{4.9797}{4.62} = 1.078$

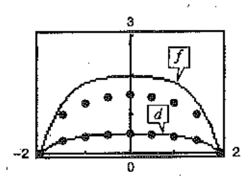
$(\bar{x}, \bar{y}) = (0, 1.078)$

(b) $f(x) = -0.1061x^4 - 0.06126x^2 + 1.9527$

$d(x) = -0.02648x^4 - 0.01497x^2 + .4862$

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{4.9133}{4.59998} = 1.068$

$(\bar{x}, \bar{y}) = (0, 1.068)$



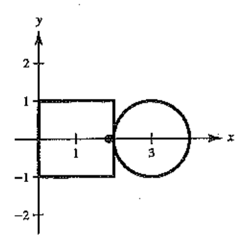
43. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Area: $A = 4 + \pi$

$\bar{x} = \dfrac{4(1) + \pi(3)}{4 + \pi} = \dfrac{4 + 3\pi}{4 + \pi}$

$\bar{y} = \dfrac{4(0) + \pi(0)}{4 + \pi} = 0$

$(\bar{x}, \bar{y}) = \left(\dfrac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$



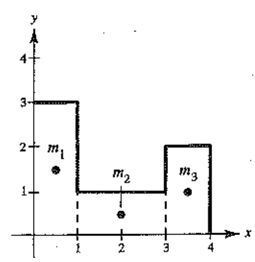
44. Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

Area: $A = 3 + 2 + 2 = 7$

$\bar{x} = \dfrac{3(1/2) + 2(2) + 2(7/2)}{7} = \dfrac{25/2}{7} = \dfrac{25}{14}$

$\bar{y} = \dfrac{3(3/2) + 2(1/2) + 2(1)}{7} = \dfrac{15/2}{7} = \dfrac{15}{14}$

$(\bar{x}, \bar{y}) = \left(\dfrac{25}{14}, \dfrac{15}{14}\right)$



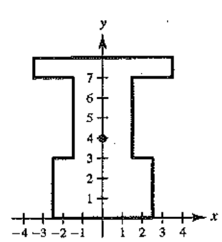
45. Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, $(0, 5)$, and $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$\bar{x} = \dfrac{15(0) + 12(0) + 7(0)}{34} = 0$

$\bar{y} = \dfrac{15(3/2) + 12(5) + 7(15/2)}{34} = \dfrac{135}{34}$

$(\bar{x}, \bar{y}) = \left(0, \dfrac{135}{34}\right) \approx (0, 3.97)$



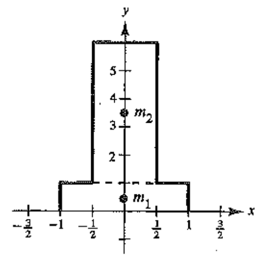
46. $m_1 = \frac{7}{8}(2) = \frac{7}{4}, P_1 = \left(0, \frac{7}{16}\right)$

$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}, P_2 = \left(0, \frac{55}{16}\right)$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16,569}{6384} = \frac{789}{304}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{789}{304}\right) \approx (0, 2.595)$$



47. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

48. Centroids of the given regions: $(3, 0)$ and $(1, 0)$

Mass: $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0\right) \approx (1.56, 0)$$

49. $r = 5$ is distance between center of circle and y-axis. $A \approx \pi(4)^2 = 16\pi$ is area of circle. Hence,

$$V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14.$$

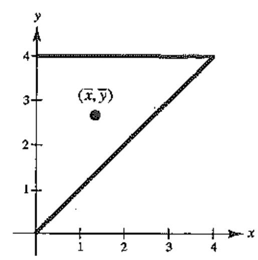
50. $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

51. $A = \frac{1}{2}(4)(4) = 8$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



52. $A = \int_2^6 2\sqrt{x - 2}\,dx = \frac{4}{3}(x - 2)^{3/2}\Big]_2^6 = \frac{32}{3}$

$$M_y = \int_2^6 (x)2\sqrt{x - 2}\,dx = 2\int_2^6 x\sqrt{x - 2}\,dx$$

Let $u = x - 2, x = u + 2, du = dx$:

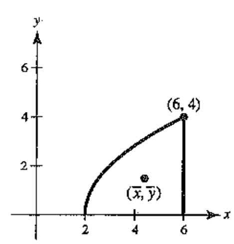
$$M_y = 2\int_0^4 (u + 2)\sqrt{u}\,du = 2\int_0^4 (u^{3/2} + 2u^{1/2})\,du = 2\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^4$$

$$= 2\left[\frac{64}{5} + \frac{32}{3}\right] = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi\left(\frac{22}{5}\right)\left(\frac{32}{3}\right) = \frac{1408\pi}{15} \approx 294.89$$



53. $m = m_1 + \cdots + m_n$

$M_y = m_1x_1 + \cdots + m_nx_n$

$M_x = m_1y_1 + \cdots + m_ny_n$

$\bar{x} = \dfrac{M_y}{m}, \bar{y} = \dfrac{M_x}{m}$

54. A planar lamina is a thin flat plate of constant density. The center of mass $(\bar{x}, \bar{y})$ is the balancing point on the lamina.

55. (a) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$

(b) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$

(c) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$

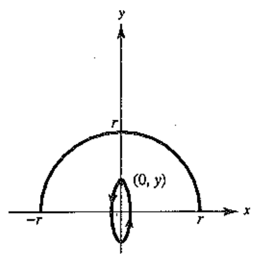
(d) No

56. Let R be a region in a plane and let L be a line such that L does not intersect the interior of R. If r is the distance between the centroid of R and L, then the volume V of the solid of revolution formed by revolving R about L is $V = 2\pi rA$ where A is the area of R.

57. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

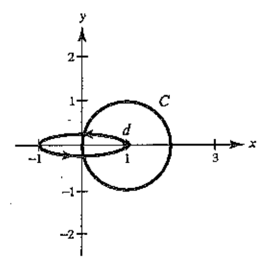
This distance is also the circumference of the circle of radius y.



$d = 2\pi y$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.

58. The centroid of the circle is $(1, 0)$. The distance traveled by the centroid is 2π. The arc length of the circle is also 2π. Therefore, $S = (2\pi)(2\pi) = 4\pi^2$.



59. $A = \displaystyle\int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$

$m = \rho A = \dfrac{\rho}{n+1}$

$M_x = \dfrac{\rho}{2}\displaystyle\int_0^1 (x^n)^2\,dx = \left[\frac{\rho}{2}\cdot\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$

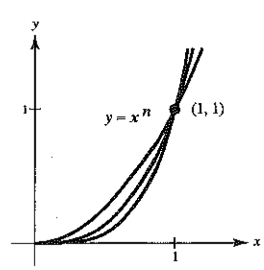
$M_y = \rho\displaystyle\int_0^1 x(x^n)\,dx = \left[\rho\cdot\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{n+1}{n+2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{n+1}{2(2n+1)} = \dfrac{n+1}{4n+2}$

Centroid: $\left(\dfrac{n+1}{n+2}, \dfrac{n+1}{4n+2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \frac{1}{4}\right)$. The graph approaches the x-axis and the line $x = 1$ as $n \to \infty$.



60. Let T be the shaded triangle with vertices $(-1, 4)$, $(1, 4)$, and $(0, 3)$. Let U be the large triangle with vertices $(-4, 4)$, $(4, 4)$, and $(0, 0)$. V consists of the region U minus the region T.

Centroid of T: $\left(0, \frac{11}{3}\right)$; Area $= 1$

Centroid of U: $\left(0, \frac{8}{3}\right)$; Area $= 16$

Area: $V = 16 - 1 = 15$

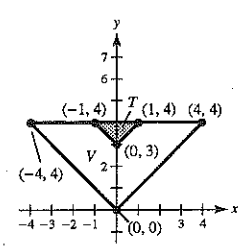
$\bar{x} = 0$ by symmetry.

