

# CHAPTER 2

## Differentiation

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# CHAPTER 2

## Differentiation

### Section 2.1 The Derivative and the Tangent Line Problem

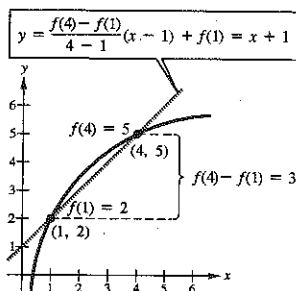
1. (a) At  $(x_1, y_1)$ , slope = 0.

At  $(x_2, y_2)$ , slope  $\approx \frac{5}{2}$ .

(b) At  $(x_1, y_1)$ , slope  $\approx -\frac{5}{2}$ .

At  $(x_2, y_2)$ , slope  $\approx 2$ .

3. (a), (b)



$$\begin{aligned} \text{(c) } y &= \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1) \\ &= \frac{3}{3}(x - 1) + 2 \\ &= 1(x - 1) + 2 \\ &= x + 1 \end{aligned}$$

5.  $f(x) = 3 - 2x$  is a line. Slope =  $-2$

$$\begin{aligned} \text{7. Slope at } (1, -3) &= \lim_{\Delta x \rightarrow 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} [2 + (\Delta x)] = 2 \end{aligned}$$

$$\begin{aligned} \text{9. Slope at } (0, 0) &= \lim_{\Delta t \rightarrow 0} \frac{f(0 + \Delta t) - f(0)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (3 - \Delta t) = 3 \end{aligned}$$

2. (a) At  $(x_1, y_1)$ , slope  $\approx \frac{2}{3}$ .

At  $(x_2, y_2)$ , slope  $\approx -\frac{2}{5}$ .

(b) At  $(x_1, y_1)$ , slope  $\approx \frac{4}{3}$ .

At  $(x_2, y_2)$ , slope  $\approx \frac{5}{4}$ .

$$\text{4. (a) } \frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$$

$$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$$

$$\text{Thus, } \frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$$

(b) The slope of the tangent line at  $(1, 2)$  equals  $f'(1)$ . This slope is steeper than the slope of the line through  $(1, 2)$  and  $(4, 5)$ . Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

6.  $g(x) = \frac{3}{2}x + 1$  is a line. Slope =  $\frac{3}{2}$

$$\begin{aligned} \text{8. Slope at } (2, 1) &= \lim_{\Delta x \rightarrow 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (-4 - \Delta x) = -4 \end{aligned}$$

$$\begin{aligned} \text{10. Slope at } (-2, 7) &= \lim_{\Delta t \rightarrow 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (-4 + \Delta t) = -4 \end{aligned}$$

11.  $f(x) = 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3 - 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

12.  $g(x) = -5$

$$\begin{aligned} g'(x) &= \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-5 - (-5)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0 \end{aligned}$$

13.  $f(x) = -5x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} -5 = -5 \end{aligned}$$

14.  $f(x) = 3x + 2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 3 = 3 \end{aligned}$$

15.  $h(s) = 3 + \frac{2}{3}s$

$$\begin{aligned} h'(s) &= \lim_{\Delta s \rightarrow 0} \frac{h(s + \Delta s) - h(s)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{3 + \frac{2}{3}(s + \Delta s) - (3 + \frac{2}{3}s)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3} \end{aligned}$$

16.  $f(x) = 9 - \frac{1}{2}x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left(-\frac{1}{2}\right) = -\frac{1}{2} \end{aligned}$$

17.  $f(x) = 2x^2 + x - 1$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (4x + 2\Delta x + 1) = 4x + 1 \end{aligned}$$

18.  $f(x) = 1 - x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-2x - \Delta x) = -2x \end{aligned}$$

19.  $f(x) = x^3 - 12x$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12
 \end{aligned}$$

20.  $f(x) = x^3 + x^2$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x
 \end{aligned}$$

21.  $f(x) = \frac{1}{x-1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x - 1} - \frac{1}{x - 1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x - 1)(x - 1)} \\
 &= -\frac{1}{(x - 1)^2}
 \end{aligned}$$

22.  $f(x) = \frac{1}{x^2}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2 x^2} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2 x^2} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2} \\
 &= \frac{-2x}{x^4} \\
 &= -\frac{2}{x^3}
 \end{aligned}$$

23.  $f(x) = \sqrt{x+1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left( \frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x [\sqrt{x + \Delta x + 1} + \sqrt{x + 1}]} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} = \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}
 \end{aligned}$$

$$24. f(x) = \frac{4}{\sqrt{x}}$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{4}{\sqrt{x + \Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x \sqrt{x} \sqrt{x + \Delta x}} \cdot \left( \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{4x - 4(x + \Delta x)}{\Delta x \sqrt{x} \sqrt{x + \Delta x} (\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-4}{\sqrt{x} \sqrt{x} \sqrt{x + \Delta x} (\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \frac{-4}{\sqrt{x} \sqrt{x} (\sqrt{x} + \sqrt{x})} = \frac{-2}{x\sqrt{x}} \end{aligned}$$

$$26. (a) f(x) = x^2 + 2x + 1$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x \Delta x + (\Delta x)^2 + 2 \Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x + 2) = 2x + 2 \end{aligned}$$

At  $(-3, 4)$ , the slope of the tangent line is  $m = 2(-3) + 2 = -4$ .  
The equation of the tangent line is

$$\begin{aligned} y - 4 &= -4(x + 3) \\ y &= -4x - 8. \end{aligned}$$

$$27. (a) f(x) = x^3$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x \Delta x + (\Delta x)^2) = 3x^2 \end{aligned}$$

At  $(2, 8)$ , the slope of the tangent is  $m = 3(2)^2 = 12$ .  
The equation of the tangent line is

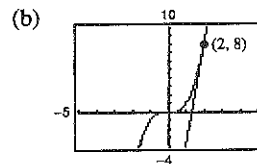
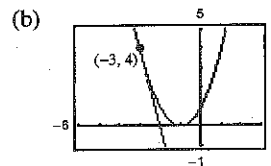
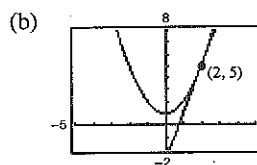
$$\begin{aligned} y - 8 &= 12(x - 2) \\ y &= 12x - 16. \end{aligned}$$

$$25. (a) f(x) = x^2 + 1$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x \Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At  $(2, 5)$ , the slope of the tangent line is  $m = 2(2) = 4$ .  
The equation of the tangent line is

$$\begin{aligned} y - 5 &= 4(x - 2) \\ y - 5 &= 4x - 8 \\ y &= 4x - 3. \end{aligned}$$



28. (a)  $f(x) = x^3 + 1$

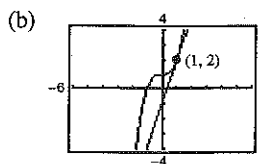
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2 \end{aligned}$$

At  $(1, 2)$ , the slope of the tangent line is  $m = 3(1)^2 = 3$ .

The equation of the tangent line is

$$y - 2 = 3(x - 1)$$

$$y = 3x - 1.$$



29. (a)  $f(x) = \sqrt{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

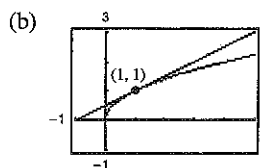
At  $(1, 1)$ , the slope of the tangent line is

$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$



30. (a)  $f(x) = \sqrt{x - 1}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left( \frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x(\sqrt{x + \Delta x - 1} + \sqrt{x - 1})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}} \end{aligned}$$

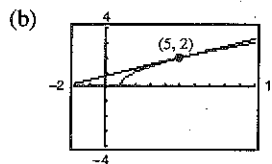
At  $(5, 2)$ , the slope of the tangent line is

$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 5)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$



31. (a)  $f(x) = x + \frac{4}{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) + \frac{4}{x + \Delta x} - \left(x + \frac{4}{x}\right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)} \\ &= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2} \end{aligned}$$

At (4, 5), the slope of the tangent line is

$$m = 1 - \frac{4}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$\begin{aligned} y - 5 &= \frac{3}{4}(x - 4) \\ y &= \frac{3}{4}x + 2. \end{aligned}$$

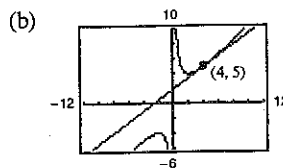
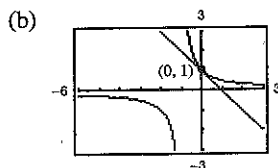
32. (a)  $f(x) = \frac{1}{x + 1}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + 1} - \frac{1}{x + 1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + 1) - (x + \Delta x + 1)}{\Delta x(x + \Delta x + 1)(x + 1)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x + 1)(x + 1)} \\ &= -\frac{1}{(x + 1)^2} \end{aligned}$$

At (0, 1), the slope of the tangent line is

$$m = \frac{-1}{(0 + 1)^2} = -1.$$

The equation of the tangent line is  $y = -x + 1$ .



33. From Exercise 27 we know that  $f'(x) = 3x^2$ . Since the slope of the given line is 3, we have

$$\begin{aligned} 3x^2 &= 3 \\ x &= \pm 1. \end{aligned}$$

Therefore, at the points (1, 1) and (-1, -1) the tangent lines are parallel to  $3x - y + 1 = 0$ . These lines have equations

$$\begin{aligned} y - 1 &= 3(x - 1) & \text{and} & & y + 1 &= 3(x + 1) \\ y &= 3x - 2 & & & y &= 3x + 2. \end{aligned}$$

34. Using the limit definition of derivative,  $f'(x) = 3x^2$ . Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points (1, 3) and (-1, 1) the tangent lines are parallel to  $3x - y - 4 = 0$ . These lines have equations

$$y - 3 = 3(x - 1) \quad \text{and} \quad y - 1 = 3(x + 1)$$

$$y = 3x \qquad \qquad y = 3x + 4.$$

35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is  $-\frac{1}{2}$ , we have

$$\begin{aligned} -\frac{1}{2x\sqrt{x}} &= -\frac{1}{2} \\ x &= 1. \end{aligned}$$

Therefore, at the point (1, 1) the tangent line is parallel to  $x + 2y - 6 = 0$ . The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

37.  $f(x) = x \Rightarrow f'(x) = 1$  Matches (b).

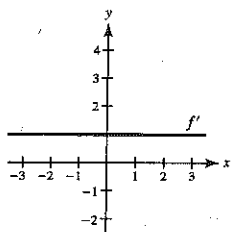
39.  $f(x) = \sqrt{x} \Rightarrow f'(x)$  Matches (a).

(decreasing slope as  $x \rightarrow \infty$ )

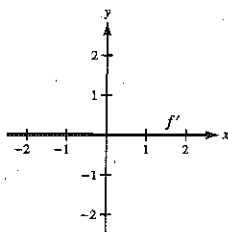
41.  $g(5) = 2$  because the tangent line passes through (5, 2).

$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

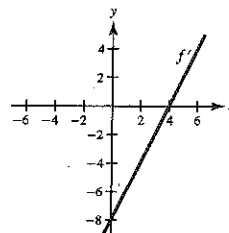
43. The slope of the graph of  $f$  is 1  $\Rightarrow f'(x) = 1$ .



44. The slope of the graph of  $f$  is 0  $\Rightarrow f'(x) = 0$ .



45. The slope of the graph of  $f$  is negative for  $x < 4$ , positive for  $x > 4$ , and 0 at  $x = 4$ .



36. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2(x-1)^{3/2}}.$$

Since the slope of the given line is  $-\frac{1}{2}$ , we have

$$\begin{aligned} \frac{-1}{2(x-1)^{3/2}} &= -\frac{1}{2} \\ 1 &= (x-1)^{3/2} \\ 1 &= x-1 \Rightarrow x = 2. \end{aligned}$$

At the point (2, 1), the tangent line is parallel to  $x + 2y + 7 = 0$ . The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2.$$

38.  $f(x) = x^2 \Rightarrow f'(x) = 2x$  Matches (d).

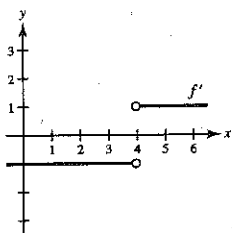
40.  $f'$  does not exist at  $x = 0$ . Matches (c).

42.  $h(-1) = 4$  because the tangent line passes through (-1, 4).

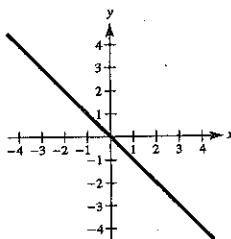
$$h'(-1) = \frac{6 - 4}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$



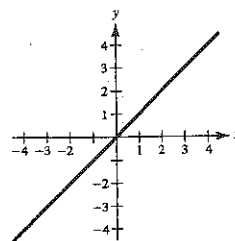
46. The slope of the graph of  $f$  is  $-1$  for  $x < 4$ ,  $1$  for  $x > 4$ , and undefined at  $x = 4$ .



47. Answers will vary.  
Sample answer:  $y = -x$

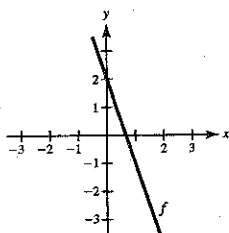


48. Answers will vary.  
Sample answer:  $y = x$

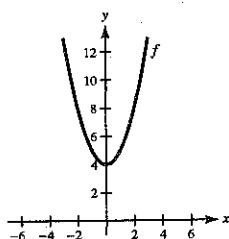


49.  $f(x) = 5 - 3x$  and  $c = 1$     50.  $f(x) = x^3$  and  $c = -2$     51.  $f(x) = -x^2$  and  $c = 6$     52.  $f(x) = 2\sqrt{x}$  and  $c = 9$

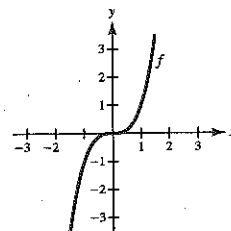
53.  $f(0) = 2$  and  $f'(x) = -3$ ,  
 $-\infty < x < \infty$   
 $f(x) = -3x + 2$



54.  $f(0) = 4$ ,  $f'(0) = 0$ ;  $f'(x) < 0$  for  
 $x < 0$ ,  $f'(x) > 0$  for  $x > 0$   
 $f(x) = x^2 + 4$



55.  $f(0) = 0$ ;  $f'(0) = 0$ ;  $f'(x) > 0$  if  
 $x \neq 0$   
 $f(x) = x^3$



56. (a) If  $f'(c) = 3$  and  $f$  is odd, then  $f'(-c) = f'(c) = 3$ .

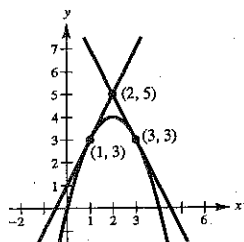
- (b) If  $f'(c) = 3$  and  $f$  is even, then  $f'(-c) = -f'(c) = -3$ .

57. Let  $(x_0, y_0)$  be a point of tangency on the graph of  $f$ . By the limit definition for the derivative,  $f'(x) = 4 - 2x$ . The slope of the line through  $(2, 5)$  and  $(x_0, y_0)$  equals the derivative of  $f$  at  $x_0$ :

$$\begin{aligned} \frac{5 - y_0}{2 - x_0} &= 4 - 2x_0 \\ 5 - y_0 &= (2 - x_0)(4 - 2x_0) \\ 5 - (4x_0 - x_0^2) &= 8 - 8x_0 + 2x_0^2 \\ 0 &= x_0^2 - 4x_0 + 3 \\ 0 &= (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3 \end{aligned}$$

Therefore, the points of tangency are  $(1, 3)$  and  $(3, 3)$ , and the corresponding slopes are  $2$  and  $-2$ . The equations of the tangent lines are:

$$\begin{aligned} y - 5 &= 2(x - 2) & y - 5 &= -2(x - 2) \\ y &= 2x + 1 & y &= -2x + 9 \end{aligned}$$

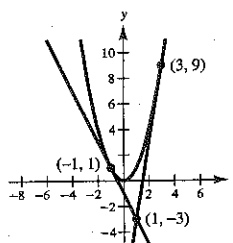


58. Let  $(x_0, y_0)$  be a point of tangency on the graph of  $f$ . By the limit definition for the derivative,  $f'(x) = 2x$ . The slope of the line through  $(1, -3)$  and  $(x_0, y_0)$  equals the derivative of  $f$  at  $x_0$ :

$$\begin{aligned} \frac{-3 - y_0}{1 - x_0} &= 2x_0 \\ -3 - y_0 &= (1 - x_0)2x_0 \\ -3 - x_0^2 &= 2x_0 - 2x_0^2 \\ x_0^2 - 2x_0 - 3 &= 0 \\ (x_0 - 3)(x_0 + 1) &= 0 \Rightarrow x_0 = 3, -1 \end{aligned}$$

Therefore, the points of tangency are  $(3, 9)$  and  $(-1, 1)$ , and the corresponding slopes are  $6$  and  $-2$ . The equations of the tangent lines are:

$$\begin{aligned} y + 3 &= 6(x - 1) & y + 3 &= -2(x - 1) \\ y &= 6x - 9 & y &= -2x - 1 \end{aligned}$$



59. (a)  $g'(0) = -3$   
 (b)  $g'(3) = 0$   
 (c) Because  $g'(1) = -\frac{8}{3}$ ,  $g$  is decreasing (falling) at  $x = 1$ .  
 (d) Because  $g'(-4) = \frac{7}{3}$ ,  $g$  is increasing (rising) at  $x = -4$ .  
 (e) Because  $g'(4)$  and  $g'(6)$  are both positive,  $g(6)$  is greater than  $g(4)$ , and  $g(6) - g(4) > 0$ .  
 (f) No, it is not possible. All you can say is that  $g$  is decreasing (falling) at  $x = 2$ .

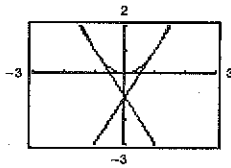
60. (a)  $f(x) = x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At  $x = -1$ ,  $f'(-1) = -2$  and the tangent line is  $y - 1 = -2(x + 1)$  or  $y = -2x - 1$ .

At  $x = 0$ ,  $f'(0) = 0$  and the tangent line is  $y = 0$ .

At  $x = 1$ ,  $f'(1) = 2$  and the tangent line is  $y = 2x - 1$ .



For this function, the slopes of the tangent lines are always distinct for different values of  $x$ .

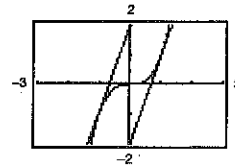
(b)  $g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$

$$\begin{aligned} &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2 \end{aligned}$$

At  $x = -1$ ,  $g'(-1) = 3$  and the tangent line is  $y + 1 = 3(x + 1)$  or  $y = 3x + 2$ .

At  $x = 0$ ,  $g'(0) = 0$  and the tangent line is  $y = 0$ .

At  $x = 1$ ,  $g'(1) = 3$  and the tangent line is  $y - 1 = 3(x - 1)$  or  $y = 3x - 2$ .

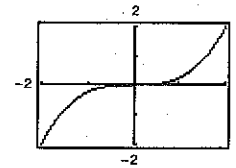


For this function, the slopes of the tangent lines are sometimes the same.

61.  $f(x) = \frac{1}{4}x^3$

By the limit definition of the derivative we have  $f'(x) = \frac{3}{4}x^2$ .