$15\bar{y} + 1\left(\frac{11}{3}\right) = 16\left(\frac{8}{3}\right)$

$15\bar{y} = \frac{117}{3}$

$\bar{y} = \frac{13}{5}$

$(\bar{x}, \bar{y}) = \left(0, \frac{13}{5}\right)$



Section 7.7 Fluid Pressure and Fluid Force

1. $F = PA = [62.4(5)](3) = 936 \text{ lb}$

2. $F = PA = [62.4(5)](16) = 4992 \text{ lb}$

3. $F = 62.4(h + 2)(6) - (62.4)(h)(6)$

$= 62.4(2)(6) = 748.8 \text{ lb}$

4. $F = 62.4(h + 4)(48) - (62.4)(h)(48)$

$= 62.4(4)(48) = 11{,}980.8 \text{ lb}$

5. $h(y) = 3 - y$

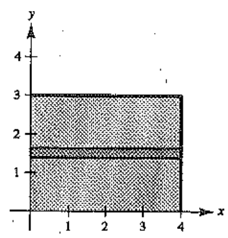
$L(y) = 4$

$F = 62.4 \int_0^3 (3 - y)(4)\, dy$

$= 249.6 \int_0^3 (3 - y)\, dy$

$= 249.6\left[3y - \frac{y^2}{2}\right]_0^3$

$= 1123.2 \text{ lb}$



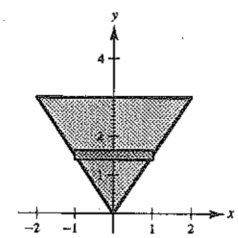
6. $h(y) = 3 - y$

$L(y) = \frac{4}{3}y$

$F = 62.4 \int_0^3 (3 - y)\left(\frac{4}{3}y\right) dy$

$= \frac{4}{3}(62.4) \int_0^3 (3y - y^2)\, dy$

$= \frac{4}{3}(62.4)\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 374.4 \text{ lb}$



Force is one-third that of Exercise 5.

7. $h(y) = 3 - y$

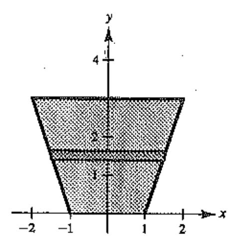
$L(y) = 2\left(\frac{y}{3} + 1\right)$

$F = 2(62.4) \int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy$

$= 124.8 \int_0^3 \left(3 - \frac{y^2}{3}\right) dy$

$= 124.8\left[3y - \frac{y^3}{9}\right]_0^3$

$= 748.8 \text{ lb}$

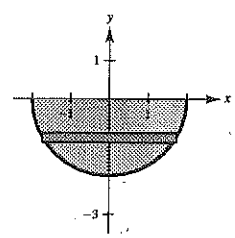


8. $h(y) = -y$

$L(y) = 2\sqrt{4 - y^2}$

$F = 62.4 \int_{-2}^0 (-y)(2)\sqrt{4 - y^2}\, dy$

$= \left[62.4\left(\frac{2}{3}\right)(4 - y^2)^{3/2}\right]_{-2}^0 = 332.8 \text{ lb}$



9. $h(y) = 4 - y$

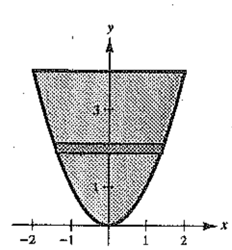
$L(y) = 2\sqrt{y}$

$F = 2(62.4) \int_0^4 (4 - y)\sqrt{y}\, dy$

$= 124.8 \int_0^4 (4y^{1/2} - y^{3/2})\, dy$

$= 124.8\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4$

$= 1064.96 \text{ lb}$



10. $h(y) = -y$

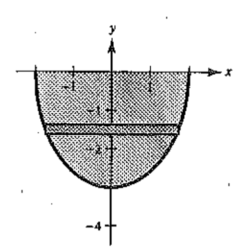
$L(y) = \frac{4}{3}\sqrt{9 - y^2}$

$F = 62.4 \int_{-3}^0 (-y)\frac{4}{3}\sqrt{9 - y^2}\, dy$

$= 62.4\left(\frac{2}{3}\right) \int_{-3}^0 (9 - y^2)^{1/2}(-2y)\, dy$

$= \left[62.4\left(\frac{4}{9}\right)(9 - y^2)^{3/2}\right]_{-3}^0$

$= 748.8 \text{ lb}$

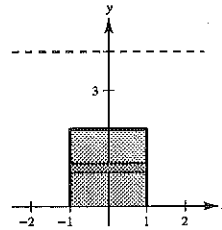


11. $h(y) = 4 - y$

$L(y) = 2$

$$F = 9800 \int_0^2 2(4 - y)\,dy$$

$$= 9800\Big[8y - y^2\Big]_0^2 = 117{,}600 \text{ newtons}$$



12. $h(y) = (1 + 3\sqrt{2}) - y$

$L_1(y) = 2y$ (lower part)

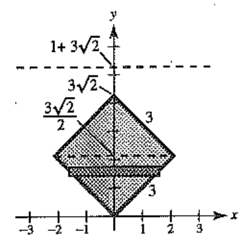
$L_2(y) = 2(3\sqrt{2} - y)$ (upper part)

$$F = 2(9800)\left[\int_0^{3\sqrt{2}/2} (1 + 3\sqrt{2} - y)y\,dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} (1 + 3\sqrt{2} - y)(3\sqrt{2} - y)\,dy\right]$$

$$= 19{,}600\left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3}\right]_0^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2}+1}{2}y\right]_{3\sqrt{2}/2}^{3\sqrt{2}}\right]$$

$$= 19{,}600\left[\frac{9(2\sqrt{2}+1)}{4} + \frac{9(\sqrt{2}+1)}{4}\right]$$

$$= 44{,}100(3\sqrt{2} + 2) \text{ newtons}$$



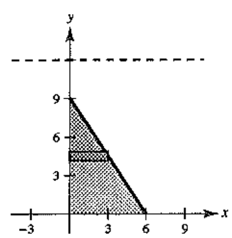
13. $h(y) = 12 - y$

$L(y) = 6 - \frac{2y}{3}$

$$F = 9800 \int_0^9 (12 - y)\left(6 - \frac{2y}{3}\right) dy$$

$$= 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_0^9$$

$$= 2{,}381{,}400 \text{ newtons}$$



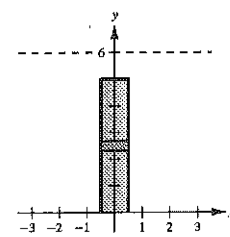
14. $h(y) = 6 - y$

$L(y) = 1$

$$F = 9800 \int_0^5 1(6 - y)\,dy$$

$$= 9800\left[6y - \frac{y^2}{2}\right]_0^5$$

$$= 171{,}500 \text{ newtons}$$



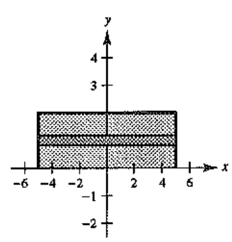
15. $h(y) = 2 - y$

$L(y) = 10$

$$F = 140.7 \int_0^2 (2 - y)(10)\,dy$$

$$= 1407 \int_0^2 (2 - y)\,dy$$

$$= 1407\left[2y - \frac{y^2}{2}\right]_0^2 = 2814 \text{ lb}$$



16. $h(y) = -y$

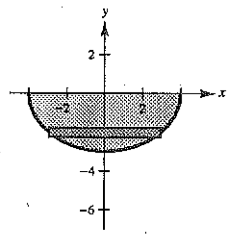
$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

$F = 140.7 \int_{-3}^{0} (-y)(2)\left(\frac{4}{3}\sqrt{9 - y^2}\right) dy$

$= \frac{(140.7)(4)}{3} \int_{-3}^{0} \sqrt{9 - y^2}\,(-2y)\, dy$

$= \left[\frac{(140.7)(4)}{3}\left(\frac{2}{3}\right)(9 - y^2)^{3/2}\right]_{-3}^{0}$

$= 3376.8$ lb



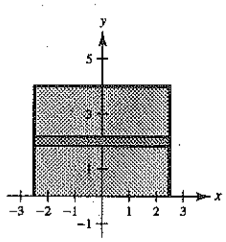
17. $h(y) = 4 - y$

$L(y) = 6$

$F = 140.7 \int_{0}^{4} (4 - y)(6)\, dy$

$= 844.2 \int_{0}^{4} (4 - y)\, dy$

$= 844.2\left[4y - \frac{y^2}{2}\right]_{0}^{4} = 6753.6$ lb



18. $h(y) = -y$

$L(y) = 5 + \frac{5}{3}y$

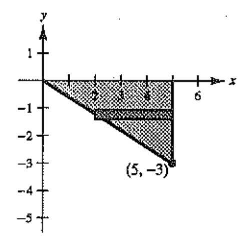
$F = 140.7 \int_{-3}^{0} (-y)\left(5 + \frac{5}{3}y\right) dy$