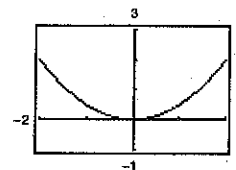
$x$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3



62.  $f(x) = \frac{1}{2}x^2$

By the limit definition of the derivative we have  $f'(x) = x$ .

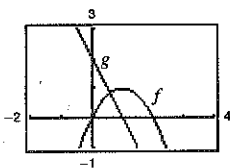
$x$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2



$$63. g(x) = \frac{f(x+0.01) - f(x)}{0.01}$$

$$= [2(x+0.01) - (x+0.01)^2 - 2x + x^2]100$$

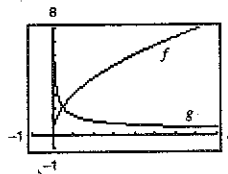
$$= 2 - 2x - 0.01$$



The graph of  $g(x)$  is approximately the graph of  $f'(x) = 2 - 2x$ .

$$64. g(x) = \frac{f(x+0.01) - f(x)}{0.01}$$

$$= (3\sqrt{x+0.01} - 3\sqrt{x})100$$



The graph of  $g(x)$  is approximately the graph of  $f'(x) = \frac{3}{2\sqrt{x}}$ .

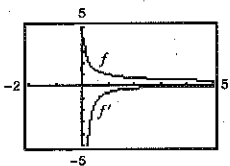
$$65. f(2) = 2(4-2) = 4, f(2.1) = 2.1(4-2.1) = 3.99$$

$$f'(2) \approx \frac{3.99-4}{2.1-2} = -0.1 \quad [\text{Exact: } f'(2) = 0]$$

$$66. f(2) = \frac{1}{4}(2^3) = 2, f(2.1) = 2.31525$$

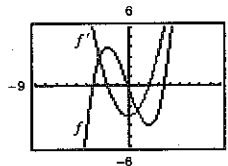
$$f'(2) \approx \frac{2.31525-2}{2.1-2} = 3.1525 \quad [\text{Exact: } f'(2) = 3]$$

$$67. f(x) = \frac{1}{\sqrt{x}} \text{ and } f'(x) = \frac{-1}{2x^{3/2}}$$



As  $x \rightarrow \infty$ ,  $f$  is nearly horizontal and thus  $f' \approx 0$ .

$$68. f(x) = \frac{x^3}{4} - 3x \text{ and } f'(x) = \frac{3}{4}x^2 - 3$$



$$69. f(x) = 4 - (x-3)^2$$

$$S_{\Delta x}(x) = \frac{f(2+\Delta x) - f(2)}{\Delta x}(x-2) + f(2)$$

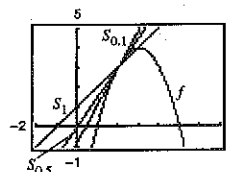
$$= \frac{4 - (2+\Delta x-3)^2 - 3}{\Delta x}(x-2) + 3 = \frac{1 - (\Delta x-1)^2}{\Delta x}(x-2) + 3 = (-\Delta x+2)(x-2) + 3$$

$$(a) \Delta x = 1: S_{\Delta x} = (x-2) + 3 = x+1$$

$$\Delta x = 0.5: S_{\Delta x} = \left(\frac{3}{2}\right)(x-2) + 3 = \frac{3}{2}x$$

$$\Delta x = 0.1: S_{\Delta x} = \left(\frac{19}{10}\right)(x-2) + 3 = \frac{19}{10}x - \frac{4}{5}$$

(b) As  $\Delta x \rightarrow 0$ , the line approaches the tangent line to  $f$  at  $(2, 3)$ .



$$70. f(x) = x + \frac{1}{x}$$

$$S_{\Delta x}(x) = \frac{f(2+\Delta x) - f(2)}{\Delta x}(x-2) + f(2) = \frac{(2+\Delta x) + \frac{1}{2+\Delta x} - \frac{5}{2}}{\Delta x}(x-2) + \frac{5}{2}$$

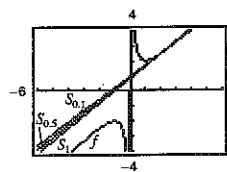
$$= \frac{2(2+\Delta x)^2 + 2 - 5(2+\Delta x)}{2(2+\Delta x)\Delta x}(x-2) + \frac{5}{2} = \frac{(2\Delta x+3)}{2(2+\Delta x)}(x-2) + \frac{5}{2}$$

$$(a) \Delta x = 1: S_{\Delta x} = \frac{5}{6}(x-2) + \frac{5}{2} = \frac{5}{6}x + \frac{5}{6}$$

$$\Delta x = 0.5: S_{\Delta x} = \frac{4}{5}(x-2) + \frac{5}{2} = \frac{4}{5}x + \frac{9}{10}$$

$$\Delta x = 0.1: S_{\Delta x} = \frac{16}{21}(x-2) + \frac{5}{2} = \frac{16}{21}x + \frac{41}{42}$$

(b) As  $\Delta x \rightarrow 0$ , the line approaches the tangent line to  $f$  at  $(2, \frac{5}{2})$ .



71.  $f(x) = x^2 - 1, c = 2$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^2 - 1) - 3}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4$$

72.  $g(x) = x(x - 1) = x^2 - x, c = 1$

$$g'(1) = \lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - x - 0}{x - 1} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{x - 1} = \lim_{x \rightarrow 1} x = 1$$

73.  $f(x) = x^3 + 2x^2 + 1, c = -2$

$$f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2} \frac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x \rightarrow -2} \frac{x^2(x + 2)}{x + 2} = \lim_{x \rightarrow -2} x^2 = 4$$

74.  $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 3)}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 3) = 5$$

75.  $g(x) = \sqrt{|x|}, c = 0$

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sqrt{|x|}}{x}. \text{ Does not exist.}$$

$$\text{As } x \rightarrow 0^-, \frac{\sqrt{|x|}}{x} = \frac{-1}{\sqrt{|x|}} \rightarrow -\infty.$$

$$\text{As } x \rightarrow 0^+, \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} \rightarrow \infty.$$

76.  $f(x) = \frac{1}{x}, c = 3$

$$f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x \rightarrow 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x \rightarrow 3} \left( -\frac{1}{3x} \right) = -\frac{1}{9}$$

77.  $f(x) = (x - 6)^{2/3}, c = 6$

$$f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6} = \lim_{x \rightarrow 6} \frac{(x - 6)^{2/3} - 0}{x - 6} = \lim_{x \rightarrow 6} \frac{1}{(x - 6)^{1/3}}$$

Does not exist.

78.  $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x \rightarrow -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x \rightarrow -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

79.  $h(x) = |x + 5|, c = -5$

$$h'(-5) = \lim_{x \rightarrow -5} \frac{h(x) - h(-5)}{x - (-5)} = \lim_{x \rightarrow -5} \frac{|x + 5| - 0}{x + 5} = \lim_{x \rightarrow -5} \frac{|x + 5|}{x + 5}$$

Does not exist.

80.  $f(x) = |x - 4|, c = 4$

$$f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

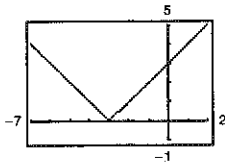
81.  $f(x)$  is differentiable everywhere except at  $x = -1$ .  
(Discontinuity)

82.  $f(x)$  is differentiable everywhere except at  $x = \pm 3$ .  
(Sharp turns in the graph)

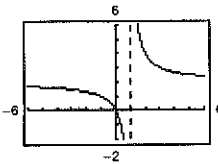
84.  $f(x)$  is differentiable everywhere except at  $x = \pm 2$ .  
(Discontinuities)

86.  $f(x)$  is differentiable everywhere except at  $x = 0$ . (Discontinuity)

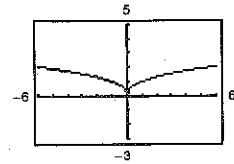
87.  $f(x) = |x + 3|$  is differentiable for all  $x \neq -3$ . There is a sharp corner at  $x = -3$ .



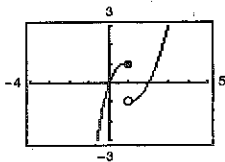
88.  $f(x) = \frac{2x}{x-1}$  is differentiable for all  $x \neq 1$ .  $f$  is not defined at  $x = 1$ . (Vertical asymptote)



89.  $f(x) = x^{2/5}$  is differentiable for all  $x \neq 0$ . There is a sharp corner at  $x = 0$ .



90.  $f$  is differentiable for all  $x \neq 1$ .  
 $f$  is not continuous at  $x = 1$ .



91.  $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore,  $f$  is not differentiable at  $x = 1$ .

92.  $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x \rightarrow 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \quad (\text{Vertical tangent})$$

The limit from the right does not exist since  $f$  is undefined for  $x > 1$ . Therefore,  $f$  is not differentiable at  $x = 1$ .

93.  $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{(x - 1)^3 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^-} (x - 1)^2 = 0. \end{aligned}$$

The derivative from the right is

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{(x - 1)^2 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^+} (x - 1) = 0. \end{aligned}$$

These one-sided limits are equal. Therefore,  $f$  is differentiable at  $x = 1$ . ( $f'(1) = 0$ )

$$94. f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore,  $f$  is not differentiable at  $x = 1$ .

$$95. \text{ Note that } f \text{ is continuous at } x = 2. f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$$

$$\text{The derivative from the left is } \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \rightarrow 2^-} (x + 2) = 4.$$

$$\text{The derivative from the right is } \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \rightarrow 2^+} 4 = 4.$$

The one-sided limits are equal. Therefore,  $f$  is differentiable at  $x = 2$ . ( $f'(2) = 4$ )

$$96. \text{ Note that } f \text{ is continuous at } x = 2. f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$$

The derivative from the left is

$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(\frac{1}{2}x + 1) - 2}{x - 2} = \lim_{x \rightarrow 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

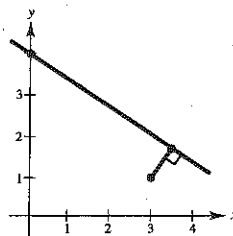
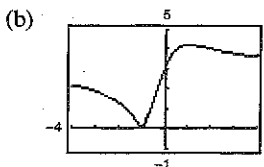
The derivative from the right is

$$\begin{aligned} \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2} \\ &= \lim_{x \rightarrow 2^+} \frac{2x - 4}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2(x - 2)}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}. \end{aligned}$$

The one-sided limits are equal. Therefore,  $f$  is differentiable at  $x = 2$ . ( $f'(2) = \frac{1}{2}$ )

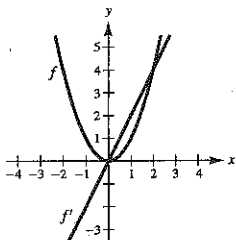
97. (a) The distance from  $(3, 1)$  to the line  $mx - y + 4 = 0$  is

$$\begin{aligned} d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\ &= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}} \end{aligned}$$

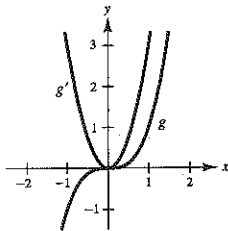


The function  $d$  is not differentiable at  $m = -1$ . This corresponds to the line  $y = -x + 4$ , which passes through the point  $(3, 1)$ .

98. (a)  $f(x) = x^2$  and  $f'(x) = 2x$



(b)  $g(x) = x^3$  and  $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If  $h(x) = x^n$ , then  $h'(x) = nx^{n-1}$ .

(d) If  $f(x) = x^4$ , then

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3. \end{aligned}$$

Hence, if  $f(x) = x^4$ , then  $f'(x) = 4x^3$  which is consistent with the conjecture. However, this is not a proof since you must verify the conjecture for all integer values of  $n$ ,  $n \geq 2$ .

99. False. The slope is  $\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$ .

100. False.  $y = |x - 2|$  is continuous at  $x = 2$ , but is not differentiable at  $x = 2$ . (Sharp turn in the graph)

101. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if  $f(x) = |x|$ , then the derivative from the left at  $x = 0$  is  $-1$  and the derivative from the right at  $x = 0$  is  $1$ . At  $x = 0$ , the derivative does not exist.

102. True—see Theorem 2.1.

103.  $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, we have  $-|x| \leq x \sin(1/x) \leq |x|$ ,  $x \neq 0$ . Thus,  $\lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$  and  $f$  is continuous at  $x = 0$ . Using the alternative form of the derivative, we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} \left( \sin \frac{1}{x} \right).$$

Since this limit does not exist ( $\sin(1/x)$  oscillates between  $-1$  and  $1$ ), the function is not differentiable at  $x = 0$ .

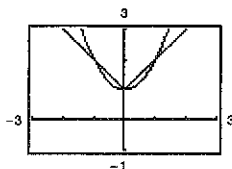
$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again, we have  $-x^2 \leq x^2 \sin(1/x) \leq x^2$ ,  $x \neq 0$ . Thus,  $\lim_{x \rightarrow 0} x^2 \sin(1/x) = 0 = g(0)$  and  $g$  is continuous at  $x = 0$ . Using the alternative form of the derivative again, we have

$$\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0.$$

Therefore,  $g$  is differentiable at  $x = 0$ ,  $g'(0) = 0$ .

104.



As you zoom in, the graph of  $y_1 = x^2 + 1$  appears to be locally the graph of a horizontal line, whereas the graph of  $y_2 = |x| + 1$  always has a sharp corner at  $(0, 1)$ .  $y_2$  is not differentiable at  $(0, 1)$ .

## Section 2.2 Basic Differentiation Rules and Rates of Change

- |   |   |   |  |
|---|---|---|--|
| 1. (a) $y = x^{1/2}$<br>$y' = \frac{1}{2}x^{-1/2}$<br>$y'(1) = \frac{1}{2}$                         | (b) $y = x^3$<br>$y' = 3x^2$<br>$y'(1) = 3$                         | 2. (a) $y = x^{-1/2}$<br>$y' = -\frac{1}{2}x^{-3/2}$<br>$y'(1) = -\frac{1}{2}$  | (b) $y = x^{-1}$<br>$y' = -x^{-2}$<br>$y'(1) = -1$                                 |
| 3. $y = 8$<br>$y' = 0$  | 4. $f(x) = -2$<br>$f'(x) = 0$                                       | 5. $y = x^6$<br>$y' = 6x^5$   | 6. $y = x^8$<br>$y' = 8x^7$  |
| 7. $y = \frac{1}{x^7} = x^{-7}$<br>$y' = -7x^{-8} = \frac{-7}{x^8}$                                 | 8. $y = \frac{1}{x^8} = x^{-8}$<br>$y' = -8x^{-9} = \frac{-8}{x^9}$ | 9. $y = \sqrt[5]{x} = x^{1/5}$<br>$y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$   | 10. $y = \sqrt[4]{x} = x^{1/4}$<br>$y' = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$ |
| 11. $f(x) = x + 1$<br>$f'(x) = 1$   | 12. $g(x) = 3x - 1$<br>$g'(x) = 3$                                  | 13. $f(t) = -2t^2 + 3t - 6$<br>$f'(t) = -4t + 3$  | 14. $y = t^2 + 2t - 3$<br>$y' = 2t + 2$  |
| 15. $g(x) = x^2 + 4x^3$<br>$g'(x) = 2x + 12x^2$   | 16. $y = 8 - x^3$<br>$y' = -3x^2$                                   | 17. $s(t) = t^3 - 2t + 4$<br>$s'(t) = 3t^2 - 2$   | 18. $f(x) = 2x^3 - x^2 + 3x$<br>$f'(x) = 6x^2 - 2x + 3$                            |
| 19. $y = \frac{\pi}{2} \sin \theta - \cos \theta$<br>$y' = \frac{\pi}{2} \cos \theta + \sin \theta$ | 20. $g(t) = \pi \cos t$<br>$g'(t) = -\pi \sin t$                    | 21. $y = x^2 - \frac{1}{2} \cos x$<br>$y' = 2x + \frac{1}{2} \sin x$  | 22. $y = 5 + \sin x$<br>$y' = \cos x$  |
| 23. $y = \frac{1}{x} - 3 \sin x$<br>$y' = -\frac{1}{x^2} - 3 \cos x$                                |   | 24. $y = \frac{5}{(2x)^3} + 2 \cos x = \frac{5}{8}x^{-3} + 2 \cos x$<br>$y' = \frac{5}{8}(-3)x^{-4} - 2 \sin x = \frac{-15}{8x^4} - 2 \sin x$ |  |

<u>Function</u>	<u>Rewrite</u>	<u>Differentiate</u>	<u>Simplify</u>
25. $y = \frac{5}{2x^2}$	$y = \frac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = \frac{-5}{x^3}$
26. $y = \frac{2}{3x^2}$	$y = \frac{2}{3}x^{-2}$	$y' = -\frac{4}{3}x^{-3}$	$y' = -\frac{4}{3x^3}$
27. $y = \frac{3}{(2x)^3}$	$y = \frac{3}{8}x^{-3}$	$y' = \frac{-9}{8}x^{-4}$	$y' = \frac{-9}{8x^4}$



<u>Function</u>	<u>Rewrite</u>	<u>Differentiate</u>	<u>Simplify</u>
28. $y = \frac{\pi}{(3x)^2}$	$y = \frac{\pi}{9}x^{-2}$	$y' = -\frac{2\pi}{9}x^{-3}$	$y' = -\frac{2\pi}{9x^3}$
29. $y = \frac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\frac{1}{2}x^{-3/2}$	$y' = -\frac{1}{2x^{3/2}}$
30. $y = \frac{4}{x^{-3}}$	$y = 4x^3$	$y' = 12x^2$	$y' = 12x^2$
31. $f(x) = \frac{3}{x^2} = 3x^{-2}, (1, 3)$ $f'(x) = -6x^{-3} = -\frac{6}{x^3}$ $f'(1) = -6$	32. $f(t) = 3 - \frac{3}{5t}, \left(\frac{3}{5}, 2\right)$ $f'(t) = \frac{3}{5t^2}$ $f'\left(\frac{3}{5}\right) = \frac{5}{3}$	33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)$ $f'(x) = \frac{21}{5}x^2$ $f'(0) = 0$	
34. $y = 3x^3 - 6, (2, 18)$ $y' = 9x^2$ $y'(2) = 36$	35. $y = (2x + 1)^2, (0, 1)$ $= 4x^2 + 4x + 1$ $y' = 8x + 4$ $y'(0) = 4$	36. $f(x) = 3(5 - x)^2, (5, 0)$ $= 3x^2 - 30x + 75$ $f'(x) = 6x - 30$ $f'(5) = 0$	
37. $f(\theta) = 4 \sin \theta - \theta, (0, 0)$ $f'(\theta) = 4 \cos \theta - 1$ $f'(0) = 4(1) - 1 = 3$	38. $g(t) = 2 + 3 \cos t, (\pi, -1)$ $g'(t) = -3 \sin t$ $g'(\pi) = 0$	39. $f(x) = x^2 + 5 - 3x^{-2}$ $f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}$	
40. $f(x) = x^2 - 3x - 3x^{-2}$ $f'(x) = 2x - 3 + 6x^{-3}$ $= 2x - 3 + \frac{6}{x^3}$	41. $g(t) = t^2 - \frac{4}{t^3} = t^2 - 4t^{-3}$ $g'(t) = 2t + 12t^{-4} = 2t + \frac{12}{t^4}$	42. $f(x) = x + x^{-2}$ $f'(x) = 1 - 2x^{-3}$ $= 1 - \frac{2}{x^3}$	
43. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$ $f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}$	44. $h(x) = \frac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$ $h'(x) = 2 - \frac{1}{x^2} = \frac{2x^2 - 1}{x^2}$	45. $y = x(x^2 + 1) = x^3 + x$ $y' = 3x^2 + 1$	
46. $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$ $y' = 36x - 45x^2$	47. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$ $f'(x) = \frac{1}{2}x^{-1/2} - 2x^{-2/3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^{2/3}}$	48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$ $f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5} = \frac{1}{3x^{2/3}} + \frac{1}{5x^{4/5}}$	
49. $h(s) = s^{4/5} - s^{2/3}$ $h'(s) = \frac{4}{5}s^{-1/5} - \frac{2}{3}s^{-1/3} = \frac{4}{5s^{1/5}} - \frac{2}{3s^{1/3}}$	50. $f(t) = t^{2/3} - t^{1/3} + 4$ $f'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{3}t^{-2/3} = \frac{2}{3t^{1/3}} - \frac{1}{3t^{2/3}}$		