$= 140.7 \int_{-3}^{0} \left(-5y - \frac{5}{3}y^2\right) dy$

$= 140.7\left[-\frac{5}{2}y^2 - \frac{5}{9}y^3\right]_{-3}^{0}$

$= 140.7\left[\frac{45}{2} - 15\right]$

$= 1055.25$ lb



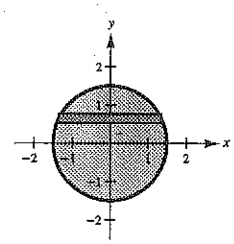
19. $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$F = 42 \int_{-3/2}^{0} (-y)\sqrt{9 - 4y^2}\, dy$

$= \frac{42}{8} \int_{-3/2}^{0} (9 - 4y^2)^{1/2}(-8y)\, dy$

$= \left[\left(\frac{21}{4}\right)\left(\frac{2}{3}\right)(9 - 4y^2)^{3/2}\right]_{-3/2}^{0} = 94.5$ lb



20. $h(y) = \frac{3}{2} - y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$F = 42 \int_{-3/2}^{3/2} \left(\frac{3}{2} - y\right)\sqrt{9 - 4y^2}\, dy = 63 \int_{-3/2}^{3/2} \sqrt{9 - 4y^2}\, dy + \frac{21}{4} \int_{-3/2}^{3/2} \sqrt{9 - 4y^2}\,(-8y)\, dy$

The second integral is zero since it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$\left(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2}\right)$

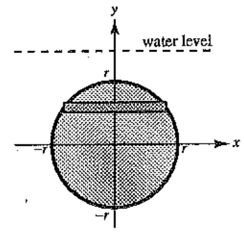
Thus, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32$ lb.

21. $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$$F = w\int_{-r}^{r} (k - y)\sqrt{r^2 - y^2}(2)\,dy$$

$$= w\left[2k\int_{-r}^{r} \sqrt{r^2 - y^2}\,dy + \int_{-r}^{r} \sqrt{r^2 - y^2}(-2y)\,dy\right]$$



The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r.

$$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$$

22. (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi$ lbs

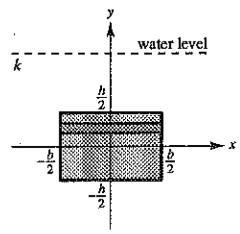
(b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi$ lbs

23. $h(y) = k - y$

$L(y) = b$

$$F = w\int_{-h/2}^{h/2} (k - y)b\,dy$$

$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



24. (a) $F = wkhb$

$= (62.4)(\frac{11}{2})(3)(5) = 5148$ lbs

(b) $F = wkhb$

$= (62.4)(\frac{17}{2})(5)(10) = 26{,}520$ lbs

25. From Exercise 23:

$F = 64(15)(1)(1) = 960$ lb

26. From Exercise 21:

$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98$ lb

27. $h(y) = 4 - y$

$$F = 62.4\int_0^4 (4 - y)L(y)\,dy$$

Using Simpson's Rule with $n = 8$ we have:

$$F \approx 62.4\left(\frac{4 - 0}{3(8)}\right)[0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0]$$

$= 3010.8$ lb

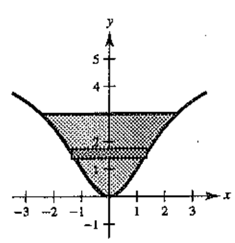
28. $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for x, you obtain $x = \sqrt{4y/(5 - y)}$.

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

$$F = 62.4(2)\int_0^3 (3 - y)\sqrt{\frac{4y}{5 - y}}\,dy$$

$$= 2(124.8)\int_0^3 (3 - y)\sqrt{\frac{y}{5 - y}}\,dy \approx 546.265 \text{ lb}$$

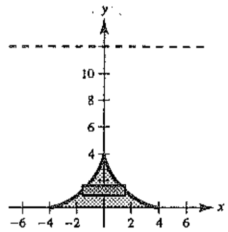


29. $h(y) = 12 - y$

$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$

$F = 62.4 \int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2}\, dy$

≈ 6448.73 lb

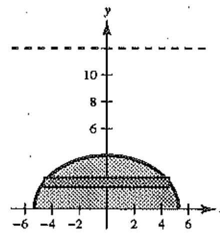


30. $h(y) = 12 - y$

$L(y) = 2\dfrac{\sqrt{7(16 - y^2)}}{2} = \sqrt{7(16 - y^2)}$

$F = 62.4 \int_0^4 (12 - y)\sqrt{7(16 - y^2)}\, dy$

$= 62.4\sqrt{7} \int_0^4 (12 - y)\sqrt{16 - y^2}\, dy \approx 21373.7$ lb



31. (a) If the fluid force is one-half of 1123.2 lb, and the height of the water is b, then

$h(y) = b - y$

$L(y) = 4$

$F = 62.4 \int_0^b (b - y)(4)\, dy = \frac{1}{2}(1123.2)$

$\int_0^b (b - y)\, dy = 2.25$

$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$

$b^2 - \frac{b^2}{2} = 2.25$

$b^2 = 4.5 \implies b \approx 2.12$ ft.

(b) The pressure increases with increasing depth.

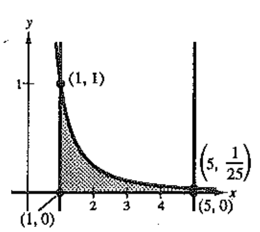
32. Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

33. $F = Fw = w\int_c^d h(y)L(y)\, dy$, see page 508.

34. The left window experiences the greater fluid force because its centroid is lower.

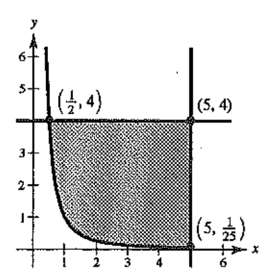
Review Exercises for Chapter 7

1. $A = \int_1^5 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$



2. $A = \int_{1/2}^5 \left(4 - \frac{1}{x^2}\right) dx$

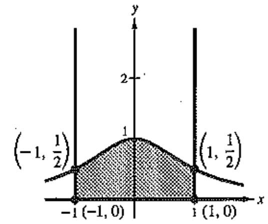
$= \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5}$



3. $A = \int_{-1}^{1} \frac{1}{x^2 + 1}\, dx$

$= \left[\arctan x\right]_{-1}^{1}$

$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

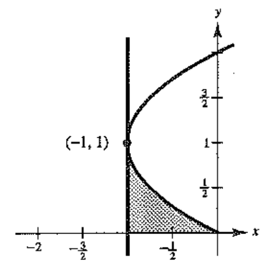


4. $A = \int_{0}^{1} [(y^2 - 2y) - (-1)]\, dy$

$= \int_{0}^{1} (y^2 - 2y + 1)\, dy$

$= \int_{0}^{1} (y - 1)^2\, dy$

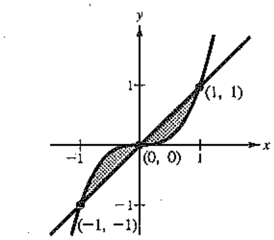
$= \left[\frac{(y-1)^3}{3}\right]_0^1 = \frac{1}{3}$