51.  $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$

$$f'(x) = 3x^{-1/2} - 5 \sin x = \frac{3}{\sqrt{x}} - 5 \sin x$$

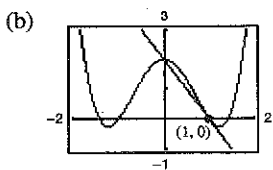
53. (a)  $y = x^4 - 3x^2 + 2$

$$y' = 4x^3 - 6x$$

At (1, 0):  $y' = 4(1)^3 - 6(1) = -2$

Tangent line:  $y - 0 = -2(x - 1)$

$$2x + y - 2 = 0$$



52.  $f(x) = \frac{2}{\sqrt[3]{x}} + 3 \cos x = 2x^{-1/3} + 3 \cos x$

$$f'(x) = \frac{-2}{3}x^{-4/3} - 3 \sin x = \frac{-2}{3x^{4/3}} - 3 \sin x$$

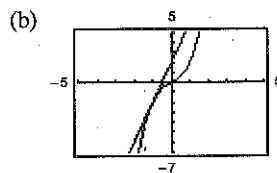
54. (a)  $y = x^3 + x$

$$y' = 3x^2 + 1$$

At (-1, -2):  $y' = 3(-1)^2 + 1 = 4$

Tangent line:  $y + 2 = 4(x + 1)$

$$4x - y + 2 = 0$$



55. (a)  $f(x) = \frac{2}{\sqrt[3]{x^3}} = 2x^{-3/4}$

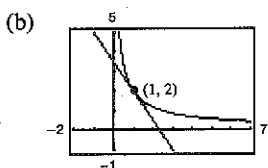
$$f'(x) = \frac{-3}{2}x^{-7/4} = \frac{-3}{2x^{7/4}}$$

At (1, 2):  $f'(1) = \frac{-3}{2}$

Tangent line:  $y - 2 = -\frac{3}{2}(x - 1)$

$$y = -\frac{3}{2}x + \frac{7}{2}$$

$$3x + 2y - 7 = 0$$



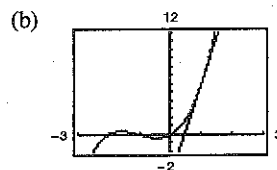
56. (a)  $y = (x^2 + 2x)(x + 1)$   
 $= x^3 + 3x^2 + 2x$

$$y' = 3x^2 + 6x + 2$$

At (1, 6):  $y' = 3(1)^2 + 6(1) + 2 = 11$

Tangent line:  $y - 6 = 11(x - 1)$

$$0 = 11x - y - 5$$



57.  $y = x^4 - 8x^2 + 2$

$$y' = 4x^3 - 16x$$

$$= 4x(x^2 - 4)$$

$$= 4x(x - 2)(x + 2)$$

$$y' = 0 \Rightarrow x = 0, \pm 2$$

Horizontal tangents: (0, 2), (2, -14), (-2, -14)

59.  $y = \frac{1}{x^2} = x^{-2}$

$$y' = -2x^{-3} = \frac{-2}{x^3} \text{ cannot equal zero.}$$

Therefore, there are no horizontal tangents.

58.  $y = x^3 + x$

$$y' = 3x^2 + 1 > 0 \text{ for all } x.$$

Therefore, there are no horizontal tangents.

60.  $y = x^2 + 1$

$$y' = 2x = 0 \Rightarrow x = 0$$

At  $x = 0, y = 1$ .

Horizontal tangent: (0, 1)

61.  $y = x + \sin x, 0 \leq x < 2\pi$

$$y' = 1 + \cos x = 0$$

$$\cos x = -1 \Rightarrow x = \pi$$

$$\text{At } x = \pi: y = \pi$$

Horizontal tangent:  $(\pi, \pi)$

62.  $y = \sqrt{3}x + 2 \cos x, 0 \leq x < 2\pi$

$$y' = \sqrt{3} - 2 \sin x = 0$$

$$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$$\text{At } x = \frac{\pi}{3}: y = \frac{\sqrt{3}\pi + 3}{3}$$

$$\text{At } x = \frac{2\pi}{3}: y = \frac{2\sqrt{3}\pi - 3}{3}$$

Horizontal tangents:  $\left(\frac{\pi}{3}, \frac{\sqrt{3}\pi + 3}{3}\right), \left(\frac{2\pi}{3}, \frac{2\sqrt{3}\pi - 3}{3}\right)$

63.  $x^2 - kx = 4x - 9$  Equate functions.

$$2x - k = 4 \quad \text{Equate derivatives.}$$

Hence,  $k = 2x - 4$  and

$$x^2 - (2x - 4)x = 4x - 9 \Rightarrow -x^2 = -9 \Rightarrow x = \pm 3.$$

For  $x = 3, k = 2$  and for  $x = -3, k = -10$ .

64.  $k - x^2 = -4x + 7$  Equate functions.

$$-2x = -4 \quad \text{Equate derivatives.}$$

Hence,  $x = 2$  and  $k - 4 = -8 + 7 \Rightarrow k = 3$ .

65.  $\frac{k}{x} = -\frac{3}{4}x + 3$  Equate functions.

$$-\frac{k}{x^2} = -\frac{3}{4} \quad \text{Equate derivatives.}$$

Hence,  $k = \frac{3}{4}x^2$  and

$$\frac{\frac{3}{4}x^2}{x} = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{4}x = -\frac{3}{4}x + 3$$

$$\Rightarrow \frac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3.$$

66.  $k\sqrt{x} = x + 4$  Equate functions.

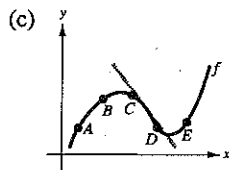
$$\frac{k}{2\sqrt{x}} = 1 \quad \text{Equate derivatives.}$$

Hence,  $k = 2\sqrt{x}$  and

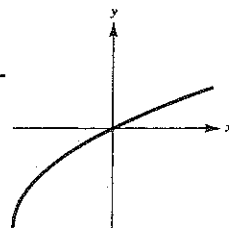
$$(2\sqrt{x})\sqrt{x} = x + 4 \Rightarrow 2x = x + 4 \Rightarrow x = 4 \Rightarrow k = 4.$$

67. (a) The slope appears to be steepest between A and B.

(b) The average rate of change between A and B is greater than the instantaneous rate of change at B.



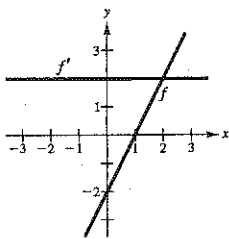
68. The graph of a function  $f$  such that  $f' > 0$  for all  $x$  and the rate of change the function is decreasing (i.e.,  $f'' < 0$ ) would, in general, look like the graph at the right.



69.  $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

70.  $g(x) = -5f(x) \Rightarrow g'(x) = -5f'(x)$

71.

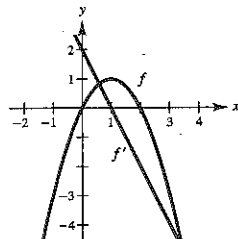


If  $f$  is linear then its derivative is a constant function.

$$f(x) = ax + b$$

$$f'(x) = a$$

72.



If  $f$  is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

73. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the points of tangency on  $y = x^2$  and  $y = -x^2 + 6x - 5$ , respectively. The derivatives of these functions are:

$$y' = 2x \Rightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Rightarrow m = -2x_2 + 6$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since  $y_1 = x_1^2$  and  $y_2 = -x_2^2 + 6x_2 - 5$ :

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (x_1^2)}{x_2 - x_1} = -2x_2 + 6$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$

$$x_2 = 1 \Rightarrow y_2 = 0, \quad x_1 = 2 \text{ and } y_1 = 4$$

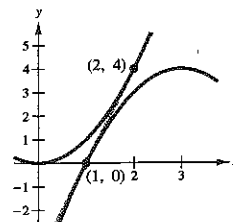
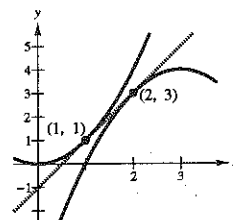
Thus, the tangent line through  $(1, 0)$  and  $(2, 4)$  is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$

$$x_2 = 2 \Rightarrow y_2 = 3, \quad x_1 = 1 \text{ and } y_1 = 1$$

Thus, the tangent line through  $(2, 3)$  and  $(1, 1)$  is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$



74.  $m_1$  is the slope of the line tangent to  $y = x$ .  $m_2$  is the slope of the line tangent to  $y = 1/x$ . Since

$$y = x \Rightarrow y' = 1 \Rightarrow m_1 = 1 \quad \text{and} \quad y = \frac{1}{x} \Rightarrow y' = \frac{-1}{x^2} \Rightarrow m_2 = \frac{-1}{x^2}.$$

The points of intersection of  $y = x$  and  $y = 1/x$  are

$$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

At  $x = \pm 1$ ,  $m_2 = -1$ . Since  $m_2 = -1/m_1$ , these tangent lines are perpendicular at the points of intersection.

75.  $f(x) = 3x + \sin x + 2$

$$f'(x) = 3 + \cos x$$

Since  $|\cos x| \leq 1$ ,  $f'(x) \neq 0$  for all  $x$  and  $f$  does not have a horizontal tangent line.

77.  $f(x) = \sqrt{x}$ ,  $(-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{xy}$$

$$4 + x = 2\sqrt{x}\sqrt{y}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

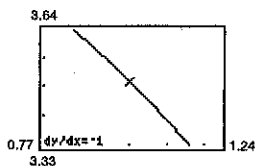
The point  $(4, 2)$  is on the graph of  $f$ .

Tangent line: 
$$y - 2 = \frac{0 - 2}{-4 - 4}(x - 4)$$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

79.  $f'(1) = -1$



81. (a) One possible secant is between  $(3.9, 7.7019)$  and  $(4, 8)$ :

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

(b)  $f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$

$$T(x) = 3(x - 4) + 8 = 3x - 4$$

$S(x)$  is an approximation of the tangent line  $T(x)$ .

76.  $f(x) = x^5 + 3x^3 + 5x$

$$f'(x) = 5x^4 + 9x^2 + 5$$

Since  $5x^4 + 9x^2 \geq 0$ ,  $f'(x) \geq 5$ . Thus,  $f$  does not have a tangent line with a slope of 3.

78.  $f(x) = \frac{2}{x}$ ,  $(5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

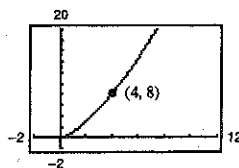
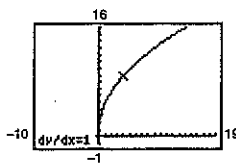
The point  $(\frac{5}{2}, \frac{4}{5})$  is on the graph of  $f$ . The slope of the tangent line is  $f'(\frac{5}{2}) = -\frac{8}{25}$ .

Tangent line: 
$$y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$$

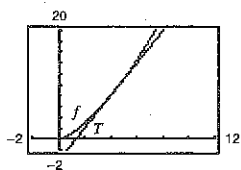
$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

80.  $f'(4) = 1$



## 81. —CONTINUED—

 (c) As you move further away from (4, 8), the accuracy of the approximation  $T$  gets worse.


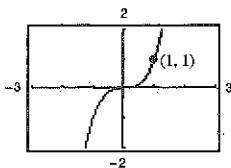
$\Delta x$	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

82. (a) Nearby point: (1.0073138, 1.0221024)

$$\text{Secant line: } y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$$

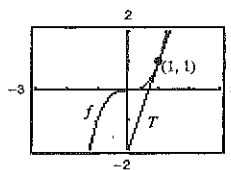
$$y = 3.022(x - 1) + 1$$

(Answers will vary.)


 (b)  $f'(x) = 3x^2$ 

$$T(x) = 3(x - 1) + 1 = 3x - 2$$

(c) The accuracy worsens as you move away from (1, 1).



$\Delta x$	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	-8	-1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	-8	-5	-2	-0.5	0.7	1	1.3	2.5	4	7	10

 The accuracy decreases more rapidly than in Exercise 81 because  $y = x^3$  is less "linear" than  $y = x^{3/2}$ .

 83. False. Let  $f(x) = x^2$  and  $g(x) = x^2 + 4$ . Then  $f'(x) = g'(x) = 2x$ , but  $f(x) \neq g(x)$ .

 84. True. If  $f(x) = g(x) + c$ , then  $f'(x) = g'(x) + 0 = g'(x)$ .

 85. False. If  $y = \pi^2$ , then  $dy/dx = 0$ . ( $\pi^2$  is a constant.)

 86. True. If  $y = x/\pi = (1/\pi) \cdot x$ , then  $dy/dx = (1/\pi)(1) = 1/\pi$ .

 87. True. If  $g(x) = 3f(x)$ , then  $g'(x) = 3f'(x)$ .

 88. False. If  $f(x) = \frac{1}{x^n} = x^{-n}$ , then  $f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}$ .

 89.  $f(t) = 2t + 7, [1, 2]$ 

$$f'(t) = 2$$

Instantaneous rate of change is the constant 2.

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

 (These are the same because  $f$  is a line of slope 2.)

 90.  $f(t) = t^2 - 3, [2, 2.1]$ 

$$f'(t) = 2t$$

Instantaneous rate of change:

$$(2, 1) \Rightarrow f'(2) = 2(2) = 4$$

$$(2.1, 1.41) \Rightarrow f'(2.1) = 4.2$$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

91.  $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \Rightarrow f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

93. (a)  $s(t) = -16t^2 + 1362$

$$v(t) = -32t$$

(b)  $\frac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48 \text{ ft/sec}$

(c)  $v(t) = s'(t) = -32t$

When  $t = 1$ :  $v(1) = -32 \text{ ft/sec}$

When  $t = 2$ :  $v(2) = -64 \text{ ft/sec}$

(d)  $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) 
$$v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right) = -8\sqrt{1362} \approx -295.242 \text{ ft/sec}$$

95.  $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + 120t$$

$$v(t) = -9.8t + 120$$

$$v(5) = -9.8(5) + 120 = 71 \text{ m/sec}$$

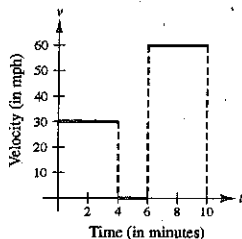
$$v(10) = -9.8(10) + 120 = 22 \text{ m/sec}$$

97. From  $(0, 0)$  to  $(4, 2)$ ,  $s(t) = \frac{1}{2}t \Rightarrow v(t) = \frac{1}{2} \text{ mi/min.}$

$$v(t) = \frac{1}{2}(60) = 30 \text{ mph for } 0 < t < 4$$

Similarly,  $v(t) = 0$  for  $4 < t < 6$ . Finally, from  $(6, 2)$  to  $(10, 6)$ ,

$$s(t) = t - 4 \Rightarrow v(t) = 1 \text{ mi/in} = 60 \text{ mph.}$$



92.  $f(x) = \sin x, \left[0, \frac{\pi}{6}\right]$

$$f'(x) = \cos x$$

Instantaneous rate of change:

$$(0, 0) \Rightarrow f'(0) = 1$$

$$\left(\frac{\pi}{6}, \frac{1}{2}\right) \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

94.

$$s(t) = -16t^2 - 22t + 220$$

$$v(t) = -32t - 22$$

$$v(3) = -118 \text{ ft/sec}$$

$$s(t) = -16t^2 - 22t + 220$$

$$= 112 \text{ (height after falling 108 ft)}$$

$$-16t^2 - 22t + 108 = 0$$

$$-2(t - 2)(8t + 27) = 0$$

$$t = 2$$

$$v(2) = -32(2) - 22$$

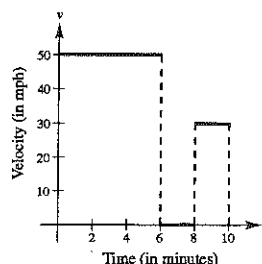
$$= -86 \text{ ft/sec}$$

96.  $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + s_0 = 0 \text{ when } t = 6.8.$$

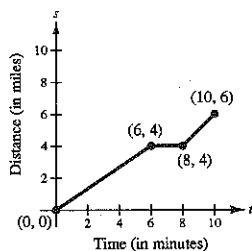
$$s_0 = 4.9t^2 = 4.9(6.8)^2 \approx 226.6 \text{ m}$$

98.

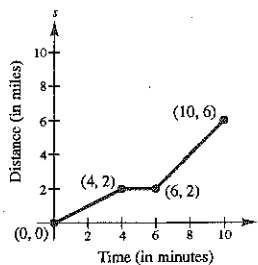


(The velocity has been converted to miles per hour.)

99.  $v = 40 \text{ mph} = \frac{2}{3} \text{ mi/min}$   
 $(\frac{2}{3} \text{ mi/min})(6 \text{ min}) = 4 \text{ mi}$   
 $v = 0 \text{ mph} = 0 \text{ mi/min}$   
 $(0 \text{ mi/min})(2 \text{ min}) = 0 \text{ mi}$   
 $v = 60 \text{ mph} = 1 \text{ mi/min}$   
 $(1 \text{ mi/min})(2 \text{ min}) = 2 \text{ mi}$



100. This graph corresponds with Exercise 97.



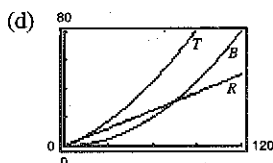
101. (a) Using a graphing utility,

$$R = 0.417v - 0.02.$$

- (b) Using a graphing utility,

$$B = 0.00557v^2 + 0.0014v + 0.04.$$

- (c)  $T = R + B = 0.00557v^2 + 0.418v + 0.02$



102.  $C = (\text{gallons of fuel used})(\text{cost per gallon})$

$$= \left(\frac{15,000}{x}\right)(1.55) = \frac{23,250}{x}$$

$$\frac{dC}{dx} = -\frac{23,250}{x^2}$$

$x$	10	15	20	25	30	35	40
$C$	2325	1550	1163	930	775	664	581
$\frac{dC}{dx}$	-233	-103	-58	-37	-26	-19	-15

The driver who gets 15 miles per gallon would benefit more. The rate of change at  $x = 15$  is larger in absolute value than that at  $x = 35$ .

103.  $V = s^3, \frac{dV}{ds} = 3s^2$

When  $s = 4 \text{ cm}$ ,  $\frac{dV}{ds} = 48 \text{ cm}^2$  per cm change in  $s$ .