5. $A = 2\int_{0}^{1} (x - x^3)\, dx$

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

$= \frac{1}{2}$

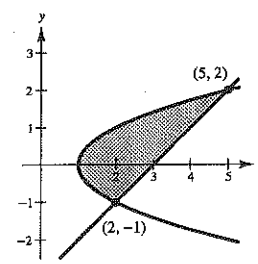


6. $A = \int_{-1}^{2} [(y + 3) - (y^2 + 1)]\, dy$

$= \int_{-1}^{2} (2 + y - y^2)\, dy$

$= \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-1}^{2}$

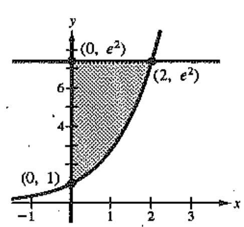
$= \frac{9}{2}$



7. $A = \int_{0}^{2} (e^2 - e^x)\, dx$

$= \left[xe^2 - e^x\right]_0^2$

$= e^2 + 1$

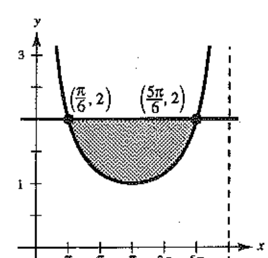


8. $A = 2\int_{\pi/6}^{\pi/2} (2 - \csc x)\, dx$

$= 2\left[2x - \ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/2}$

$= 2\left([\pi - 0] - \left[\frac{\pi}{3} - \ln\left(2 - \sqrt{3}\right)\right]\right)$

$= 2\left[\frac{2\pi}{3} + \ln\left(2 - \sqrt{3}\right)\right] \approx 1.555$

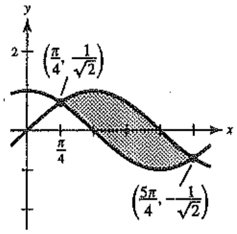


9. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$

$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

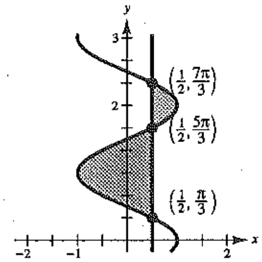
$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$



10. $A = \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y\right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2}\right) dy$

$= \left[\frac{y}{2} - \sin y\right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2}\right]_{5\pi/3}^{7\pi/3}$

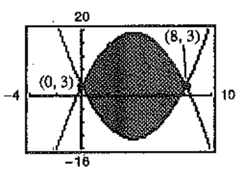
$= \frac{\pi}{3} + 2\sqrt{3}$



11. $A = \int_0^8 [(3 + 8x - x^2) - (x^2 - 8x + 3)]\, dx$

$= \int_0^8 (16x - 2x^2)\, dx$

$= \left[8x^2 - \frac{2}{3}x^3\right]_0^8 = \frac{512}{3} \approx 170.667$



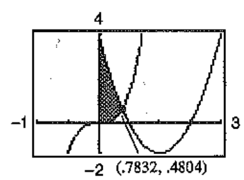
12. Point of intersection is given by:

$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$

$A \approx \int_0^{0.783} (3 - 4x + x^2 - x^3)\, dx$

$= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^{0.783}$

≈ 1.189

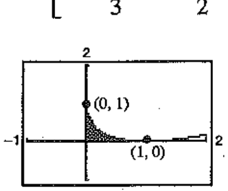


13. $y = \left(1 - \sqrt{x}\right)^2$

$A = \int_0^1 \left(1 - \sqrt{x}\right)^2 dx$

$= \int_0^1 (1 - 2x^{1/2} + x)\, dx$

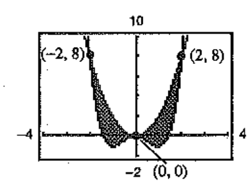
$= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6} \approx 0.1667$



14. $A = 2\int_0^2 [2x^2 - (x^4 - 2x^2)]\, dx$

$= 2\int_0^2 (4x^2 - x^4)\, dx$

$= 2\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \frac{128}{15} \approx 8.5333$



15. $x = y^2 - 2y \Rightarrow x + 1 = (y-1)^2 \Rightarrow y = 1 \pm \sqrt{x+1}$

$$A = \int_{-1}^{0} \left[\left(1 + \sqrt{x+1}\right) - \left(1 - \sqrt{x+1}\right)\right] dx$$

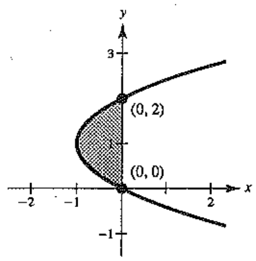
$$= \int_{-1}^{0} 2\sqrt{x+1}\, dx$$

$$A = \int_{0}^{2} [0 - (y^2 - 2y)]\, dy$$

$$= \int_{0}^{2} (2y - y^2)\, dy$$

$$= \left[y^2 - \frac{1}{3}y^3\right]_0^2$$

$$= \frac{4}{3}$$



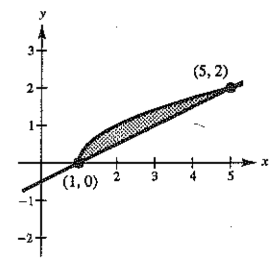
16. $y = \sqrt{x-1} \Rightarrow x = y^2 + 1$

$$y = \frac{x-1}{2} \Rightarrow x = 2y + 1$$

$$A = \int_{0}^{2} [(2y+1) - (y^2+1)]\, dy$$

$$= \int_{1}^{5} \left[\sqrt{x-1} - \frac{x-1}{2}\right] dx$$

$$= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{4}(x-1)^2\right]_1^5 = \frac{4}{3}$$



17. $A = \int_{0}^{2} \left[1 - \left(1 - \frac{x}{2}\right)\right] dx + \int_{2}^{3} [1 - (x-2)]\, dx$

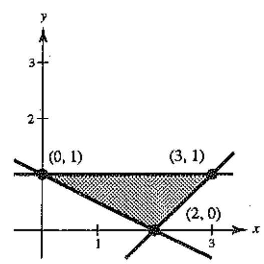
$$= \int_{0}^{2} \frac{x}{2}\, dx + \int_{2}^{3} (3 - x)\, dx$$

$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$

$$y = x - 2 \Rightarrow x = y + 2,\ y = 1$$

$$A = \int_{0}^{1} [(y+2) - (2-2y)]\, dy$$

$$= \int_{0}^{1} 3y\, dy = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$$

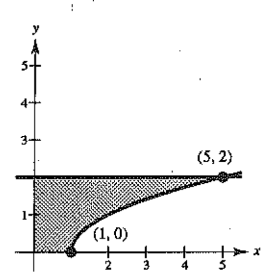


18. $A = \int_{0}^{1} 2\, dx + \int_{1}^{5} \left[2 - \sqrt{x-1}\right] dx$

$$x = y^2 + 1$$

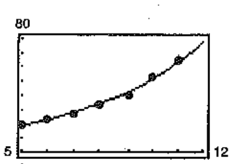
$$A = \int_{0}^{2} (y^2 + 1)\, dy$$

$$= \left[\frac{1}{3}y^3 + y\right]_0^2 = \frac{14}{3}$$



19. Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

20. (a) $y = (6.8335)(1.2235)^t = (6.8335)e^{0.2017t}$

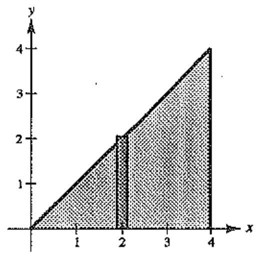


(b) $R_2 = 5 + 6.83e^{0.2t}$

Difference: $\int_{15}^{20} (R_2 - y)\, dt \approx 12.06$ billion dollars

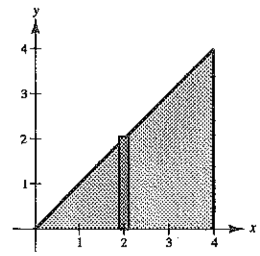
21. (a) **Disk**

$$V = \pi \int_0^4 x^2\,dx = \left[\frac{\pi x^3}{3}\right]_0^4 = \frac{64\pi}{3}$$



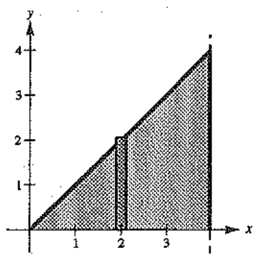
(b) **Shell**

$$V = 2\pi \int_0^4 x^2\,dx = \left[\frac{2\pi}{3}x^3\right]_0^4 = \frac{128\pi}{3}$$