104.  $A = s^2, \frac{dA}{ds} = 2s$

When  $s = 4 \text{ m}$ ,

$$\frac{dA}{ds} = 8 \text{ square meters per meter change in } s.$$

- (e)  $\frac{dT}{dv} = 0.01114v + 0.418$

For  $v = 40$ ,  $T'(40) \approx 0.86$ .

For  $v = 80$ ,  $T'(80) \approx 1.31$ .

For  $v = 100$ ,  $T'(100) \approx 1.53$ .

- (f) For increasing speeds, the total stopping distance increases.



105.  $s(t) = -\frac{1}{2}at^2 + c$  and  $s'(t) = -at$

$$\begin{aligned} \text{Average velocity: } \frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} &= \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c]}{2\Delta t} \\ &= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t} \\ &= \frac{-2at_0\Delta t}{2\Delta t} \\ &= -at_0 \\ &= s'(t_0) \quad \text{instantaneous velocity at } t = t_0 \end{aligned}$$

106.  $C = \frac{1,008,000}{Q} + 6.3Q$

$$\frac{dC}{dQ} = -\frac{1,008,000}{Q^2} + 6.3$$

$$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$$

$$\text{When } Q = 350, \frac{dC}{dQ} \approx -\$1.93.$$

107.  $N = f(p)$

(a)  $f'(1.479)$  is the rate of change of gallons of gasoline sold when the price is \$1.479 per gallon.

(b)  $f'(1.479)$  is usually negative. As prices go up, sales go down.

108.  $\frac{dT}{dt} = K(T - T_a)$

109.  $y = ax^2 + bx + c$

Since the parabola passes through  $(0, 1)$  and  $(1, 0)$ , we have:

$$(0, 1): 1 = a(0)^2 + b(0) + c \Rightarrow c = 1$$

$$(1, 0): 0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1$$

Thus,  $y = ax^2 + (-a - 1)x + 1$ . From the tangent line  $y = x - 1$ , we know that the derivative is 1 at the point  $(1, 0)$ .

$$y' = 2ax + (-a - 1)$$

$$1 = 2a(1) + (-a - 1)$$

$$1 = a - 1$$

$$a = 2$$

$$b = -a - 1 = -3$$

Therefore,  $y = 2x^2 - 3x + 1$ .

110.  $y = \frac{1}{x}, x > 0$

$$y' = -\frac{1}{x^2}$$

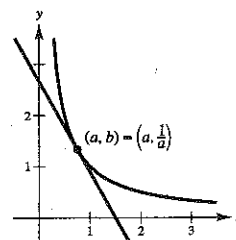
At  $(a, b)$ , the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}$$

The  $x$ -intercept is  $(2a, 0)$ .

The  $y$ -intercept is  $(0, \frac{2}{a})$ .

The area of the triangle is  $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$ .



111.  $y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through  $(1, -9)$ :

$$y + 9 = (3x^2 - 9)(x - 1)$$

$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$

$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

The points of tangency are  $(0, 0)$  and  $(\frac{3}{2}, -\frac{81}{8})$ . At  $(0, 0)$ , the slope is  $y'(0) = -9$ . At  $(\frac{3}{2}, -\frac{81}{8})$ , the slope is  $y'(\frac{3}{2}) = -\frac{9}{4}$ .

Tangent lines:

$$y - 0 = -9(x - 0) \quad \text{and} \quad y + \frac{81}{8} = -\frac{9}{4}(x - \frac{3}{2})$$

$$y = -9x \quad \quad \quad y = -\frac{9}{4}x - \frac{27}{4}$$

$$9x + y = 0$$

$$9x + 4y + 27 = 0$$

112.  $y = x^2$

$y' = 2x$

(a) Tangent lines through  $(0, a)$ :

$$y - a = 2x(x - 0)$$

$$x^2 - a = 2x^2$$

$$-a = x^2$$

$$\pm\sqrt{-a} = x$$

The points of tangency are  $(\pm\sqrt{-a}, -a)$ . At  $(\sqrt{-a}, -a)$ , the slope is  $y'(\sqrt{-a}) = 2\sqrt{-a}$ . At  $(-\sqrt{-a}, -a)$ , the slope is  $y'(-\sqrt{-a}) = -2\sqrt{-a}$ .Tangent lines:  $y + a = 2\sqrt{-a}(x - \sqrt{-a})$  and  $y + a = -2\sqrt{-a}(x + \sqrt{-a})$ 

$$y = 2\sqrt{-a}x + a$$

$$y = -2\sqrt{-a}x + a$$

**Restriction:**  $a$  must be negative.(b) Tangent lines through  $(a, 0)$ :

$$y - 0 = 2x(x - a)$$

$$x^2 = 2x^2 - 2ax$$

$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are  $(0, 0)$  and  $(2a, 4a^2)$ . At  $(0, 0)$ , the slope is  $y'(0) = 0$ . At  $(2a, 4a^2)$ , the slope is  $y'(2a) = 4a$ .Tangent lines:  $y - 0 = 0(x - 0)$  and  $y - 4a^2 = 4a(x - 2a)$ 

$$y = 0$$

$$y = 4ax - 4a^2$$

**Restriction:** None,  $a$  can be any real number.

$$113. f(x) = \begin{cases} ax^3, & x \leq 2 \\ x^2 + b, & x > 2 \end{cases}$$

$f$  must be continuous at  $x = 2$  to be differentiable at  $x = 2$ .

$$\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} ax^3 = 8a \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 + b) = 4 + b \end{aligned} \right\} \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For  $f$  to be differentiable at  $x = 2$ , the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

$$b = 8a - 4 = -\frac{4}{3}$$

$$114. f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \geq 0 \end{cases}$$

$$f(0) = b = \cos(0) = 1 \Rightarrow b = 1$$

$$f'(x) = \begin{cases} -\sin x, & x < 0 \\ a, & x > 0 \end{cases}$$

Hence,  $a = 0$ .

Answer:  $a = 0, b = 1$

115.  $f_1(x) = |\sin x|$  is differentiable for all  $x \neq n\pi, n$  an integer.

$f_2(x) = \sin|x|$  is differentiable for all  $x \neq 0$ .

You can verify this by graphing  $f_1$  and  $f_2$  and observing the locations of the sharp turns.

116. Let  $f(x) = \cos x$ .

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \sin x \left( \frac{\sin \Delta x}{\Delta x} \right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

## Section 2.3 Product and Quotient Rules and Higher-Order Derivatives

$$1. g(x) = (x^2 + 1)(x^2 - 2x)$$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

$$2. f(x) = (6x + 5)(x^3 - 2)$$

$$\begin{aligned} f'(x) &= (6x + 5)(3x^2) + (x^3 - 2)(6) \\ &= 18x^3 + 15x^2 + 6x^3 - 12 \\ &= 24x^3 + 15x^2 - 12 \end{aligned}$$

$$3. h(t) = \sqrt[3]{t^2 + 4} = t^{1/3}(t^2 + 4)$$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4) \frac{1}{3} t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

$$4. g(s) = \sqrt{s(4 - s^2)} = s^{1/2}(4 - s^2)$$

$$\begin{aligned} g'(s) &= s^{1/2}(-2s) + (4 - s^2) \frac{1}{2} s^{-1/2} \\ &= -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}} \\ &= \frac{4 - 5s^2}{2s^{1/2}} \end{aligned}$$

5.  $f(x) = x^3 \cos x$

$$f'(x) = x^3(-\sin x) + \cos x(3x^2)$$

$$= 3x^2 \cos x - x^3 \sin x$$

7.  $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

9.  $h(x) = \frac{\sqrt[3]{x}}{x^3 + 1} = \frac{x^{1/3}}{x^3 + 1}$

$$h'(x) = \frac{(x^3 + 1)^{1/3} x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2}$$

$$= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2}$$

$$= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2}$$

11.  $g(x) = \frac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

13.  $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$f'(x) = (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3)$$

$$= 10x^4 + 12x^3 - 3x^2 - 18x - 15$$

$$f'(0) = -15$$

15.  $f(x) = \frac{x^2 - 4}{x - 3}$

$$f'(x) = \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2}$$

$$= \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2}$$

$$= \frac{x^2 - 6x + 4}{(x - 3)^2}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

17.  $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

6.  $g(x) = \sqrt{x} \sin x$

$$g'(x) = \sqrt{x} \cos x + \sin x \left(\frac{1}{2\sqrt{x}}\right) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$$

8.  $g(t) = \frac{t^2 + 2}{2t - 7}$

$$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$$

10.  $h(s) = \frac{s}{\sqrt{s} - 1}$

$$h'(s) = \frac{(\sqrt{s} - 1)(1) - s(\frac{1}{2}s^{-1/2})}{(\sqrt{s} - 1)^2}$$

$$= \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{(\sqrt{s} - 1)^2} = \frac{\sqrt{s} - 2}{2(\sqrt{s} - 1)^2}$$

12.  $f(t) = \frac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t \sin t + 3 \cos t}{t^4}$$

14.  $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$f'(x) = (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2)$$

$$= 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1)$$

$$= (x - 1)^2(5x^2 + 2x + 2)$$

$$f'(1) = 0$$

16.  $f(x) = \frac{x + 1}{x - 1}$

$$f'(x) = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}$$

$$= \frac{x - 1 - x - 1}{(x - 1)^2}$$

$$= -\frac{2}{(x - 1)^2}$$

$$f'(2) = -\frac{2}{(2 - 1)^2} = -2$$

$$\begin{aligned}
 18. \quad f(x) &= \frac{\sin x}{x} \\
 f'(x) &= \frac{(x)(\cos x) - (\sin x)(1)}{x^2} \\
 &= \frac{x \cos x - \sin x}{x^2} \\
 f'\left(\frac{\pi}{6}\right) &= \frac{(\pi/6)(\sqrt{3}/2) - (1/2)}{\pi^2/36} \\
 &= \frac{3\sqrt{3}\pi - 18}{\pi^2} \\
 &= \frac{3(\sqrt{3}\pi - 6)}{\pi^2}
 \end{aligned}$$

<u>Function</u>	<u>Rewrite</u>	<u>Differentiate</u>	<u>Simplify</u>
19. $y = \frac{x^2 + 2x}{3}$	$y = \frac{1}{3}x^2 + \frac{2}{3}x$	$y' = \frac{2}{3}x + \frac{2}{3}$	$y' = \frac{2x + 2}{3}$
20. $y = \frac{5x^2 - 3}{4}$	$y = \frac{5}{4}x^2 - \frac{3}{4}$	$y' = \frac{10}{4}x$	$y' = \frac{5x}{2}$
21. $y = \frac{7}{3x^3}$	$y = \frac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\frac{7}{x^4}$
22. $y = \frac{4}{5x^2}$	$y = \frac{4}{5}x^{-2}$	$y' = -\frac{8}{5}x^{-3}$	$y' = -\frac{8}{5x^3}$
23. $y = \frac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, \quad x > 0$	$y' = 2x^{-1/2}$	$y' = \frac{2}{\sqrt{x}}$
24. $y = \frac{3x^2 - 5}{7}$	$y = \frac{3}{7}x^2 - \frac{5}{7}$	$y' = \frac{6x}{7}$	$y' = \frac{6}{7}x$

$$\begin{aligned}
 25. \quad f(x) &= \frac{3 - 2x - x^2}{x^2 - 1} \\
 f'(x) &= \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2} \\
 &= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2} \\
 &= \frac{2}{(x + 1)^2}, \quad x \neq 1
 \end{aligned}$$

$$\begin{aligned}
 26. \quad f(x) &= \frac{x^3 + 3x + 2}{x^2 - 1} \\
 f'(x) &= \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2} \\
 &= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 27. \quad f(x) &= x\left(1 - \frac{4}{x + 3}\right) = x - \frac{4x}{x + 3} \\
 f'(x) &= 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} \\
 &= \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2} \\
 &= \frac{x^2 + 6x - 3}{(x + 3)^2}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad f(x) &= x^4\left[1 - \frac{2}{x + 1}\right] = x^4\left[\frac{x - 1}{x + 1}\right] \\
 f'(x) &= x^4\left[\frac{(x + 1) - (x - 1)}{(x + 1)^2}\right] + \left[\frac{x - 1}{x + 1}\right](4x^3) \\
 &= x^4\left[\frac{2}{(x + 1)^2}\right] + \left[\frac{x^2 - 1}{(x + 1)^2}\right](4x^3) \\
 &= 2x^3\left[\frac{2x^2 + x - 2}{(x + 1)^2}\right]
 \end{aligned}$$

$$29. f(x) = \frac{2x+5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$$

$$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2} \left[ x - \frac{5}{2} \right] = \frac{2x-5}{2x\sqrt{x}} = \frac{2x-5}{2x^{3/2}}$$

$$30. f(x) = \sqrt[3]{x}(\sqrt{x} + 3) = x^{1/3}(x^{1/2} + 3)$$

Alternate solution:

$$f'(x) = x^{1/3} \left( \frac{1}{2}x^{-1/2} \right) + (x^{1/2} + 3) \left( \frac{1}{3}x^{-2/3} \right)$$

$$f(x) = \sqrt[3]{x}(\sqrt{x} + 3) \\ = x^{5/6} + 3x^{1/3}$$

$$= \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

$$31. h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$$

$$32. h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$$

$$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$$

$$h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

$$33. f(x) = \frac{2 - (1/x)}{x - 3} = \frac{2x - 1}{x(x - 3)} = \frac{2x - 1}{x^2 - 3x}$$

$$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$$

$$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$$

$$34. g(x) = x^2 \left( \frac{2}{x} - \frac{1}{x+1} \right) = 2x - \frac{x^2}{x+1}$$

$$g'(x) = 2 - \frac{(x+1)2x - x^2(1)}{(x+1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x+1)^2} = \frac{x^2 + 2x + 2}{(x+1)^2}$$

$$35. f(x) = (3x^3 + 4x)(x - 5)(x + 1)$$

$$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1) \\ = (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x \\ = 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x \\ = 15x^4 - 48x^3 - 33x^2 - 32x - 20$$

$$36. f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$$

$$f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1) \\ = (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1) \\ = 2x^5 + x^4 + 3x^3 + x^2 + 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x \\ = 6x^5 + 4x^3 - 3x^2 - 1$$

37.  $f(x) = \frac{x^2 + c^2}{x^2 - c^2}$

$$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$$

$$= \frac{-4xc^2}{(x^2 - c^2)^2}$$

39.  $f(t) = t^2 \sin t$

$$f'(t) = t^2 \cos t + 2t \sin t$$

$$= t(t \cos t + 2 \sin t)$$

41.  $f(t) = \frac{\cos t}{t}$

$$f'(t) = \frac{-t \sin t - \cos t}{t^2} = -\frac{t \sin t + \cos t}{t^2}$$

43.  $f(x) = -x + \tan x$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

45.  $g(t) = \sqrt[3]{t} + 8 \sec t = t^{1/3} + 8 \sec t$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8 \sec t \tan t = \frac{1}{4t^{3/4}} + 8 \sec t \tan t$$

47.  $y = \frac{3(1 - \sin x)}{2 \cos x} = \frac{3}{2}(\sec x - \tan x)$

$$y' = \frac{3}{2}(\sec x \tan x - \sec^2 x) = \frac{3}{2} \sec x (\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x \tan x - \tan^2 x - 1)$$

49.  $y = -\csc x - \sin x$

$$y' = \csc x \cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x \cot^2 x$$

51.  $f(x) = x^2 \tan x$

$$f'(x) = x^2 \sec^2 x + 2x \tan x$$

$$= x(x \sec^2 x + 2 \tan x)$$

53.  $y = 2x \sin x + x^2 \cos x$

$$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$$

$$= 4x \cos x + 2 \sin x - x^2 \sin x$$

38.  $f(x) = \frac{c^2 - x^2}{c^2 + x^2}$

$$f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$$

$$= \frac{-4xc^2}{(c^2 + x^2)^2}$$

40.  $f(\theta) = (\theta + 1) \cos \theta$

$$f'(\theta) = (\theta + 1)(-\sin \theta) + (\cos \theta)(1)$$

$$= \cos \theta - (\theta + 1) \sin \theta$$

42.  $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

44.  $y = x + \cot x$

$$y' = 1 - \csc^2 x = -\cot^2 x$$

46.  $h(s) = \frac{1}{s} - 10 \csc s$

$$h'(s) = -\frac{1}{s^2} + 10 \csc s \cot s$$

48.  $y = \frac{\sec x}{x}$

$$y' = \frac{x \sec x \tan x - \sec x}{x^2}$$

$$= \frac{\sec x(x \tan x - 1)}{x^2}$$

50.  $y = x \sin x + \cos x$

$$y' = x \cos x + \sin x - \sin x = x \cos x$$

52.  $f(x) = \sin x \cos x$

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos 2x$$

54.  $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

$$h'(\theta) = 5\theta \sec \theta \tan \theta + 5 \sec \theta + \theta \sec^2 \theta + \tan \theta$$

55.  $g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2} \quad (\text{Form of answer may vary.})$$

57.  $g(\theta) = \frac{\theta}{1 - \sin \theta}$

$$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2} \quad (\text{Form of answer may vary.})$$

59.  $y = \frac{1 + \csc x}{1 - \csc x}$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1-2)^2} = -4\sqrt{3}$$

60.  $f(x) = \tan x \cot x = 1$

$f'(x) = 0$

$f'(1) = 0$

62.  $f(x) = \sin x(\sin x + \cos x)$

$$\begin{aligned} f'(x) &= \sin x(\cos x - \sin x) + (\sin x + \cos x) \cos x \\ &= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x \\ &= \sin 2x + \cos 2x \end{aligned}$$

$$f'\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1$$

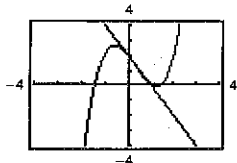
64. (a)  $f(x) = (x-1)(x^2-2)$ ,  $(0, 2)$

$f'(x) = (x-1)(2x) + (x^2-2)(1) = 3x^2 - 2x - 2$

$f'(0) = -2$ ; Slope at  $(0, 2)$

Tangent line:  $y - 2 = -2x \Rightarrow y = -2x + 2$

(b)



56.  $f(x) = \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$$f'(x) = 2 \frac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2} \quad (\text{Form of answer may vary.})$$

58.  $f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$

$$f'(\theta) = \frac{1}{\cos \theta - 1} = \frac{\cos \theta - 1}{(1 - \cos \theta)^2}$$

(Form of answer may vary.)