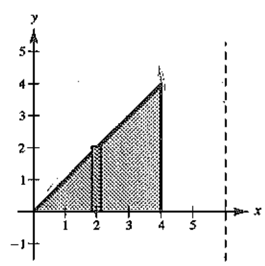
(c) **Shell**

$$V = 2\pi \int_0^4 (4 - x)x\,dx$$
$$= 2\pi \int_0^4 (4x - x^2)\,dx$$
$$= 2\pi\left[2x^2 - \frac{x^3}{3}\right]_0^4 = \frac{64\pi}{3}$$



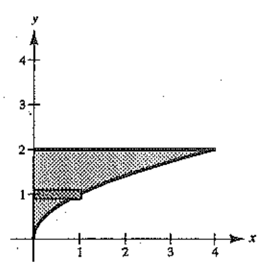
(d) **Shell**

$$V = 2\pi \int_0^4 (6 - x)x\,dx$$
$$= 2\pi \int_0^4 (6x - x^2)\,dx$$
$$= 2\pi\left[3x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{160\pi}{3}$$



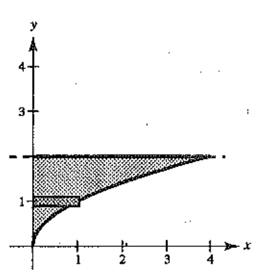
22. (a) **Shell**

$$V = 2\pi \int_0^2 y^3\,dy = \left[\frac{\pi}{2}y^4\right]_0^2 = 8\pi$$



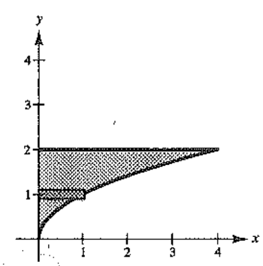
(b) **Shell**

$$V = 2\pi \int_0^2 (2 - y)y^2\,dy$$
$$= 2\pi \int_0^2 (2y^2 - y^3)\,dy$$
$$= 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2$$
$$= \frac{8\pi}{3}$$



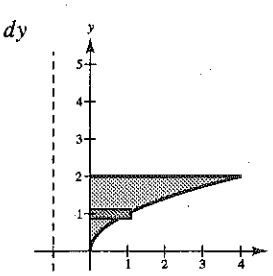
(c) **Disk**

$$V = \pi \int_0^2 y^4\,dy = \left[\frac{\pi}{5}y^5\right]_0^2 = \frac{32\pi}{5}$$



(d) **Disk**

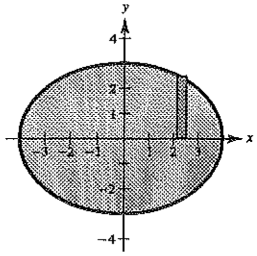
$$V = \pi \int_0^2 \left[(y^2 + 1)^2 - 1^2\right] dy$$
$$= \pi \int_0^2 (y^4 + 2y^2)\,dy$$
$$= \pi\left[\frac{1}{5}y^5 + \frac{2}{3}y^3\right]_0^2$$
$$= \frac{176\pi}{15}$$



23. (a) **Shell**

$$V = 4\pi \int_0^4 x\left(\frac{3}{4}\right)\sqrt{16 - x^2}\,dx$$

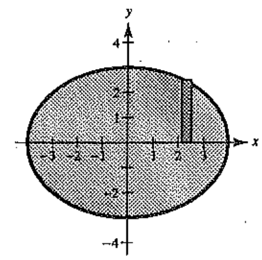
$$= \left[3\pi\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16 - x^2)^{3/2}\right]_0^4 = 64\pi$$



(b) **Disk**

$$V = 2\pi \int_0^4 \left[\frac{3}{4}\sqrt{16 - x^2}\right]^2 dx$$

$$= \frac{9\pi}{8}\left[16x - \frac{x^3}{3}\right]_0^4 = 48\pi$$



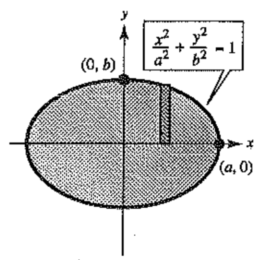
24. (a) **Shell**

$$V = 4\pi \int_0^a (x)\frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{-2\pi b}{a}\int_0^a (a^2 - x^2)^{1/2}(-2x)\,dx$$

$$= \left[\frac{-4\pi b}{3a}(a^2 - x^2)^{3/2}\right]_0^a$$

$$= \frac{4}{3}\pi a^2 b$$

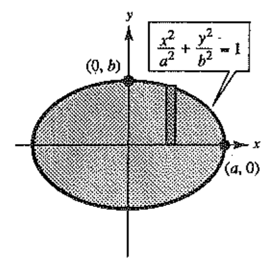


(b) **Disk**

$$V = 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{4}{3}\pi a b^2$$



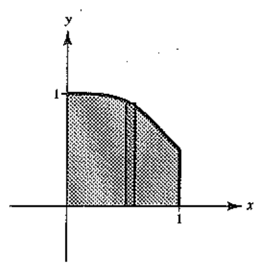
25. Shell

$$V = 2\pi \int_0^1 \frac{x}{x^4 + 1}\,dx$$

$$= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1}\,dx$$

$$= \left[\pi \arctan(x^2)\right]_0^1$$

$$= \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$



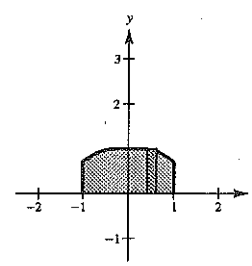
26. Disk

$$V = 2\pi \int_0^1 \left[\frac{1}{\sqrt{1 + x^2}}\right]^2 dx$$

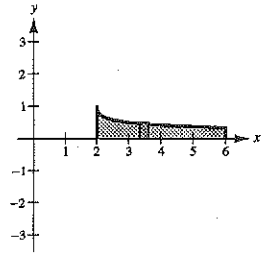
$$= \left[2\pi \arctan x\right]_0^1$$

$$= 2\pi\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi^2}{2}$$



27. Shell: $V = 2\pi\int_2^6 \frac{x}{1+\sqrt{x-2}}\,dx$



$u = \sqrt{x-2}$

$x = u^2 + 2$

$dx = 2u\,du$

$$V = 2\pi\int_2^6 \frac{x}{1+\sqrt{x-2}}\,dx = 4\pi\int_0^2 \frac{(u^2+2)u}{1+u}\,du$$

$$= 4\pi\int_0^2 \frac{u^3+2u}{1+u}\,du = 4\pi\int_0^2\left(u^2 - u + 3 - \frac{3}{1+u}\right)du$$

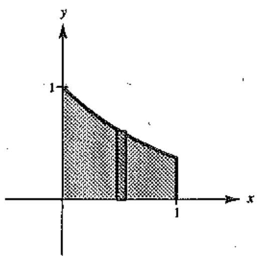
$$= 4\pi\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3\ln(1+u)\right]_0^2 = \frac{4\pi}{3}(20 - 9\ln 3) \approx 42.359$$

28. Disk

$$V = \pi\int_0^1 (e^{-x})^2\,dx$$

$$= \pi\int_0^1 e^{-2x}\,dx = \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$

$$= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2}\right) = \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$$



29. Since $y \le 0$, $A = -\int_{-1}^0 x\sqrt{x+1}\,dx.$

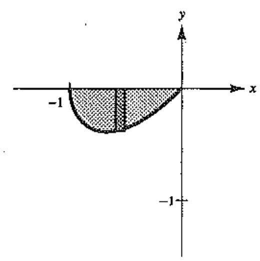
$u = x + 1$

$x = u - 1$

$dx = du$

$$A = -\int_0^1 (u-1)\sqrt{u}\,du = -\int_0^1 (u^{3/2} - u^{1/2})\,du$$

$$= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^1 = \frac{4}{15}$$

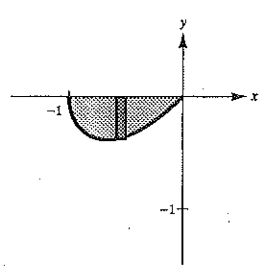


30. (a) **Disk**

$$V = \pi\int_{-1}^0 x^2(x+1)\,dx$$

$$= \pi\int_{-1}^0 (x^3 + x^2)\,dx$$

$$= \pi\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0 = \frac{\pi}{12}$$

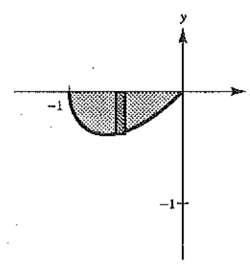


(b) **Shell**

$u = \sqrt{x+1}$

$x = u^2 - 1$

$dx = 2u\,du$



$$V = 2\pi\int_{-1}^0 x^2\sqrt{x+1}\,dx$$

$$= 4\pi\int_0^1 (u^2-1)^2u^2\,du$$

$$= 4\pi\int_0^1 (u^6 - 2u^4 + u^2)\,du$$

$$= 4\pi\left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_0^1 = \frac{32\pi}{105}$$

31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disk:

$$\pi\int_{-3}^{y_0}\frac{16}{9}(9-y^2)\,dy = 16\pi$$

$$\frac{1}{9}\int_{-3}^{y_0}(9-y^2)\,dy = 1$$

$$\left[9y-\frac{1}{3}y^3\right]_{-3}^{y_0} = 9$$

$$\left(9y_0-\frac{1}{3}y_0^3\right)-(-27+9) = 9$$

$$y_0^3-27y_0-27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ ft.