61.  $h(t) = \frac{\sec t}{t}$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2} = \frac{\sec t(t \tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

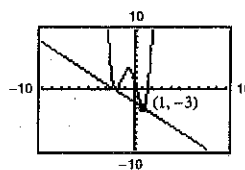
63. (a)  $f(x) = (x^3 - 3x + 1)(x + 2)$ ,  $(1, -3)$

$$\begin{aligned} f'(x) &= (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3) \\ &= 4x^3 + 6x^2 - 6x - 5 \end{aligned}$$

$f'(1) = -1$ ; Slope at  $(1, -3)$

Tangent line:  $y + 3 = -1(x - 1) \Rightarrow y = -x - 2$

(b)



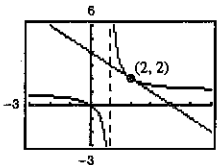
65. (a)  $f(x) = \frac{x}{x-1}$ ,  $(2, 2)$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$f'(2) = \frac{-1}{(2-1)^2} = -1$ ; Slope at  $(2, 2)$

Tangent line:  $y - 2 = -1(x - 2) \Rightarrow y = -x + 4$

(b)



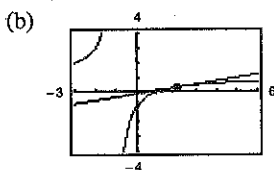


66. (a)  $f(x) = \frac{x-1}{x+1}, \left(2, \frac{1}{3}\right)$

$$f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$f'(2) = \frac{2}{9}; \text{ Slope at } \left(2, \frac{1}{3}\right)$$

$$\text{Tangent line: } y - \frac{1}{3} = \frac{2}{9}(x-2) \Rightarrow y = \frac{2}{9}x - \frac{1}{9}$$



68. (a)  $f(x) = \sec x, \left(\frac{\pi}{3}, 2\right)$

$$f'(x) = \sec x \tan x$$

$$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3}; \text{ Slope at } \left(\frac{\pi}{3}, 2\right)$$

$$\text{Tangent line: } y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$

69.  $f(x) = \frac{8}{x^2+4}, (2, 1)$

$$f'(x) = \frac{(x^2+4)(0) - 8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$$

$$f'(2) = \frac{-16(2)}{(4+4)^2} = -\frac{1}{2}$$

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2$$

$$2y + x - 4 = 0$$

71.  $f(x) = \frac{16x}{x^2+16}, \left(-2, -\frac{8}{5}\right)$

$$f'(x) = \frac{(x^2+16)(16) - 16x(2x)}{(x^2+16)^2} = \frac{256-16x^2}{(x^2+16)^2}$$

$$f'(-2) = \frac{256-16(4)}{20^2} = \frac{12}{25}$$

$$y + \frac{8}{5} = \frac{12}{25}(x+2)$$

$$y = \frac{12}{25}x - \frac{16}{25}$$

$$25y - 12x + 16 = 0$$

67. (a)  $f(x) = \tan x, \left(\frac{\pi}{4}, 1\right)$

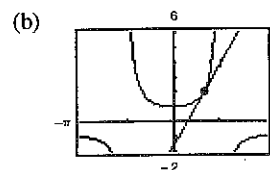
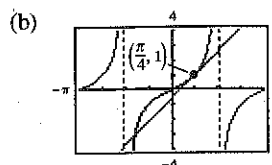
$$f'(x) = \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2; \text{ Slope at } \left(\frac{\pi}{4}, 1\right)$$

$$\text{Tangent line: } y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$



70.  $f(x) = \frac{27}{x^2+9}, \left(-3, \frac{3}{2}\right)$

$$f'(x) = \frac{(x^2+9)(0) - 27(2x)}{(x^2+9)^2} = \frac{-54x}{(x^2+9)^2}$$

$$f'(-3) = \frac{-54(-3)}{(9+9)^2} = \frac{1}{2}$$

$$y - \frac{3}{2} = \frac{1}{2}(x+3)$$

$$y = \frac{1}{2}x + 3$$

$$2y - x - 6 = 0$$

72.  $f(x) = \frac{4x}{x^2+6}, \left(2, \frac{4}{5}\right)$

$$f'(x) = \frac{(x^2+6)(4) - 4x(2x)}{(x^2+6)^2} = \frac{24-4x^2}{(x^2+6)^2}$$

$$f'(2) = \frac{24-16}{10^2} = \frac{2}{25}$$

$$y - \frac{4}{5} = \frac{2}{25}(x-2)$$

$$y = \frac{2}{25}x + \frac{16}{25}$$

$$25y - 2x - 16 = 0$$

$$73. f(x) = \frac{x^2}{x-1}$$

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$f'(x) = 0 \text{ when } x = 0 \text{ or } x = 2.$$

Horizontal tangents are at (0, 0) and (2, 4).

$$74. f(x) = \frac{x^2}{x^2+1}$$

$$f'(x) = \frac{(x^2+1)(2x) - (x^2)(2x)}{(x^2+1)^2}$$

$$= \frac{2x}{(x^2+1)^2}$$

$$f'(x) = 0 \text{ when } x = 0.$$

Horizontal tangent is at (0, 0).

$$75. f(x) = \frac{4x-2}{x^2}$$

$$f'(x) = \frac{x^2(4) - (4x-2)(2x)}{x^4}$$

$$= \frac{4x^2 - 8x^2 + 4x}{x^4}$$

$$= \frac{4-4x}{x^3}$$

$$f'(x) = 0 \text{ for } 4-4x = 0 \Rightarrow x = 1.$$

$$f(1) = 2$$

$f$  has a horizontal tangent at (1, 2).

$$76. f(x) = \frac{x-4}{x^2-7}$$

$$f'(x) = \frac{(x^2-7)(1) - (x-4)(2x)}{(x^2-7)^2} = \frac{x^2-7-2x^2+8x}{(x^2-7)^2} = \frac{-x^2+8x-7}{(x^2-7)^2} = \frac{-(x-7)(x-1)}{(x^2-7)^2}$$

$$f'(x) = 0 \text{ for } x = 1, 7; f(1) = \frac{1}{2}, f(7) = \frac{1}{14}$$

$f$  has horizontal tangents at  $(1, \frac{1}{2})$  and  $(7, \frac{1}{14})$ .

$$77. f(x) = \frac{x+1}{x-1}$$

$$f'(x) = \frac{(x-1) - (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$2y + x = 6 \Rightarrow y = -\frac{1}{2}x + 3; \text{ Slope: } -\frac{1}{2}$$

$$\frac{-2}{(x-1)^2} = -\frac{1}{2}$$

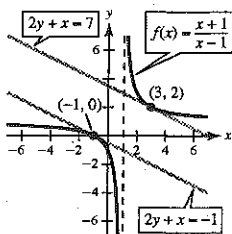
$$(x-1)^2 = 4$$

$$x-1 = \pm 2$$

$$x = -1, 3; f(-1) = 0, f(3) = 2$$

$$y - 0 = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}$$

$$y - 2 = -\frac{1}{2}(x - 3) \Rightarrow y = -\frac{1}{2}x + \frac{7}{2}$$



$$78. f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

Let  $(x, y) = (x, x/(x-1))$  be a point of tangency on the graph of  $f$ .

$$\frac{5 - (x/(x-1))}{-1-x} = \frac{-1}{(x-1)^2}$$

$$\frac{4x-5}{(x-1)(x+1)} = \frac{1}{(x-1)^2}$$

$$(4x-5)(x-1) = x+1$$

$$4x^2 - 10x + 4 = 0$$

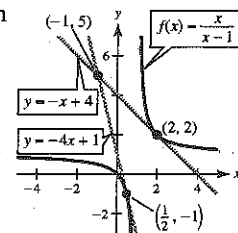
$$(x-2)(2x-1) = 0 \Rightarrow x = \frac{1}{2}, 2$$

$$f\left(\frac{1}{2}\right) = -1, f(2) = 2; f'\left(\frac{1}{2}\right) = -4, f'(2) = -1$$

Two tangent lines:

$$y + 1 = -4\left(x - \frac{1}{2}\right) \Rightarrow y = -4x + 1$$

$$y - 2 = -1(x - 2) \Rightarrow y = -x + 4$$



$$79. f'(x) = \frac{(x+2)3 - 3x(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$$

$$g'(x) = \frac{(x+2)5 - (5x+4)(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$$

$$g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$$

$f$  and  $g$  differ by a constant.

$$81. (a) p'(x) = f'(x)g(x) + f(x)g'(x)$$

$$p'(1) = f'(1)g(1) + f(1)g'(1) = 1(4) + 6\left(-\frac{1}{2}\right) = 1$$

$$(b) q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$q'(4) = \frac{3(-1) - 7(0)}{3^2} = -\frac{1}{3}$$

$$83. \text{Area} = A(t) = (2t+1)\sqrt{t} = 2t^{3/2} + t^{1/2}$$

$$A'(t) = 2\left(\frac{3}{2}t^{1/2}\right) + \frac{1}{2}t^{-1/2}$$

$$= 3t^{1/2} + \frac{1}{2}t^{-1/2}$$

$$= \frac{6t+1}{2\sqrt{t}} \text{ cm}^2/\text{sec}$$

$$85. C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad 1 \leq x$$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

(a) When  $x = 10$ :  $\frac{dC}{dx} = -\$38.13$

(b) When  $x = 15$ :  $\frac{dC}{dx} = -\$10.37$

(c) When  $x = 20$ :  $\frac{dC}{dx} = -\$3.80$

As the order size increases, the cost per item decreases.

$$87. P(t) = 500\left[1 + \frac{4t}{50+t^2}\right]$$

$$P'(t) = 500\left[\frac{(50+t^2)(4) - (4t)(2t)}{(50+t^2)^2}\right]$$

$$= 500\left[\frac{200-4t^2}{(50+t^2)^2}\right]$$

$$= 2000\left[\frac{50-t^2}{(50+t^2)^2}\right]$$

$P'(2) \approx 31.55$  bacteria per hour

$$80. f'(x) = \frac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$$

$f$  and  $g$  differ by a constant.

$$82. (a) p'(x) = f'(x)g(x) + f(x)g'(x)$$

$$p'(4) = \frac{1}{2}(8) + 1(0) = 4$$

$$(b) q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$q'(7) = \frac{4(2) - 4(-1)}{4^2} = \frac{12}{16} = \frac{3}{4}$$

$$84. V = \pi r^2 h = \pi(t+2)\left(\frac{1}{2}\sqrt{t}\right)$$

$$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$$

$$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t+2}{4t^{1/2}}\pi \text{ cubic inches/sec}$$

$$86. P = \frac{k}{V}$$

$$\frac{dP}{dV} = -\frac{k}{V^2}$$

$$88. F = \frac{Gm_1m_2}{d^2} = Gm_1m_2d^{-2}$$

$$\frac{dF}{dd} = F'(d) = \frac{-2Gm_1m_2}{d^3}$$

89. (a)  $\sec x = \frac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(b)  $\csc x = \frac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

(c)  $\cot x = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

90.  $f(x) = \sec x$

$g(x) = \csc x, [0, 2\pi)$

$f'(x) = g'(x)$

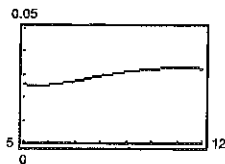
$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = -1 \Rightarrow \frac{\sin^2 x}{\cos^2 x} = -1 \Rightarrow \tan^2 x = -1 \Rightarrow \tan x = -1$$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

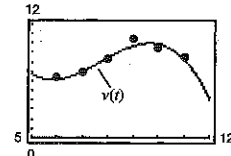
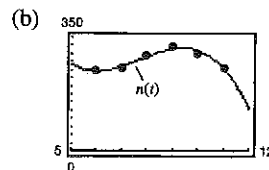
91. (a)  $n(t) = -3.5806t^3 + 82.577t^2 - 603.60t + 1667.5$

$v(t) = -0.1361t^3 + 3.165t^2 - 23.02t + 59.8$

(c)  $A = \frac{v(t)}{n(t)} = \frac{-0.1361t^3 + 3.165t^2 - 23.02t + 59.8}{-3.5806t^3 + 82.577t^2 - 603.60t + 1667.5}$



A represents the average retail value (in millions of dollars) per 1000 motor homes.



(d)  $A'(t)$  represents the rate of change of the average retail value per 1000 motor homes.

92. (a)  $\sin \theta = \frac{r}{r+h}$

$r+h = r \csc \theta$

$h = r \csc \theta - r = r(\csc \theta - 1)$

(b)  $h'(\theta) = r(-\csc \theta \cdot \cot \theta)$

$h'(30^\circ) = h'\left(\frac{\pi}{6}\right)$

$= -3960(2 \cdot \sqrt{3}) = -7920\sqrt{3} \text{ mi/radian}$

93.  $f(x) = 4x^{3/2}$

$f'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$

94.  $f(x) = x + \frac{32}{x^2}$

$f'(x) = 1 - \frac{64}{x^3}$

$f''(x) = \frac{192}{x^4}$

95.  $f(x) = \frac{x}{x-1}$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

97.  $f(x) = 3 \sin x$

$$f'(x) = 3 \cos x$$

$$f''(x) = -3 \sin x$$

99.  $f'(x) = x^2$

$$f''(x) = 2x$$

100.  $f''(x) = 2 - 2x^{-1}$

$$f'''(x) = 2x^{-2} = \frac{2}{x^2}$$

96.  $f(x) = \frac{x^2 + 2x - 1}{x} = x + 2 - \frac{1}{x}$

$$f'(x) = 1 + \frac{1}{x^2}$$

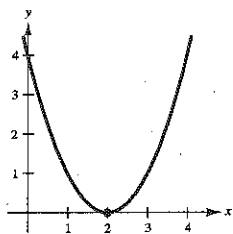
$$f''(x) = -\frac{2}{x^3}$$

98.  $f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

$$\begin{aligned} f''(x) &= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\ &= \sec x(\sec^2 x + \tan^2 x) \end{aligned}$$

103.



$$f(2) = 0$$

One such function is  $f(x) = (x-2)^2$ .

105.  $f(x) = 2g(x) + h(x)$

$$f'(x) = 2g'(x) + h'(x)$$

$$f'(2) = 2g'(2) + h'(2)$$

$$= 2(-2) + 4$$

$$= 0$$

107.  $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

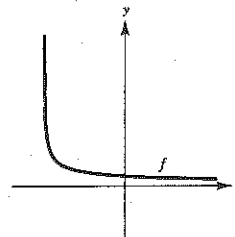
101.  $f'''(x) = 2\sqrt{x}$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$

102.  $f^{(4)}(x) = 2x + 1$

$$f^{(5)}(x) = 2$$

$$f^{(6)}(x) = 0$$

104. The graph of a differentiable function  $f$  such that  $f > 0$  and  $f' < 0$  for all real numbers  $x$  would, in general, look like the graph below.

106.  $f(x) = 4 - h(x)$

$$f'(x) = -h'(x)$$

$$f'(2) = -h'(2) = -4$$

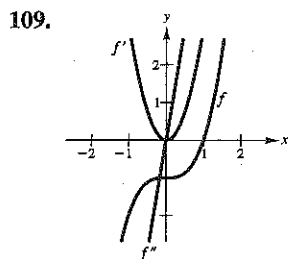
108.  $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

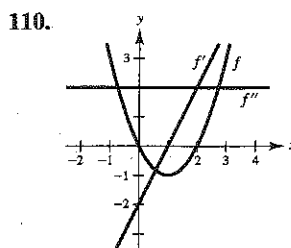
$$f'(2) = g(2)h'(2) + h(2)g'(2)$$

$$= (3)(4) + (-1)(-2)$$

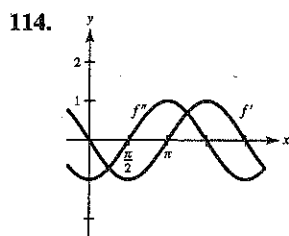
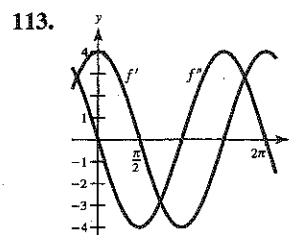
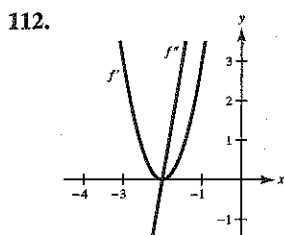
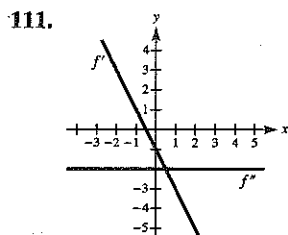
$$= 14$$



It appears that  $f$  is cubic; so  $f'$  would be quadratic and  $f''$  would be linear.



It appears that  $f$  is quadratic; so  $f'$  would be linear and  $f''$  would be constant.



115.  $v(t) = 36 - t^2$ ,  $0 \leq t \leq 6$

$$a(t) = -2t$$

$$v(3) = 27 \text{ m/sec}$$

$$a(3) = -6 \text{ m/sec}^2$$

The speed of the object is decreasing.

116.  $v(t) = \frac{100t}{2t + 15}$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2}$$

$$= \frac{1500}{(2t + 15)^2}$$

(a)  $a(5) = \frac{1500}{[2(5) + 15]^2} = 2.4 \text{ ft/sec}^2$

(b)  $a(10) = \frac{1500}{[2(10) + 15]^2} \approx 1.2 \text{ ft/sec}^2$

(c)  $a(20) = \frac{1500}{[2(20) + 15]^2} \approx 0.5 \text{ ft/sec}^2$

117.  $s(t) = -8.25t^2 + 66t$

$$v(t) = -16.50t + 66$$

$$a(t) = -16.50$$

$t(\text{sec})$	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft/sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft/sec <sup>2</sup> )	-16.5	-16.5	-16.5	-16.5	-16.5

Average velocity on:

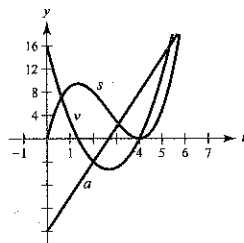
$$[0, 1] \text{ is } \frac{57.75 - 0}{1 - 0} = 57.75.$$

$$[1, 2] \text{ is } \frac{99 - 57.75}{2 - 1} = 41.25.$$

$$[2, 3] \text{ is } \frac{123.75 - 99}{3 - 2} = 24.75.$$

$$[3, 4] \text{ is } \frac{132 - 123.75}{4 - 3} = 8.25.$$

118. (a)

 $s$  position function $v$  velocity function $a$  acceleration function(b) The particle speeds up on the intervals  $(\frac{4}{3}, \frac{8}{3})$  and  $(4, 6)$ .

Answers will vary.

119.  $f(x) = x^n$ 

$$f^{(n)}(x) = n(n-1)(n-2) \cdots (2)(1) = n!$$

**Note:**  $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$  (read “ $n$  factorial”)120.  $f(x) = \frac{1}{x}$ 

$$\begin{aligned} f^{(n)}(x) &= \frac{(-1)^n(n-1)(n-2) \cdots (2)(1)}{x^{n+1}} \\ &= \frac{(-1)^n n!}{x^{n+1}} \end{aligned}$$

121.  $f(x) = g(x)h(x)$ 

(a)  $f'(x) = g(x)h'(x) + h(x)g'(x)$

$$\begin{aligned} f''(x) &= g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x) \\ &= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x) \end{aligned}$$

$$\begin{aligned} f'''(x) &= g(x)h'''(x) + g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x) \\ &= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x) \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x) + 3g^{(4)}(x)h(x) \\ &\quad + g'''(x)h'(x) + g^{(4)}(x)h(x) \\ &= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x) \end{aligned}$$

$$\begin{aligned} \text{(b) } f^{(n)}(x) &= g(x)h^{(n)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{1[(n-1)(n-2) \cdots (2)(1)]} g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{(2)(1)[(n-2)(n-3) \cdots (2)(1)]} g''(x)h^{(n-2)}(x) \\ &\quad + \frac{n(n-1)(n-2) \cdots (2)(1)}{(3)(2)(1)[(n-3)(n-4) \cdots (2)(1)]} g'''(x)h^{(n-3)}(x) + \cdots \\ &\quad + \frac{n(n-1)(n-2) \cdots (2)(1)}{[(n-1)(n-2) \cdots (2)(1)](1)} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x) \\ &= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!} g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!} g''(x)h^{(n-2)}(x) + \cdots \\ &\quad + \frac{n!}{(n-1)!1!} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x) \end{aligned}$$

**Note:**  $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$  (read “ $n$  factorial”)122.  $[xf(x)]' = xf'(x) + f(x)$ 

$$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$$

$$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$$

In general,  $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$ .

123.  $f(x) = x^n \sin x$

$$f'(x) = x^n \cos x + nx^{n-1} \sin x$$

$$= x^{n-1}(x \cos x + n \sin x)$$

When  $n = 1$ :  $f'(x) = x \cos x + \sin x$

When  $n = 2$ :  $f'(x) = x(x \cos x + 2 \sin x)$

When  $n = 3$ :  $f'(x) = x^2(x \cos x + 3 \sin x)$

When  $n = 4$ :  $f'(x) = x^3(x \cos x + 4 \sin x)$

For general  $n$ ,  $f'(x) = x^{n-1}(x \cos x + n \sin x)$ .

124.  $f(x) = \frac{\cos x}{x^n} = x^{-n} \cos x$

$$f'(x) = -x^{-n} \sin x - nx^{-n-1} \cos x$$

$$= -x^{-n-1}(x \sin x + n \cos x)$$

$$= -\frac{x \sin x + n \cos x}{x^{n+1}}$$

When  $n = 1$ :  $f'(x) = -\frac{x \sin x + \cos x}{x^2}$

When  $n = 2$ :  $f'(x) = -\frac{x \sin x + 2 \cos x}{x^3}$

When  $n = 3$ :  $f'(x) = -\frac{x \sin x + 3 \cos x}{x^4}$

When  $n = 4$ :  $f'(x) = -\frac{x \sin x + 4 \cos x}{x^5}$

For general  $n$ ,  $f'(x) = -\frac{x \sin x + n \cos x}{x^{n+1}}$ .

125.  $y = \frac{1}{x}, y' = -\frac{1}{x^2}, y'' = \frac{2}{x^3}$

$$x^3 y'' + 2x^2 y' = x^3 \left[ \frac{2}{x^3} \right] + 2x^2 \left[ -\frac{1}{x^2} \right] = 2 - 2 = 0$$

127.  $y = 2 \sin x + 3$

$y' = 2 \cos x$

$y'' = -2 \sin x$

$y'' + y = -2 \sin x + (2 \sin x + 3) = 3$

126.  $y = 2x^3 - 6x + 10$

$y' = 6x^2 - 6$

$y'' = 12x$

$y''' = 12$

$-y''' - xy'' - 2y' = -12 - x(12x) - 2(6x^2 - 6) = -24x^2$

128.  $y = 3 \cos x + \sin x$

$y' = -3 \sin x + \cos x$

$y'' = -3 \cos x - \sin x$

$y'' + y = (-3 \cos x - \sin x) + (3 \cos x + \sin x) = 0$

129. False. If  $y = f(x)g(x)$ , then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

130. True.  $y$  is a fourth-degree polynomial.

$$\frac{d^n y}{dx^n} = 0 \text{ when } n > 4.$$

131. True

$$h'(c) = f(c)g'(c) + g(c)f'(c)$$

$$= f(c)(0) + g(c)(0)$$

$$= 0$$

132. True

133. True

134. True.

If  $v(t) = c$  then  $a(t) = v'(t) = 0$ .

135.  $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$x$ -intercept at  $(1, 0)$ :  $0 = a + b + c$

$(2, 7)$  on graph:  $7 = 4a + 2b + c$

Slope 10 at  $(2, 7)$ :  $10 = 4a + b$

Subtracting the third equation from the second,  $-3 = b + c$ . Subtracting this equation from the first,  $3 = a$ . Then,  $10 = 4(3) + b \Rightarrow b = -2$ . Finally,  $-3 = (-2) + c \Rightarrow c = -1$ .

$f(x) = 3x^2 - 2x - 1$



136.  $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f'(x) = 0 \Rightarrow x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}$$

(a) No horizontal tangents:  $f'(x) \neq 0$ 

$$4b^2 - 12ac < 0$$

Example:  $a = c = 1, b = 0$ :

$$f(x) = x^3 + x$$

(b) Exactly one horizontal tangent

$$4b^2 - 12ac = 0$$

Example:  $a = 1, b = 3, c = 3$ :

$$f(x) = x^3 + 3x^2 + 3x$$

(c) Exactly two horizontal tangents

$$4b^2 - 12ac > 0$$

Example:  $b = 1, a = 1, c = 0$ :

$$f(x) = x^3 + x^2$$

137.  $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

 $f''(0)$  does not exist since the left and right derivatives are not equal.

138. (a)  $(fg' - f'g)' = fg'' + f'g'' - f'g'' - f''g$   
 $= fg'' - f''g$  True

(b)  $(fg)'' = (fg' + f'g)'$   
 $= fg'' + f'g'' + f'g'' + f''g$   
 $= fg'' + 2f'g'' + f''g$   
 $\neq fg'' + f''g$  False

## Section 2.4 The Chain Rule

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

1.  $y = (6x - 5)^4$

$$u = 6x - 5$$

$$y = u^4$$

2.  $y = \frac{1}{\sqrt{x+1}}$

$$u = x + 1$$

$$y = u^{-1/2}$$

3.  $y = \sqrt{x^2 - 1}$

$$u = x^2 - 1$$

$$y = \sqrt{u}$$

4.  $y = 3 \tan(\pi x^2)$

$$u = \pi x^2$$

$$y = 3 \tan u$$

5.  $y = \csc^3 x$

$$u = \csc x$$

$$y = u^3$$

6.  $y = \cos \frac{3x}{2}$

$$u = \frac{3x}{2}$$

$$y = \cos u$$

7.  $y = (2x - 7)^3$

$$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$$

8.  $y = 3(4 - x^2)^5$

$$y' = 15(4 - x^2)^4(-2x) = -30x(4 - x^2)^4$$

9.  $g(x) = 3(4 - 9x)^4$

$$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$$

10.  $f(t) = (9t + 2)^{2/3}$

$$f'(t) = \frac{2}{3}(9t + 2)^{-1/3}(9) = \frac{6}{\sqrt[3]{9t + 2}}$$

11.  $f(t) = (1 - t)^{1/2}$

$$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$$

13.  $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

15.  $y = 2(4 - x^2)^{1/4}$

$$y' = 2\left(\frac{1}{4}\right)(4 - x^2)^{-3/4}(-2x) = \frac{-x}{\sqrt[4]{(4 - x^2)^3}}$$

17.  $y = (x - 2)^{-1}$

$$y' = -1(x - 2)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

20.  $y = -\frac{5}{(t + 3)^3}$

$$y = -5(t + 3)^{-3}$$

$$y' = 15(t + 3)^{-4} = \frac{15}{(t + 3)^4}$$

23.  $f(x) = x^2(x - 2)^4$

$$\begin{aligned} f'(x) &= x^2[4(x - 2)^3(1)] + (x - 2)^4(2x) \\ &= 2x(x - 2)^3[2x + (x - 2)] \\ &= 2x(x - 2)^3(3x - 2) \end{aligned}$$

25.  $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$\begin{aligned} y' &= x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1) \\ &= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2} \\ &= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)] \\ &= \frac{1 - 2x^2}{\sqrt{1 - x^2}} \end{aligned}$$

12.  $g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$

$$g'(x) = \frac{1}{2}(5 - 3x)^{-1/2}(-3) = \frac{-3}{2\sqrt{5 - 3x}}$$

14.  $g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$

$$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

16.  $f(x) = -3\sqrt[4]{2 - 9x}$

$$f(x) = -3(2 - 9x)^{1/4}$$

$$f'(x) = -\frac{3}{4}(2 - 9x)^{-3/4}(-9) = \frac{27}{4(2 - 9x)^{3/4}}$$

18.  $s(t) = \frac{1}{t^2 + 3t - 1}$

$$s(t) = (t^2 + 3t - 1)^{-1}$$

$$\begin{aligned} s'(t) &= -1(t^2 + 3t - 1)^{-2}(2t + 3) \\ &= \frac{-(2t + 3)}{(t^2 + 3t - 1)^2} \end{aligned}$$

19.  $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

21.  $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2}$$

$$= -\frac{1}{2(x + 2)^{3/2}}$$

22.  $g(t) = \sqrt{\frac{1}{t^2 - 2}}$

$$g(t) = (t^2 - 2)^{-1/2}$$

$$\begin{aligned} g'(t) &= -\frac{1}{2}(t^2 - 2)^{-3/2}(2t) \\ &= -\frac{t}{(t^2 - 2)^{3/2}} \end{aligned}$$

24.  $f(x) = x(3x - 9)^3$

$$\begin{aligned} f'(x) &= x[3(3x - 9)^2(3)] + (3x - 9)^3(1) = 9x(3x - 9)^2 + (3x - 9)^3 \\ &= (3x - 9)^2[9x + 3x - 9] \\ &= 27(x - 3)^2(4x - 3) \end{aligned}$$

26.  $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

$$\begin{aligned} y' &= \frac{1}{2}x^2\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2} \\ &= \frac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2} \\ &= \frac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}} \end{aligned}$$

$$27. y = \frac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$$

$$\begin{aligned} y' &= x \left[ -\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2}(1) \\ &= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2} \\ &= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)] \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

$$29. g(x) = \left( \frac{x+5}{x^2+2} \right)^2$$

$$\begin{aligned} g'(x) &= 2 \left( \frac{x+5}{x^2+2} \right) \left( \frac{(x^2+2) - (x+5)(2x)}{(x^2+2)^2} \right) \\ &= \frac{2(x+5)(2-10x-x^2)}{(x^2+2)^3} \end{aligned}$$

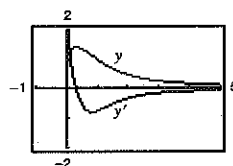
$$31. f(v) = \left( \frac{1-2v}{1+v} \right)^3$$

$$\begin{aligned} f'(v) &= 3 \left( \frac{1-2v}{1+v} \right)^2 \left( \frac{(1+v)(-2) - (1-2v)}{(1+v)^2} \right) \\ &= \frac{-9(1-2v)^2}{(1+v)^4} \end{aligned}$$

$$33. y = \frac{\sqrt{x} + 1}{x^2 + 1}$$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

The zero of  $y'$  corresponds to the point on the graph of  $y$  where the tangent line is horizontal.



$$35. y = \sqrt{\frac{x+1}{x}}$$

$$y' = -\frac{\sqrt{(x+1)/x}}{2x(x+1)}$$

$y'$  has no zeros.



$$28. y = \frac{x}{\sqrt{x^4 + 4}}$$

$$\begin{aligned} y' &= \frac{(x^4 + 4)^{1/2}(1) - x \frac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4} \\ &= \frac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} \\ &= \frac{4 - x^4}{(x^4 + 4)^{3/2}} \end{aligned}$$

$$30. h(t) = \left( \frac{t^2}{t^3 + 2} \right)^2$$

$$\begin{aligned} h'(t) &= 2 \left( \frac{t^2}{t^3 + 2} \right) \left( \frac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2} \right) \\ &= \frac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \frac{2t^3(4 - t^3)}{(t^3 + 2)^3} \end{aligned}$$

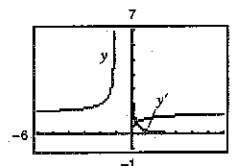
$$32. g(x) = \left( \frac{3x^2 - 2}{2x + 3} \right)^3$$

$$\begin{aligned} g'(x) &= 3 \left( \frac{3x^2 - 2}{2x + 3} \right)^2 \left( \frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2} \right) \\ &= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} \\ &= \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4} \end{aligned}$$

$$34. y = \sqrt{\frac{2x}{x+1}}$$

$$y' = \frac{1}{\sqrt{2x}(x+1)^{3/2}}$$

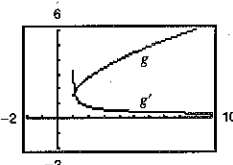
$y'$  has no zeros.



$$36. g(x) = \sqrt{x-1} + \sqrt{x+1}$$

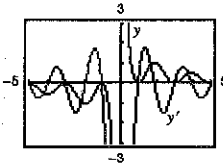
$$g'(x) = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$$

$g'$  has no zeros.



$$37. y = \frac{\cos \pi x + 1}{x}$$

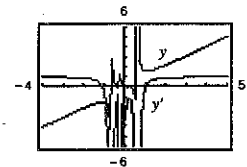
$$\begin{aligned} \frac{dy}{dx} &= \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2} \\ &= \frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2} \end{aligned}$$



The zeros of  $y'$  correspond to the points on the graph of  $y$  where the tangent lines are horizontal.

$$38. y = x^2 \tan \frac{1}{x}$$

$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$



The zeros of  $y'$  correspond to the points on the graph of  $y$  where the tangent lines are horizontal.

$$39. (a) y = \sin x$$

$$y' = \cos x$$

$$y'(0) = 1$$

1 cycle in  $[0, 2\pi]$

$$(b) y = \sin 2x$$

$$y' = 2 \cos 2x$$

$$y'(0) = 2$$

2 cycles in  $[0, 2\pi]$

The slope of  $\sin ax$  at the origin is  $a$ .

$$40. (a) y = \sin 3x$$

$$y' = 3 \cos 3x$$

$$y'(0) = 3$$

3 cycles in  $[0, 2\pi]$

$$(b) y = \sin\left(\frac{x}{2}\right)$$

$$y' = \left(\frac{1}{2}\right) \cos\left(\frac{x}{2}\right)$$

$$y'(0) = \frac{1}{2}$$

Half cycle in  $[0, 2\pi]$

The slope of  $\sin ax$  at the origin is  $a$ .

$$41. y = \cos 3x$$

$$\frac{dy}{dx} = -3 \sin 3x$$

$$42. y = \sin \pi x$$

$$\frac{dy}{dx} = \pi \cos \pi x$$

$$43. g(x) = 3 \tan 4x$$

$$g'(x) = 12 \sec^2 4x$$

$$44. h(x) = \sec(x^2)$$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

$$45. y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$$

$$y' = \cos(\pi^2 x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$$

$$46. y = \cos(1 - 2x)^2 = \cos((1 - 2x)^2)$$

$$\begin{aligned} y' &= -\sin(1 - 2x)^2(2(1 - 2x)(-2)) \\ &= 4(1 - 2x) \sin(1 - 2x)^2 \end{aligned}$$

$$47. h(x) = \sin 2x \cos 2x$$

$$\begin{aligned} h'(x) &= \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x) \\ &= 2 \cos^2 2x - 2 \sin^2 2x \\ &= 2 \cos 4x \end{aligned}$$

**Alternate solution:**  $h(x) = \frac{1}{2} \sin 4x$

$$h'(x) = \frac{1}{2} \cos 4x(4) = 2 \cos 4x$$

$$49. f(x) = \frac{\cot x}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$\begin{aligned} f'(x) &= \frac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x} \\ &= \frac{-\sin^2 x - 2 \cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x} \end{aligned}$$

$$50. g(v) = \frac{\cos v}{\csc v} = \cos v \cdot \sin v$$

$$\begin{aligned} g'(v) &= \cos v(\cos v) + \sin v(-\sin v) \\ &= \cos^2 v - \sin^2 v = \cos 2v \end{aligned}$$

$$51. y = 4 \sec^2 x$$

$$\begin{aligned} y' &= 8 \sec x \cdot \sec x \tan x \\ &= 8 \sec^2 x \tan x \end{aligned}$$

$$52. g(t) = 5 \cos^2 \pi t = 5(\cos \pi t)^2$$

$$\begin{aligned} g'(t) &= 10 \cos \pi t(-\sin \pi t)(\pi) \\ &= -10\pi(\sin \pi t)(\cos \pi t) \\ &= -5\pi \sin 2\pi t \end{aligned}$$

$$53. f(\theta) = \frac{1}{4} \sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$$

$$\begin{aligned} f'(\theta) &= 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2) \\ &= \sin 2\theta \cos 2\theta = \frac{1}{2} \sin 4\theta \end{aligned}$$

54.  $h(t) = 2 \cot^2(\pi t + 2)$

$$h'(t) = 4 \cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi))$$

$$= -4\pi \cot(\pi t + 2) \csc^2(\pi t + 2)$$

56.  $y = 3x - 5 \cos(\pi x)^2$

$$= 3x - 5 \cos(\pi^2 x^2)$$

$$\frac{dy}{dx} = 3 + 5 \sin(\pi^2 x^2)(2\pi^2 x)$$

$$= 3 + 10\pi^2 x \sin(\pi x)^2$$

58.  $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$y' = \cos x^{1/3} \left( \frac{1}{3} x^{-2/3} \right) + \frac{1}{3} (\sin x)^{-2/3} \cos x$$

$$= \frac{1}{3} \left[ \frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}} \right]$$

60.  $y = (3x^3 + 4x)^{1/5}, (2, 2)$

$$y' = \frac{1}{5} (3x^3 + 4x)^{-4/5} (9x^2 + 4)$$

$$= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$$

$$y'(2) = \frac{1}{2}$$

62.  $f(x) = \frac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \left(4, \frac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

64.  $f(x) = \frac{x+1}{2x-3}, (2, 3)$

$$f'(x) = \frac{(2x-3)(1) - (x+1)(2)}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$$

$$f'(2) = -5$$

55.  $f(t) = 3 \sec^2(\pi t - 1)$

$$f'(t) = 6 \sec(\pi t - 1) \sec(\pi t - 1) \tan(\pi t - 1)(\pi)$$

$$= 6\pi \sec^2(\pi t - 1) \tan(\pi t - 1) = \frac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)}$$

57.  $y = \sqrt{x} + \frac{1}{4} \sin(2x)^2$

$$= \sqrt{x} + \frac{1}{4} \sin(4x^2)$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} + \frac{1}{4} \cos(4x^2)(8x)$$

$$= \frac{1}{2\sqrt{x}} + 2x \cos(2x)^2$$

59.  $s(t) = (t^2 + 2t + 8)^{1/2}, (2, 4)$

$$s'(t) = \frac{1}{2} (t^2 + 2t + 8)^{-1/2} (2t + 2)$$

$$= \frac{t + 1}{\sqrt{t^2 + 2t + 8}}$$

$$s'(2) = \frac{3}{4}$$

61.  $f(x) = \frac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \left(-1, -\frac{3}{5}\right)$

$$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\frac{9x^2}{(x^3 - 4)^2}$$

$$f'(-1) = -\frac{9}{25}$$

63.  $f(t) = \frac{3t+2}{t-1}, (0, -2)$

$$f'(t) = \frac{(t-1)(3) - (3t+2)(1)}{(t-1)^2} = \frac{-5}{(t-1)^2}$$

$$f'(0) = -5$$

65.  $y = 37 - \sec^3(2x), (0, 36)$

$$y' = -3 \sec^2(2x)[2 \sec(2x) \tan(2x)]$$

$$= -6 \sec^3(2x) \tan(2x)$$

$$y'(0) = 0$$

$$66. y = \frac{1}{x} + \sqrt{\cos x}, \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$  is undefined.