32. $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{a^2-x^2}\right)\left(\sqrt{3}\sqrt{a^2-x^2}\right)$

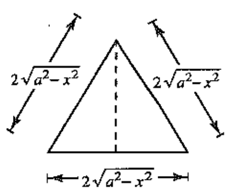
$= \sqrt{3}(a^2-x^2)$

$V = \sqrt{3}\int_{-a}^{a}(a^2-x^2)\,dx = \sqrt{3}\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a}$

$= \sqrt{3}\left(\frac{4a^3}{3}\right)$

Since $\left(4\sqrt{3}\,a^3\right)/3 = 10$, we have $a^3 = \left(5\sqrt{3}\right)/2$. Thus,

$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630$ meters.



33.

$f(x) = \frac{4}{5}x^{5/4}$

$f'(x) = x^{1/4}$

$1 + [f'(x)]^2 = 1 + \sqrt{x}$

$u = 1 + \sqrt{x}$

$x = (u-1)^2$

$dx = 2(u-1)\,du$

$s = \int_0^4\sqrt{1+\sqrt{x}}\,dx = 2\int_1^3\sqrt{u}(u-1)\,du$

$= 2\int_1^3\left(u^{3/2}-u^{1/2}\right)du$

$= 2\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^3 = \frac{4}{15}\left[u^{3/2}(3u-5)\right]_1^3$

$= \frac{8}{15}\left(1+6\sqrt{3}\right) \approx 6.076$

34.

$y = \frac{x^3}{6}+\frac{1}{2x}$

$y' = \frac{1}{2}x^2-\frac{1}{2x^2}$

$1+(y')^2 = \left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)^2$

$s = \int_1^3\left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)dx = \left[\frac{1}{6}x^3-\frac{1}{2x}\right]_1^3 = \frac{14}{3}$

35. $y = 300\cosh\left(\frac{x}{2000}\right) - 280,\ -2000 \le x \le 2000$

$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$

$s = \int_{-2000}^{2000}\sqrt{1+\left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\,dx$

$= \frac{1}{20}\int_{-2000}^{2000}\sqrt{400+9\sinh^2\left(\frac{x}{2000}\right)}\,dx$

≈ 4018.2 ft (by Simpson's Rule or graphing utility)

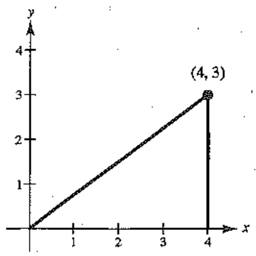
36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answers (b).

37. $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^3 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx = 4\pi \int_0^3 \sqrt{x+1}\,dx$$

$$= 4\pi\left[\left(\frac{2}{3}\right)(x+1)^{3/2}\right]_0^3 = \frac{56\pi}{3}$$

39. $F = kx$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x\,dx = \Big[2x^2\Big]_0^5$$

$$= 50 \text{ in.} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

40. $F = kx$

$$50 = k(9) \implies k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x\,dx = \left[\frac{25}{9}x^2\right]_0^9$$

$$= 225 \text{ in.} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

41. Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $175 - y$

$$W = \frac{62.4\pi}{9}\int_0^{150} (175 - y)\,dy = \frac{62.4\pi}{9}\left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104{,}000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3\text{/min}$$

$$V = \pi r^2 h = \pi\left(\frac{1}{9}\right)h$$

$$\frac{dV}{dt} = \frac{\pi}{9}\left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{9}{\pi}\left(\frac{dV}{dt}\right) = \frac{9}{\pi}\left(-\frac{8}{7.481}\right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49$ minutes

$(49)(12) = 588$ gallons pumped

Volume of water pumped in Exercise 41: 391.7 gallons

$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi$ ft · ton

43. Weight of section of chain: $5\,\Delta x$

Distance moved: $10 - x$

$$W = 5\int_0^{10} (10 - x)\,dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10} = 250 \text{ ft} \cdot \text{lb}$$

44. (a) Weight of section of cable: $4\,\Delta x$

Distance: $200 - x$

$$W = 4\int_0^{200} (200 - x)\,dx = \left[-2(200 - x)^2\right]_0^{200} = 80{,}000 \text{ ft} \cdot \text{lb} = 40 \text{ ft} \cdot \text{ton}$$

(b) Work to move 300 pounds 200 feet vertically: $200(300) = 60{,}000 \text{ ft} \cdot \text{lb} = 30 \text{ ft} \cdot \text{ton}$

Total work = work for drawing up the cable + work of lifting the load

$= 40 \text{ ft} \cdot \text{ton} + 30 \text{ ft} \cdot \text{ton} = 70 \text{ ft} \cdot \text{ton}$

45. $W = \int_a^b F(x)\,dx$

$$80 = \int_0^4 ax^2\,dx = \frac{ax^3}{3}\Big]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

46. $W = \int_a^b F(x)\,dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \le x \le 9 \\ -(4/3)x + 16, & 9 \le x \le 12 \end{cases}$$

$$W = \int_0^9 \left(-\frac{2}{9}x + 6\right)dx + \int_9^{12}\left(-\frac{4}{3}x + 16\right)dx$$

$$= \left[-\frac{1}{9}x^2 + 6x\right]_0^9 + \left[-\frac{2}{3}x^2 + 16x\right]_9^{12}$$

$$= (-9 + 54) + (-96 + 192 + 54 - 144)$$

$$= 51 \text{ ft} \cdot \text{lbs}$$

47. $A = \int_0^a (\sqrt{a} - \sqrt{x})^2\,dx = \int_0^a \left(a - 2\sqrt{a}x^{1/2} + x\right)dx = \left[ax - \frac{4}{3}\sqrt{a}x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$

$$\frac{1}{A} = \frac{6}{a^2}$$

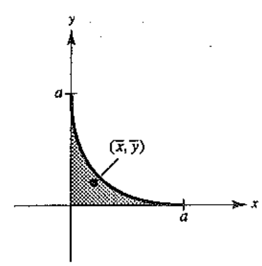
$$\bar{x} = \frac{6}{a^2}\int_0^a x(\sqrt{a} - \sqrt{x})^2\,dx = \frac{6}{a^2}\int_0^a \left(ax - 2\sqrt{a}x^{3/2} + x^2\right)dx = \frac{a}{5}$$

$$\bar{y} = \left(\frac{6}{a^2}\right)\frac{1}{2}\int_0^a (\sqrt{a} - \sqrt{x})^4\,dx$$

$$= \frac{3}{a^2}\int_0^a (a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2)\,dx$$

$$= \frac{3}{a^2}\left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



48. $A = \int_{-1}^{3} [(2x + 3) - x^2]\,dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^{3} = \frac{32}{3}$

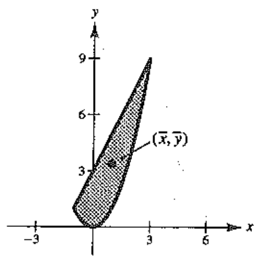
$\frac{1}{A} = \frac{3}{32}$

$\bar{x} = \frac{3}{32}\int_{-1}^{3} x(2x + 3 - x^2)\,dx = \frac{3}{32}\int_{-1}^{3} (3x + 2x^2 - x^3)\,dx = \frac{3}{32}\left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{-1}^{3} = 1$