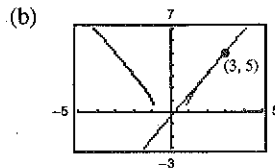
$$67. (a) f(x) = \sqrt{3x^2 - 2}, (3, 5)$$

$$f'(x) = \frac{1}{2}(3x^2 - 2)^{-1/2}(6x) = \frac{3x}{\sqrt{3x^2 - 2}}$$

$$f'(3) = \frac{9}{5}$$

Tangent line:

$$y - 5 = \frac{9}{5}(x - 3) \Rightarrow 9x - 5y - 2 = 0$$



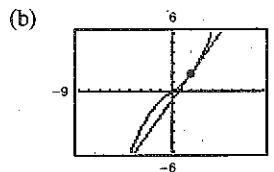
$$68. (a) f(x) = \frac{1}{3}x\sqrt{x^2 + 5}, (2, 2)$$

$$f'(x) = \frac{1}{3}x \left[ \frac{1}{2}(x^2 + 5)^{-1/2}(2x) \right] + \frac{1}{3}(x^2 + 5)^{1/2}$$

$$= \frac{x^2}{3\sqrt{x^2 + 5}} + \frac{1}{3}\sqrt{x^2 + 5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:  $y - 2 = \frac{13}{9}(x - 2) \Rightarrow 13x - 9y - 8 = 0$



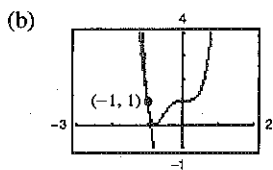
$$69. (a) y = (2x^3 + 1)^2, (-1, 1)$$

$$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$$

$$y'(-1) = 12(-1) = -12$$

Tangent line:  $y - 1 = -12(x + 1)$

$$y = -12x - 11$$



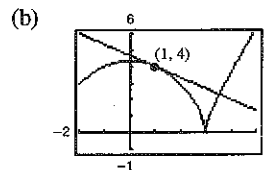
$$70. (a) f(x) = (9 - x^2)^{2/3}, (1, 4)$$

$$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = \frac{-4x}{3(9 - x^2)^{1/3}}$$

$$f'(1) = \frac{-4}{3(8)^{1/3}} = -\frac{2}{3}$$

Tangent line:  $y - 4 = -\frac{2}{3}(x - 1)$

$$y = -\frac{2}{3}x + \frac{14}{3}$$



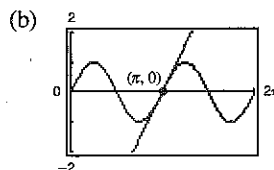
$$71. (a) f(x) = \sin 2x, (\pi, 0)$$

$$f'(x) = 2 \cos 2x$$

$$f'(\pi) = 2$$

Tangent line:

$$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$$



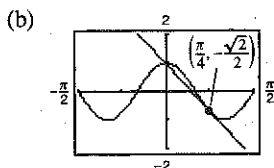
72. (a)  $y = \cos 3x, \left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$

$$y' = -3 \sin 3x$$

$$y'\left(\frac{\pi}{4}\right) = -3 \sin\left(\frac{3\pi}{4}\right) = \frac{-3\sqrt{2}}{2}$$

Tangent line:  $y + \frac{\sqrt{2}}{2} = \frac{-3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$

$$y = \frac{-3\sqrt{2}}{2}x + \frac{3\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}$$



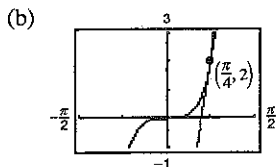
74. (a)  $y = 2 \tan^3 x, \left(\frac{\pi}{4}, 2\right)$

$$y' = 6 \tan^2 x \cdot \sec^2 x$$

$$y'\left(\frac{\pi}{4}\right) = 6(1)(2) = 12$$

Tangent line:  $y - 2 = 12\left(x - \frac{\pi}{4}\right)$

$$y = 12x + 2 - 3\pi$$



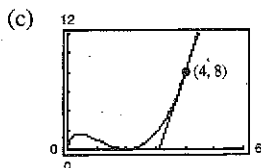
76. (a)  $f(x) = \sqrt{x}(2-x)^2, (4, 8)$

$$f'(x) = \frac{(x-2)(5x-2)}{2\sqrt{x}}$$

$$f'(4) = 9$$

(b)  $y - 8 = 9(x - 4)$

$$y = 9x - 28$$



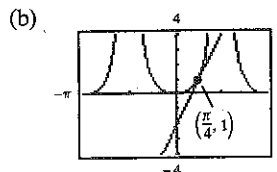
73. (a)  $f(x) = \tan^2 x, \left(\frac{\pi}{4}, 1\right)$

$$f'(x) = 2 \tan x \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$$



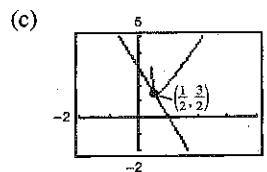
75. (a)  $g(t) = \frac{3t^2}{\sqrt{t^2 + 2t - 1}}, \left(\frac{1}{2}, \frac{3}{2}\right)$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

$$g'\left(\frac{1}{2}\right) = -3$$

(b)  $y - \frac{3}{2} = -3\left(x - \frac{1}{2}\right)$

$$y = -3x + 3$$



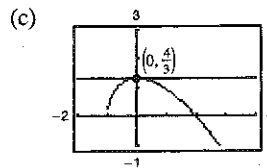
77. (a)  $s(t) = \frac{(4-2t)\sqrt{1+t}}{3}, \left(0, \frac{4}{3}\right)$

$$s'(t) = \frac{-2\sqrt{1+t}}{3} + \frac{2-t}{3\sqrt{1+t}}$$

$$s'(0) = 0$$

(b)  $y - \frac{4}{3} = 0(x - 0)$

$$y = \frac{4}{3}$$



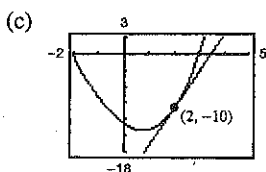
78. (a)  $y = (t^2 - 9)\sqrt{t + 2}, (2, -10)$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t + 2}}$$

$$y'(2) = \frac{27}{4}$$

(b)  $y + 10 = \frac{27}{4}(t - 2)$

$$y = \frac{27}{4}t - \frac{47}{2}$$



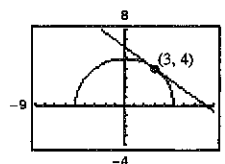
79.  $f(x) = \sqrt{25 - x^2}, (3, 4)$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

$$f'(3) = -\frac{3}{4}$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{25}{4}; \text{ Tangent line}$$



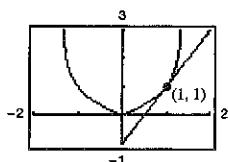
80.  $f(x) = \frac{|x|}{\sqrt{2 - x^2}}, (1, 1)$

$$f'(x) = \frac{2}{(2 - x^2)^{3/2}} \text{ for } x > 0$$

$$f'(1) = 2$$

$$y - 1 = 2(x - 1)$$

$$y = 2x - 1; \text{ Tangent line}$$



81.  $f(x) = 2 \cos x + \sin 2x, 0 < x < 2\pi$

$$f'(x) = -2 \sin x + 2 \cos 2x$$

$$= -2 \sin x + 2 - 4 \sin^2 x = 0$$

$$2 \sin^2 x + \sin x - 1 = 0$$

$$(\sin x + 1)(2 \sin x - 1) = 0$$

$$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$$

$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Horizontal tangents at  $x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}$

Horizontal tangent at the points  $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right), \left(\frac{3\pi}{2}, 0\right)$ , and

$$\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$$

82.  $f(x) = \frac{x}{\sqrt{2x - 1}}$

$$f'(x) = \frac{(2x - 1)^{1/2} - x(2x - 1)^{-1/2}}{2x - 1}$$

$$= \frac{2x - 1 - x}{(2x - 1)^{3/2}}$$

$$= \frac{x - 1}{(2x - 1)^{3/2}}$$

$$\frac{x - 1}{(2x - 1)^{3/2}} = 0 \Rightarrow x = 1$$

Horizontal tangent at (1, 1)

83.  $f(x) = 2(x^2 - 1)^3$

$$f'(x) = 6(x^2 - 1)^2(2x)$$

$$= 12x(x^4 - 2x^2 + 1)$$

$$= 12x^5 - 24x^3 + 12x$$

$$f''(x) = 60x^4 - 72x^2 + 12$$

$$= 12(5x^2 - 1)(x^2 - 1)$$



84.  $f(x) = (x - 2)^{-1}$

$$f'(x) = -(x - 2)^{-2} = \frac{-1}{(x - 2)^2}$$

$$f''(x) = 2(x - 2)^{-3} = \frac{2}{(x - 2)^3}$$

86.  $f(x) = \sec^2 \pi x$

$$f'(x) = 2 \sec \pi x (\pi \sec \pi x \tan \pi x)$$

$$= 2\pi \sec^2 \pi x \tan \pi x$$

$$f''(x) = 2\pi \sec^2 \pi x (\sec^2 \pi x)(\pi) + 2\pi \tan \pi x (2\pi \sec^2 \pi x \tan \pi x)$$

$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$= 2\pi^2 \sec^2 \pi x (\sec^2 \pi x + 2 \tan^2 \pi x)$$

$$= 2\pi^2 \sec^2 \pi x (3 \sec^2 \pi x - 2)$$

87.  $h(x) = \frac{1}{9}(3x + 1)^3, \left(1, \frac{64}{9}\right)$

$$h'(x) = \frac{1}{9}3(3x + 1)^2(3) = (3x + 1)^2$$

$$h''(x) = 2(3x + 1)(3) = 6(3x + 1)$$

$$h''(1) = 24$$

89.  $f(x) = \cos(x^2), (0, 1)$

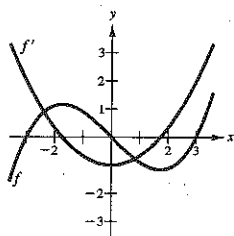
$$f'(x) = -\sin(x^2)(2x) = -2x \sin(x^2)$$

$$f''(x) = -2x \cos(x^2)(2x) - 2 \sin(x^2)$$

$$= -4x^2 \cos(x^2) - 2 \sin(x^2)$$

$$f''(0) = 0$$

91.



The zeros of  $f'$  correspond to the points where the graph of  $f$  has horizontal tangents.

85.  $f(x) = \sin x^2$

$$f'(x) = 2x \cos x^2$$

$$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2 \\ = 2[\cos x^2 - 2x^2 \sin x^2]$$

88.  $f(x) = \frac{1}{\sqrt{x+4}} = (x+4)^{-1/2}, \left(0, \frac{1}{2}\right)$

$$f'(x) = -\frac{1}{2}(x+4)^{-3/2}$$

$$f''(x) = \frac{3}{4}(x+4)^{-5/2} = \frac{3}{4(x+4)^{5/2}}$$

$$f''(0) = \frac{3}{128}$$

90.  $g(t) = \tan(2t), \left(\frac{\pi}{6}, \sqrt{3}\right)$

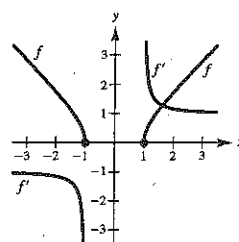
$$g'(t) = 2 \sec^2(2t)$$

$$g''(t) = 4 \sec(2t) \cdot \sec(2t) \tan(2t) \cdot 2$$

$$= 8 \sec^2(2t) \tan(2t)$$

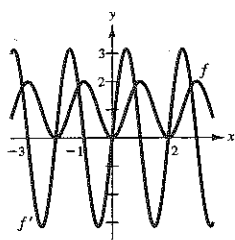
$$g''\left(\frac{\pi}{6}\right) = 32\sqrt{3}$$

92.



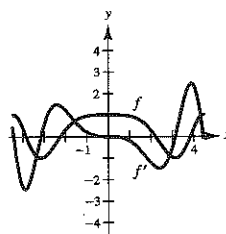
$f$  is decreasing on  $(-\infty, -1)$  so  $f'$  must be negative there.  
 $f$  is increasing on  $(1, \infty)$  so  $f'$  must be positive there.

93.



The zeros of  $f'$  correspond to the points where the graph of  $f$  has horizontal tangents.

94.



The zeros of  $f'$  correspond to the points where the graph of  $f$  has horizontal tangents.

95.  $g(x) = f(3x)$

$$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$$

97. (a)  $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$

$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(c)  $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

98. (a)  $g(x) = f(x) - 2 \Rightarrow g'(x) = f'(x)$

(b)  $h(x) = 2f(x) \Rightarrow h'(x) = 2f'(x)$

(c)  $r(x) = f(-3x) \Rightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know  $f'(-3x)$ .

$$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$$

$$r'(-1) = -3f'(3) = (-3)(-4) = 12$$

(d)  $s(x) = f(x+2) \Rightarrow s'(x) = f'(x+2)$

Hence, you need to know  $f'(x+2)$ .

$$s'(-2) = f'(0) = -\frac{1}{3}, \text{ etc.}$$

99. (a)  $h(x) = f(g(x)), g(1) = 4, g'(1) = -\frac{1}{2}, f'(4) = -1, h'(x) = f'(g(x))g'(x)$

$$h'(1) = f'(g(1))g'(1)$$

$$= f'(4)g'(1)$$

$$= (-1)\left(-\frac{1}{2}\right) = \frac{1}{2}$$

(b)  $s(x) = g(f(x)), f(5) = 6, f'(5) = -1, g'(6)$  does not exist.

$$s'(x) = g'(f(x))f'(x)$$

$$s'(5) = g'(f(5))f'(5) = g'(6)(-1)$$

Since  $g'(6)$  does not exist,  $s'(5)$  is not defined.

96.  $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \Rightarrow g'(x) = 2xf'(x^2)$$

(b)  $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$

$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need  $g'(3)$  to find  $f'(5)$ .

(d)  $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2g'(x)$$

$$f'(5) = 3(-3)^2(6) = 162$$

$x$	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

100. (a)  $h(x) = f(g(x))$

$$h'(x) = f'(g(x))g'(x)$$

$$h'(3) = f'(g(3))g'(3)$$

$$= f'(5)(1)$$

$$= \frac{1}{2}$$

(b)  $s(x) = g(f(x))$

$$s'(x) = g'(f(x))f'(x)$$

$$s'(9) = g'(f(9))f'(9)$$

$$= g'(8)(2)$$

$$= (-1)(2)$$

$$= -2$$

101. (a)  $F = 132,400(331 - v)^{-1}$

$$F' = (-1)(132,400)(331 - v)^{-2}(-1)$$

$$= \frac{132,400}{(331 - v)^2}$$

$$\text{When } v = 30, F' \approx 1.461.$$

(b)  $F = 132,400(331 + v)^{-1}$

$$F' = (-1)(132,400)(331 + v)^{-2}(1)$$

$$= \frac{-132,400}{(331 + v)^2}$$

$$\text{When } v = 30, F' \approx -1.016.$$

102.  $y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$

$$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$$

$$= -4 \sin 12t - 3 \cos 12t$$

When  $t = \pi/8$ ,  $y = 0.25$  feet and  $v = 4$  feet per second.

103.  $\theta = 0.2 \cos 8t$

The maximum angular displacement is  $\theta = 0.2$  (since  $-1 \leq \cos 8t \leq 1$ ).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When  $t = 3$ ,  $d\theta/dt = -1.6 \sin 24 \approx 1.4489$  radians per second.

104.  $y = A \cos \omega t$

(a) Amplitude:  $A = \frac{3.5}{2} = 1.75$

$$y = 1.75 \cos \omega t$$

Period:  $10 \Rightarrow \omega = \frac{2\pi}{10} = \frac{\pi}{5}$

$$y = 1.75 \cos \frac{\pi t}{5}$$

(b)  $v = y' = 1.75 \left[ -\frac{\pi}{5} \sin \frac{\pi t}{5} \right] = -0.35\pi \sin \frac{\pi t}{5}$

105.  $S = C(R^2 - r^2)$

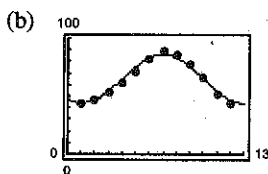
$$\frac{dS}{dt} = C \left( 2R \frac{dR}{dt} - 2r \frac{dr}{dt} \right)$$

Since  $r$  is constant, we have  $dr/dt = 0$  and

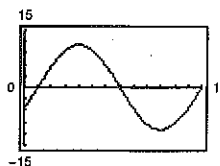
$$\begin{aligned} \frac{dS}{dt} &= (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5}) \\ &= 4.224 \times 10^{-2} = 0.04224. \end{aligned}$$

106. (a) Using a graphing utility, or trial and error, you obtain a model similar to

$$T(t) = 65 + 21 \sin\left(\frac{\pi t}{6} - 2.1\right).$$



(c)  $T'(t) \approx 11 \cos\left(\frac{\pi t}{6} - 2.1\right)$



(d) The temperature changes most rapidly in the spring (April–June) and fall (September–November).

107. (a)  $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6162$

(b)  $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6162) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function  $dC/dt$  is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

108.  $f(x) = \sin \beta x$

(a)  $f'(x) = \beta \cos \beta x$

$$f''(x) = -\beta^2 \sin \beta x$$

$$f'''(x) = -\beta^3 \cos \beta x$$

$$f^{(4)}(x) = \beta^4 \sin \beta x$$

(b)  $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2 (\sin \beta x) = 0$

(c)  $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$$

110. (a)  $r'(x) = f'(g(x))g'(x)$

$$r'(1) = f'(g(1))g'(1)$$

Note that  $g(1) = 4$  and  $f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$ .

Also,  $g'(1) = 0$ . Thus,  $r'(1) = 0$ .

111. (a)  $g(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow g'(x) = 0$

$$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$$

112. (a) If  $f(-x) = -f(x)$ , then

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[-f(x)]$$

$$f'(-x)(-1) = -f'(x)$$

$$f'(-x) = f'(x).$$

Thus,  $f'(x)$  is even.

113.  $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}[\sqrt{u^2}] = \frac{1}{2}(u^2)^{-1/2}(2uu')$$

$$= \frac{uu'}{\sqrt{u^2}} = u' \frac{u}{|u|}, \quad u \neq 0$$

109.  $f(x+p) = f(x)$  for all  $x$ .

(a) Yes,  $f'(x+p) = f'(x)$ , which shows that  $f'$  is periodic as well.

(b) Yes, let  $g(x) = f(2x)$ , so  $g'(x) = 2f'(2x)$ . Since  $f'$  is periodic, so is  $g'$ .

(b)  $s'(x) = g'(f(x))f'(x)$

$$s'(4) = g'(f(4))f'(4)$$

Note that  $f(4) = \frac{5}{2}$ ,  $g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2}$  and

$$f'(4) = \frac{5}{4}. \text{ Thus, } s'(4) = \frac{1}{2} \left(\frac{5}{4}\right) = \frac{5}{8}.$$

(b)  $\tan^2 x + 1 = \sec^2 x$

$$g(x) + 1 = f(x)$$

Taking derivatives of both sides,  $g'(x) = f'(x)$ . Equivalently,  $f'(x) = 2 \sec x \cdot \sec x \cdot \tan x$  and  $g'(x) = 2 \tan x \cdot \sec^2 x$ , which are the same.

(b) If  $f(-x) = f(x)$ , then

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[f(x)]$$

$$f'(-x)(-1) = f'(x)$$

$$f'(-x) = -f'(x).$$

Thus,  $f'$  is odd.