$\bar{y} = \left(\frac{3}{32}\right)\frac{1}{2}\int_{-1}^{3} [(2x + 3)^2 - x^4]\,dx = \frac{3}{64}\int_{-1}^{3} (9 + 12x + 4x^2 - x^4)\,dx$

$= \frac{3}{64}\left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{3} = \frac{17}{5}$

$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5}\right)$



49. By symmetry, $x = 0$.

$A = 2\int_{0}^{1} (a^2 - x^2)\,dx = 2\left[a^2x - \frac{x^3}{3}\right]_{0}^{a} = \frac{4a^3}{3}$

$\frac{1}{A} = \frac{3}{4a^3}$

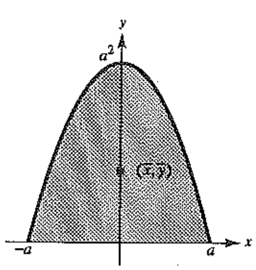
$\bar{y} = \left(\frac{3}{4a^3}\right)\frac{1}{2}\int_{-a}^{a} (a^2 - x^2)^2\,dx$

$= \frac{6}{8a^3}\int_{0}^{a} (a^4 - 2a^2x^2 + x^4)\,dx$

$= \frac{6}{8a^3}\left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5\right]_{0}^{a}$

$= \frac{6}{8a^3}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) = \frac{2a^2}{5}$

$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5}\right)$



50. $A = \int_{0}^{8} \left(x^{2/3} - \frac{1}{2}x\right)dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2\right]_{0}^{8} = \frac{16}{5}$

$\frac{1}{A} = \frac{5}{16}$

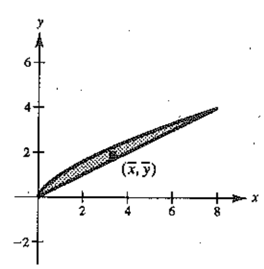
$\bar{x} = \frac{5}{16}\int_{0}^{8} x\left(x^{2/3} - \frac{1}{2}x\right)dx$

$= \frac{5}{16}\left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3\right]_{0}^{8} = \frac{10}{3}$

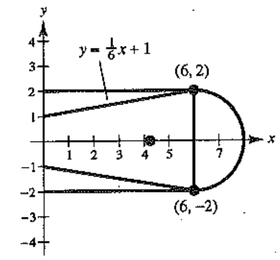
$\bar{y} = \left(\frac{5}{16}\right)\frac{1}{2}\int_{0}^{8} \left(x^{4/3} - \frac{1}{4}x^2\right)dx$

$= \frac{1}{2}\left(\frac{5}{16}\right)\left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3\right]_{0}^{8} = \frac{40}{21}$

$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21}\right)$



51. $\bar{y} = 0$ by symmetry.



For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$M_y = \rho \int_0^6 x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right] dx$$

$$= \rho \int_0^6 \left(\frac{1}{3}x^2 + 2x\right) dx = \rho\left[\frac{x^3}{9} + x^2\right]_0^6 = 60\rho$$

For the semicircle:

$$m = \left(\frac{1}{2}\right)(\pi)(2)^2\rho = 2\pi\rho$$

$$M_y = \rho \int_6^8 x\left[\sqrt{4 - (x-6)^2} - \left(-\sqrt{4 - (x-6)^2}\right)\right] dx = 2\rho \int_6^8 x\sqrt{4 - (x-6)^2}\, dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$M_y = 2\rho \int_0^2 (u + 6)\sqrt{4 - u^2}\, du = 2\rho \int_0^2 u\sqrt{4 - u^2}\, du + 12\rho \int_0^2 \sqrt{4 - u^2}\, du$$

$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(4 - u^2)^{3/2}\right]_0^2 + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3}$$

Thus, we have:

$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4 + 9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3} \cdot \frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\frac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$.

52. Wall at shallow end:

$$F = 62.4 \int_0^5 y(20)\, dy = \left[(1248)\frac{y^2}{2}\right]_0^5 = 15{,}600 \text{ lb}$$

Wall at deep end:

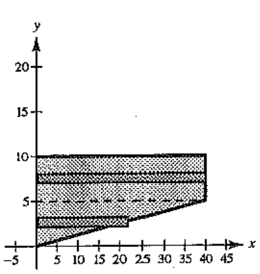
$$F = 62.4 \int_0^{10} y(20)\, dy = \left[(624)y^2\right]_0^{10} = 62{,}400 \text{ lb}$$

Side wall:

$$F_1 = 62.4 \int_0^5 y(40)\, dy = \left[(1248)y^2\right]_0^5 = 31{,}200 \text{ lb}$$

$$F_2 = 62.4 \int_0^5 (10 - y)8y\, dy = 62.4 \int_0^5 (80y - 8y^2)\, dy$$

$$F = F_1 + F_2 = 72{,}800 \text{ lb}$$



53. Let D = surface of liquid; ρ = weight per cubic volume.

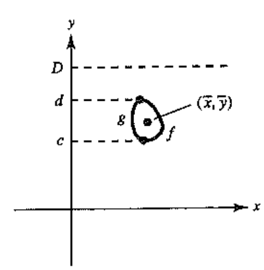
$$F = \rho \int_c^d (D - y)[f(y) - g(y)]\, dy$$

$$= \rho\left[\int_c^d D[f(y) - g(y)]\, dy - \int_c^d y[f(y) - g(y)]\, dy\right]$$

$$= \rho\left[\int_c^d [f(y) - g(y)]\, dy\right]\left[D - \frac{\int_c^d y[f(y) - g(y)]\, dy}{\int_c^d [f(y) - g(y)]\, dy}\right]$$

$$= \rho(\text{Area})(D - \bar{y})$$

$$= \rho(\text{Area})(\text{depth of centroid})$$



54. $F = 62.4(16\pi)5 = 4992\pi$ lb

Problem Solving for Chapter 7

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2)\,dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c\to 0^+} \frac{T}{R} = \lim_{c\to 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

2. $R = \int_0^1 x(1 - x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Let (c, mc) be the intersection of the line and the parabola.

Then, $mc = c(1 - c) \implies m = 1 - c$ or $c = 1 - m$.

$$\frac{1}{2}\left(\frac{1}{6}\right) = \int_0^{1-m} (x - x^2 - mx)\,dx$$

$$\frac{1}{12} = \left[\frac{x^2}{2} - \frac{x^3}{3} - m\frac{x^2}{2}\right]_0^{1-m}$$

$$= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m\frac{(1-m)^2}{2}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2(6 - 4(1-m) - 6m)$$

$$= (1-m)^2(2 - 2m)$$

$$\frac{1}{2} = (1-m)^3$$

$$\left(\frac{1}{2}\right)^{1/3} = 1 - m$$

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.2063$$

3. (a) $\frac{1}{2}V = \int_0^1 \left[\pi\left(2 + \sqrt{1 - y^2}\right)^2 - \pi\left(2 - \sqrt{1 - y^2}\right)^2\right] dy$

$$= \pi\int_0^1 \left[\left(4 + 4\sqrt{1-y^2} + (1 - y^2)\right) - \left(4 - 4\sqrt{1-y^2} + (1-y^2)\right)\right] dy$$

$$= 8\pi\int_0^1 \sqrt{1 - y^2}\,dy \quad \text{(Integral represents 1/4 (area of circle))}$$

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2 \implies V = 4\pi^2.$$

(b) $(x - R)^2 + y^2 = r^2 \implies x = R \pm \sqrt{r^2 - y^2}$

$$\frac{1}{2}V = \int_0^r \left[\pi\left(R + \sqrt{r^2 - y^2}\right)^2 - \pi\left(R - \sqrt{r^2 - y^2}\right)^2\right] dy$$

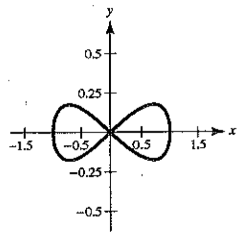
$$= \pi\int_0^r 4R\sqrt{r^2 - y^2}\,dy$$

$$= \pi(4R)\frac{1}{4}\pi r^2 = \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

4. $8y^2 = x^2(1 - x^2)$

$$y = \pm\frac{|x|\sqrt{1-x^2}}{2\sqrt{2}}$$



For $x > 0$, $y' = \dfrac{1-2x^2}{2\sqrt{2}\sqrt{1-x^2}}$

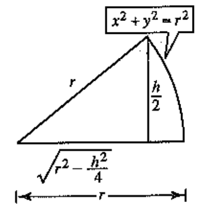
$$S = 2(2\pi)\int_0^1 x\sqrt{1+\left(\frac{1-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\right)^2}\,dx$$

$$= \frac{5\sqrt{2}\pi}{3}$$

5. $V = 2(2\pi)\displaystyle\int_{\sqrt{r^2-(h^2/4)}}^{r} x\sqrt{r^2-x^2}\,dx$

$$= -2\pi\left[\frac{2}{3}(r^2-x^2)^{3/2}\right]_{\sqrt{r^2-(h^2/4)}}^{r}$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!$$



6. By the Theorem of Pappus,

$$V = 2\pi r A$$

$$= 2\pi\left[d + \frac{1}{2}\sqrt{w^2+l^2}\right]lw$$

7. (a) Tangent at A: $y = x^3$, $y' = 3x^2$

$$y - 1 = 3(x-1)$$

$$y = 3x - 2$$

To find point B: $x^3 = 3x - 2$

$$x^3 - 3x + 2 = 0$$

$$(x-1)^2(x+2) = 0 \implies B = (-2, -8)$$

Tangent at B: $y = x^3$, $y' = 3x^2$

$$y + 8 = 12(x+2)$$

$$y = 12x + 16$$

To find point C: $x^3 = 12x + 16$

$$x^3 - 12x - 16 = 0$$

$$(x+2)^2(x-4) = 0 \implies C = (4, 64)$$

Area of $R = \displaystyle\int_{-2}^{1}(x^3 - 3x + 2)\,dx = \frac{27}{4}$

Area of $S = \displaystyle\int_{-2}^{4}(12x + 16 - x^3)\,dx = 108$

Area of S = 16(area of R) $\left[\dfrac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x-a)^2(x+2a) = 0$$

$$\implies B = (-2a, -8a^3)$$

Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x+2a)^2(x-4a) = 0$$

$$\implies C = (4a, 64a^3)$$

Area of $R = \displaystyle\int_{-2a}^{a}[x^3 - 3a^2x + 2a^3]\,dx = \frac{27}{4}a^4$

Area of $S = \displaystyle\int_{-2a}^{4a}[12a^2x + 16a^3 - x^3]\,dx = 108a^4$

Area of S = 16(area of R)

8. $f'(x)^2 = e^x$

$f'(x) = e^{x/2}$

$f(x) = 2e^{x/2} + C$

$f(0) = 0 \Rightarrow C = -2$

$f(x) = 2e^{x/2} - 2$

9. $s(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dt$

(a) $s'(x) = \dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\, dx$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2}\, dt = \int_1^x \sqrt{1 + \frac{9}{4}t}\, dt$

(d) $s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t}\, dt = \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_1^2 = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$

This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

10. Let ρ_f be the density of the fluid and ρ_0 the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^{0} A(y)\, dy$$

where $A(y)$ is a typical cross section and g is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\, dy.$$

$F = W$

$$\rho_f g \int_{-h}^{0} A(y)\, dy = \rho_0 g \int_{-h}^{L-h} A(y)\, dy$$

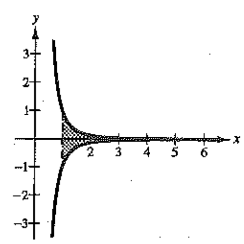
$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}} = \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

11. (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_1^6 x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right) dx = \int_1^6 \frac{2}{x^2}\, dx = \left[-2\frac{1}{x}\right]_1^6 = \frac{5}{3}$$

$$m = 2\int_1^6 \frac{1}{x^3}\, dx = \left[-\frac{1}{x^2}\right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$



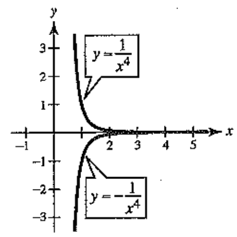
(b) $m = 2\int_1^b \frac{1}{x^3}\, dx = \dfrac{b^2 - 1}{b^2}$

$$M_y = 2\int_1^6 \frac{1}{x^2}\, dx = \frac{2(b-1)}{b}$$

$$\bar{x} = \frac{2(b-1)/b}{(b^2-1)/b^2} = \frac{2b}{b+1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b+1}, 0\right)$$

(c) $\lim\limits_{b\to\infty} \bar{x} = \lim\limits_{b\to\infty} \dfrac{2b}{b+1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$

12. (a) $\bar{y} = 0$ by symmetry



$$M_y = 2\int_1^6 x\frac{1}{x^4}\,dx = 2\int_1^6 \frac{1}{x^3}\,dx = \frac{35}{36}$$

$$m = 2\int_1^6 \frac{1}{x^4}\,dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \qquad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0\right)$$

(b) $$M_y = 2\int_1^b \frac{1}{x^3}\,dx = \frac{b^2 - 1}{b^2}$$

$$m = 2\int_1^b \frac{1}{x^4}\,dx = \frac{2(b^3 - 1)}{3b^3}$$

$$\bar{x} = \frac{(b^2 - 1)/b^2}{2(b^3 - 1)/3b^3} = \frac{3b(b + 1)}{2(b^2 + b + 1)} \qquad (\bar{x}, \bar{y}) = \left(\frac{3b(b + 1)}{2(b^2 + b + 1)}, 0\right)$$

$$\lim_{b\to\infty} \bar{x} = \frac{3}{2} \qquad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0\right)$$

13. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

14. (a) Trapezoidal: Area $\approx \frac{160}{2(8)}[0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0] = 9920$ sq ft

(b) Simpson's: Area $\approx \frac{160}{3(8)}[0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0] = 10{,}413\frac{1}{3}$ sq ft

15. Point of equilibrium: $50 - 0.5x = 0.125x$

$x = 80, p = 10$

$(P_0, x_0) = (10, 80)$

Consumer surplus $= \int_0^{80} [(50 - 0.5x) - 10]\,dx = 1600$

Producer surplus $= \int_0^{80} [10 - 0.125x]\,dx = 400$

16. Point of equilibrium: $1000 - 0.4x^2 = 42x$

$x = 20, p = 840$

$(P_0, x_0) = (840, 20)$

Consumer surplus $= \int_0^{20} [(1000 - 0.4x^2) - 840]\,dx = 2133.33$

Producer surplus $= \int_0^{20} [840 - 42x]\,dx = 8400$

17. We use Exercise 23, Section 7.7, which gives $F = wkhb$ for a rectangle plate.

Wall at shallow end

From Exercise 23: $F = 62.4(2)(4)(20) = 9984$ lb

Wall at deep end

From Exercise 23: $F = 62.4(4)(8)(20) = 39{,}936$ lb

Side wall

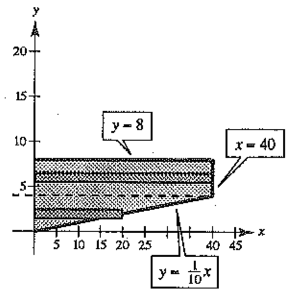
From Exercise 23: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$$F_2 = 62.4\int_0^4 (8 - y)(10y)\,dy$$

$$= 624\int_0^4 (8y - y^2)\,dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$$

$$= 26{,}624 \text{ lb}$$

Total force: $F_1 + F_2 = 46{,}592$ lb



18. (a) Answers will vary.

$f_1(x) = 6(x - x^2)$

$f_2(x) = \dfrac{\pi}{2}\sin(\pi x)$

(b) f_1 arc length ≈ 3.2490

f_2 arc length ≈ 3.3655

(c) See the article by Professor Larson Riddle at http://ecademy.agnesscott.edu/lriddle/arc/contest.htm
One such function is

$$f_3(x) = \frac{8}{\pi}\sqrt{x - x^2} \qquad \text{(arc length} \approx 2.9195)$$

CHAPTER 8
Integration Techniques, L'Hôpital's Rule, and Improper Integrals