114.  $g(x) = |2x - 3|$

$$g'(x) = 2 \left( \frac{2x-3}{|2x-3|} \right), \quad x \neq \frac{3}{2}$$

115.  $f(x) = |x^2 - 4|$

$$f'(x) = 2x \left( \frac{x^2 - 4}{|x^2 - 4|} \right), \quad x \neq \pm 2$$

116.  $h(x) = |x| \cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|} \cos x, \quad x \neq 0$$

117.  $f(x) = |\sin x|$

$$f'(x) = \cos x \left( \frac{\sin x}{|\sin x|} \right), \quad x \neq k\pi$$

118. (a)  $f(x) = \tan \frac{\pi x}{4}$

$$f(1) = 1$$

$$f'(x) = \frac{\pi}{4} \sec^2 \frac{\pi x}{4}$$

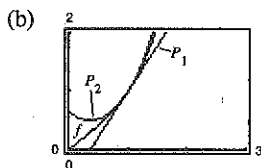
$$f'(1) = \frac{\pi}{4}(2) = \frac{\pi}{2}$$

$$f''(x) = \frac{\pi}{2} \sec^2 \frac{\pi x}{4} \cdot \tan \frac{\pi x}{4} \left( \frac{\pi}{4} \right)$$

$$f''(1) = \frac{\pi^2}{8}(2)(1) = \frac{\pi^2}{4}$$

$$P_1(x) = f'(1)(x - 1) + f(1) = \frac{\pi}{2}(x - 1) + 1$$

$$P_2(x) = \frac{1}{2} \left( \frac{\pi^2}{4} \right) (x - 1)^2 + f'(1)(x - 1) + f(1) = \frac{\pi^2}{8}(x - 1)^2 + \frac{\pi}{2}(x - 1) + 1$$

(c)  $P_2$  is a better approximation than  $P_1$ .(d) The accuracy worsens as you move away from  $x = c = 1$ .

119. (a)  $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$

$$\begin{aligned} f''(x) &= 2[2(\sec 2x)(\tan 2x)] \tan 2x + 2(\sec 2x)(\sec^2 2x)(2) \\ &= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x] \end{aligned}$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

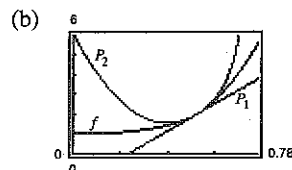
$$f'\left(\frac{\pi}{6}\right) = 2 \sec\left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(c)  $P_2$  is a better approximation than  $P_1$ .(d) The accuracy worsens as you move away from  $x = \pi/6$ .120. False. If  $y = (1 - x)^{1/2}$ , then  $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$ .121. False. If  $f(x) = \sin^2 2x$ , then  $f'(x) = 2(\sin 2x)(2 \cos 2x)$ .

122. True

$$123. \quad \begin{aligned} f(x) &= a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx \\ f'(x) &= a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx \\ f'(0) &= a_1 + 2a_2 + \cdots + na_n \end{aligned}$$

$$\begin{aligned} |a_1 + 2a_2 + \cdots + na_n| &= |f'(0)| \\ &= \lim_{x \rightarrow 0} \left| \frac{f(x) - f(0)}{x - 0} \right| \\ &= \lim_{x \rightarrow 0} \left| \frac{f(x)}{\sin x} \right| \cdot \left| \frac{\sin x}{x} \right| \\ &= \lim_{x \rightarrow 0} \left| \frac{f(x)}{\sin x} \right| \leq 1 \end{aligned}$$

$$124. \quad \frac{d}{dx} \left[ \frac{P_n(x)}{(x^k - 1)^{n+1}} \right] = \frac{(x^k - 1)^{n+1} P_n'(x) - P_n(x)(n+1)(x^k - 1)^n kx^{k-1}}{(x^k - 1)^{2n+2}}$$

$$= \frac{(x^k - 1)P_n'(x) - (n+1)kx^{k-1}P_n(x)}{(x^k - 1)^{n+2}}$$

$$P_n(x) = (x^k - 1)^{n+1} \frac{d^n}{dx^n} \left[ \frac{1}{x^k - 1} \right] \Rightarrow$$

$$P_{n+1}(x) = (x^k - 1)^{n+2} \frac{d}{dx} \left[ \frac{d^n}{dx^n} \left[ \frac{1}{x^k - 1} \right] \right] = (x^k - 1)P_n'(x) - (n+1)kx^{k-1}P_n(x)$$

$$P_{n+1}(1) = -(n+1)kP_n(1)$$

$$\text{For } n=1, \frac{d}{dx} \left[ \frac{1}{x^k - 1} \right] = \frac{-kx^{k-1}}{(x^k - 1)^2} = \frac{P_1(x)}{(x^k - 1)^2} \Rightarrow P_1(1) = -k. \text{ Also, } P_0(1) = 1.$$

We now use mathematical induction to verify that  $P_n(1) = (-k)^n n!$  for  $n \geq 0$ . Assume true for  $n$ . Then

$$\begin{aligned} P_{n+1}(1) &= -(n+1)kP_n(1) \\ &= -(n+1)k(-k)^n n! \\ &= (-k)^{n+1}(n+1)!. \end{aligned}$$

## Section 2.5 Implicit Differentiation

$$1. \quad x^2 + y^2 = 36$$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$2. \quad x^2 - y^2 = 16$$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$3. \quad x^{1/2} + y^{1/2} = 9$$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = -\frac{x^{-1/2}}{y^{-1/2}}$$

$$= -\sqrt{\frac{y}{x}}$$

$$4. \quad x^3 + y^3 = 8$$

$$3x^2 + 3y^2y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

$$5. \quad x^3 - xy + y^2 = 4$$

$$3x^2 - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 3x^2$$

$$y' = \frac{y - 3x^2}{2y - x}$$

6.  $x^2y + y^2x = -2$

$$x^2y' + 2xy + y^2 + 2yxy' = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

7.  $x^3y^3 - y - x = 0$

$$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$$

$$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$$

$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

8.  $(xy)^{1/2} - x + 2y = 0$

$$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$$

$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$$

$$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}y' = 0$$

$$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$$

9.  $x^3 - 3x^2y + 2xy^2 = 12$

$$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$$

$$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$$

$$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$$

10.  $2 \sin x \cos y = 1$

$$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

$$= \cot x \cot y$$

11.  $\sin x + 2 \cos 2y = 1$

$$\cos x - 4(\sin 2y)y' = 0$$

$$y' = \frac{\cos x}{4 \sin 2y}$$

12.  $(\sin \pi x + \cos \pi y)^2 = 2$

$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$

$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$

$$y' = \frac{\cos \pi x}{\sin \pi y}$$

13.  $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

14.  $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

15.  $y = \sin(xy)$

$$y' = [xy' + y] \cos(xy)$$

$$y' - x \cos(xy)y' = y \cos(xy)$$

$$y' = \frac{y \cos(xy)}{1 - x \cos(xy)}$$

16.  $x = \sec \frac{1}{y}$

$$1 = -\frac{y'}{y^2} \sec \frac{1}{y} \tan \frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y) \tan(1/y)}$$

$$= -y^2 \cos\left(\frac{1}{y}\right) \cot\left(\frac{1}{y}\right)$$

17. (a)  $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

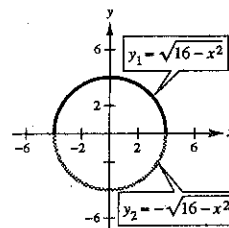
$$y = \pm \sqrt{16 - x^2}$$

(c) Explicitly:

$$\frac{dy}{dx} = \pm \frac{1}{2}(16 - x^2)^{-1/2}(-2x)$$

$$= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm \sqrt{16 - x^2}} = \frac{-x}{y}$$

(b)

(d) Implicitly:  $2x + 2yy' = 0$ 

$$y' = -\frac{x}{y}$$

18. (a)  $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$$(x - 2)^2 + (y + 3)^2 = 4, \text{ Circle}$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(c) Explicitly:

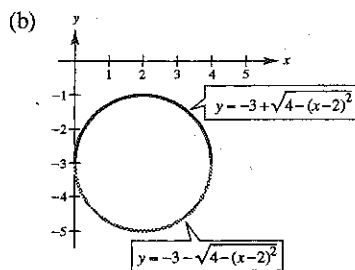
$$\frac{dy}{dx} = \pm \frac{1}{2} [4 - (x - 2)^2]^{-1/2} (-2)(x - 2)$$

$$= \frac{\mp(x - 2)}{\sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{\pm \sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$

$$= \frac{-(x - 2)}{y + 3}$$



(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

$$(2y + 6)y' = -2(x - 2)$$

$$y' = \frac{-(x - 2)}{y + 3}$$

19. (a)  $16y^2 = 144 - 9x^2$

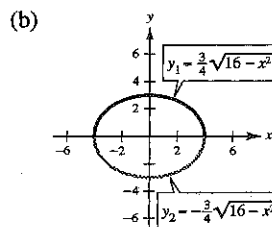
$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm \frac{3}{4} \sqrt{16 - x^2}$$

(c) Explicitly:

$$\frac{dy}{dx} = \pm \frac{3}{8} (16 - x^2)^{-1/2} (-2x)$$

$$= \mp \frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y}$$



(d) Implicitly:  $18x + 32yy' = 0$

$$y' = \frac{-9x}{16y}$$

20. (a)  $9y^2 = x^2 + 9$

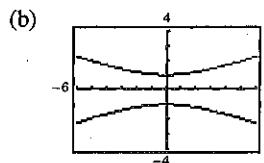
$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm \sqrt{x^2 + 9}}{3}$$

(c) Explicitly:

$$\frac{dy}{dx} = \frac{\pm \frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3}$$

$$= \frac{\pm x}{3\sqrt{x^2 + 9}} = \frac{\pm x}{3(\pm 3y)} = \frac{x}{9y}$$



(d) Implicitly:  $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

21.  $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

At  $(-4, -1)$ :  $y' = -\frac{1}{4}$

22.  $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At  $(1, 1)$ :  $y' = \frac{2}{3}$



$$23. \quad y^2 = \frac{x^2 - 4}{x^2 + 4}$$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At (2, 0):  $y'$  is undefined.

$$25. \quad x^{2/3} + y^{2/3} = 5$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

$$\text{At (8, 1): } y' = -\frac{1}{2}$$

$$27. \quad \tan(x + y) = x$$

$$(1 + y') \sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1}$$

$$= -\sin^2(x + y)$$

$$= -\frac{x^2}{x^2 + 1}$$

At (0, 0):  $y' = 0$

$$29. \quad (x^2 + 4)y = 8$$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

$$\text{At (2, 1): } y' = \frac{-32}{64} = -\frac{1}{2}$$

(Or, you could just solve for  $y$ :  $y = \frac{8}{x^2 + 4}$ )

$$24. \quad (x + y)^3 = x^3 + y^3$$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At (-1, 1):  $y' = -1$

$$26. \quad x^3 + y^3 = 4xy + 1$$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

$$\text{At (2, 1): } y' = \frac{4 - 12}{3 - 8} = \frac{8}{5}$$

$$28. \quad x \cos y = 1$$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x} \cot y$$

$$= \frac{\cot y}{x}$$

$$\text{At } \left(2, \frac{\pi}{3}\right): y' = \frac{1}{2\sqrt{3}}$$

$$30. \quad (4 - x)y^2 = x^3$$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At (2, 2):  $y' = 2$

$$\begin{aligned}
 31. \quad & (x^2 + y^2)^2 = 4x^2y \\
 & 2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x) \\
 & 4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy \\
 & 4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2 \\
 & 4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2) \\
 & y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}
 \end{aligned}$$

$$\text{At } (1, 1): y' = 0$$

$$\begin{aligned}
 33. \quad & (y - 2)^2 = 4(x - 3), \quad (4, 0) \\
 & 2(y - 2)y' = 4 \\
 & y' = \frac{2}{y - 2}
 \end{aligned}$$

$$\text{At } (4, 0): y' = -1$$

$$\begin{aligned}
 \text{Tangent line: } & y - 0 = -1(x - 4) \\
 & y = -x + 4
 \end{aligned}$$

$$\begin{aligned}
 35. \quad & xy = 1, \quad (1, 1) \\
 & xy' + y = 0 \\
 & y' = \frac{-y}{x}
 \end{aligned}$$

$$\text{At } (1, 1): y' = -1$$

$$\begin{aligned}
 \text{Tangent line: } & y - 1 = -1(x - 1) \\
 & y = -x + 2
 \end{aligned}$$

$$\begin{aligned}
 37. \quad & x^2y^2 - 9x^2 - 4y^2 = 0, \quad (-4, 2\sqrt{3}) \\
 & x^2 2yy' + 2xy^2 - 18x - 8yy' = 0 \\
 & y' = \frac{18x - 2xy^2}{2x^2y - 8y}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } (-4, 2\sqrt{3}): y' &= \frac{18(-4) - 2(-4)(12)}{2(16)(2\sqrt{3}) - 16\sqrt{3}} \\
 &= \frac{24}{48\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}
 \end{aligned}$$

$$\text{Tangent line: } y - 2\sqrt{3} = \frac{\sqrt{3}}{6}(x + 4)$$

$$y = \frac{\sqrt{3}}{6}x + \frac{8}{3}\sqrt{3}$$

$$\begin{aligned}
 32. \quad & x^3 + y^3 - 6xy = 0 \\
 & 3x^2 + 3y^2y' - 6xy' - 6y = 0 \\
 & y'(3y^2 - 6x) = 6y - 3x^2 \\
 & y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}
 \end{aligned}$$

$$\text{At } \left(\frac{4}{3}, \frac{8}{3}\right): y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$$

$$\begin{aligned}
 34. \quad & (x + 1)^2 + (y - 2)^2 = 20, \quad (3, 4) \\
 & 2(x + 1) + 2(y - 2)y' = 0 \\
 & 2(y - 2)y' = -2(x + 1) \\
 & y' = \frac{-(x + 1)}{y - 2}
 \end{aligned}$$

$$\text{At } (3, 4): y' = -2$$

$$\begin{aligned}
 \text{Tangent line: } & y - 4 = -2(x - 3) \\
 & y = -2x + 10
 \end{aligned}$$

$$\begin{aligned}
 36. \quad & 7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0, \quad (\sqrt{3}, 1) \\
 & 14x - 6\sqrt{3}xy' - 6\sqrt{3}y + 26yy' = 0 \\
 & y' = \frac{6\sqrt{3}y - 14x}{26y - 6\sqrt{3}x}
 \end{aligned}$$

$$\text{At } (\sqrt{3}, 1): y' = \frac{6\sqrt{3} - 14\sqrt{3}}{26 - 6\sqrt{3}\sqrt{3}} = \frac{-8\sqrt{3}}{8} = -\sqrt{3}$$

$$\begin{aligned}
 \text{Tangent line: } & y - 1 = -\sqrt{3}(x - \sqrt{3}) \\
 & y = -\sqrt{3}x + 4
 \end{aligned}$$

$$38. \quad x^{2/3} + y^{2/3} = 5, \quad (8, 1)$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\text{At } (8, 1): y' = -\frac{1}{2}$$

$$\text{Tangent line: } y - 1 = -\frac{1}{2}(x - 8)$$

$$y = -\frac{1}{2}x + 5$$

39.  $3(x^2 + y^2)^2 = 100(x^2 - y^2), (4, 2)$

$$6(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy')$$

At (4, 2):

$$6(16 + 4)(8 + 4y') = 100(8 - 4y')$$

$$960 + 480y' = 800 - 400y'$$

$$880y' = -160$$

$$y' = -\frac{2}{11}$$

Tangent line:  $y - 2 = -\frac{2}{11}(x - 4)$

$$11y + 2x - 30 = 0$$

$$y = -\frac{2}{11}x + \frac{30}{11}$$

40.  $y^2(x^2 + y^2) = 2x^2, (1, 1)$

$$y^2x^2 + y^4 = 2x^2$$

$$2yy'x^2 + 2xy^2 + 4y^3y' = 4x$$

At (1, 1):

$$2y' + 2 + 4y' = 4$$

$$6y' = 2$$

$$y' = \frac{1}{3}$$

Tangent line:  $y - 1 = \frac{1}{3}(x - 1)$

$$y = \frac{1}{3}x + \frac{2}{3}$$

41. (a)  $\frac{x^2}{2} + \frac{y^2}{8} = 1, (1, 2)$

$$x + \frac{yy'}{4} = 0$$

$$y' = -\frac{4x}{y}$$

At (1, 2):  $y' = -2$

Tangent line:  $y - 2 = -2(x - 1)$

$$y = -2x + 4$$

Note: From part (a),  $\frac{1(x)}{2} + \frac{2(y)}{8} = 1 \Rightarrow \frac{1}{4}y = -\frac{1}{2}x + 1 \Rightarrow y = -2x + 4$ , Tangent line.

(b)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{-b^2x}{a^2y}$

$$y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0), \text{ Tangent line at } (x_0, y_0)$$

$$\frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = \frac{-x_0x}{a^2} + \frac{x_0^2}{a^2}$$

Since  $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$ , you have  $\frac{y_0y}{b^2} + \frac{x_0x}{a^2} = 1$ .

42. (a)  $\frac{x^2}{6} - \frac{y^2}{8} = 1, (3, -2)$

$$\frac{x}{3} - \frac{y}{4}y' = 0$$

$$\frac{y}{4}y' = \frac{x}{3}$$

$$y' = \frac{4x}{3y}$$

At (3, -2):  $y' = \frac{4(3)}{3(-2)} = -2$

Tangent line:  $y + 2 = -2(x - 3)$

$$y = -2x + 4$$

Note: From part (a),  $\frac{3x}{6} - \frac{(-2)y}{8} = 1 \Rightarrow \frac{1}{2}x + \frac{y}{4} = 1 \Rightarrow y = -2x + 4$ , Tangent line.

(b)  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{xb^2}{ya^2}$

$$y - y_0 = \frac{x_0b^2}{y_0a^2}(x - x_0), \text{ Tangent line at } (x_0, y_0)$$

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2}$$

Since  $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$ , you have  $\frac{x_0x}{a^2} - \frac{yy_0}{b^2} = 1$ .

43.  $\tan y = x$

$$y' \sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

44.  $\cos y = x$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{-1}{\sin y}, \quad 0 < y < \pi$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

45.  $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x(-x/y)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

46.

$$x^2y^2 - 2x = 3$$

$$2x^2yy' + 2xy^2 - 2 = 0$$

$$x^2yy' + xy^2 - 1 = 0$$

$$y' = \frac{1 - xy^2}{x^2y}$$

$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$

$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$

$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$

$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$

$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$

$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

47.  $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

48.  $1 - xy = x - y$

$$y - xy = x - 1$$

$$y = \frac{x - 1}{1 - x} = -1$$

$$y' = 0$$

$$y'' = 0$$

49.  $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy}$$

$$= \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

50.  $y^2 = 4x$

$2yy' = 4$

$y' = \frac{2}{y}$

$y'' = -2y^{-2}y' = \left[\frac{-2}{y^2}\right] \cdot \frac{2}{y} = \frac{-4}{y^3}$

51.  $\sqrt{x} + \sqrt{y} = 4$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

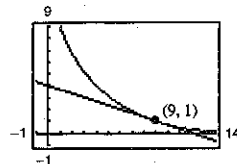
$y' = \frac{-\sqrt{y}}{\sqrt{x}}$

At (9, 1):  $y' = -\frac{1}{3}$

Tangent line:  $y - 1 = -\frac{1}{3}(x - 9)$

$y = -\frac{1}{3}x + 4$

$x + 3y - 12 = 0$



52.  $y^2 = \frac{x-1}{x^2+1}$

$2yy' = \frac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2}$

$= \frac{x^2+1-2x^2+2x}{(x^2+1)^2}$

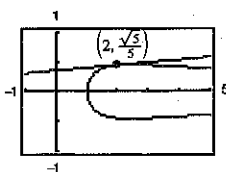
$y' = \frac{1+2x-x^2}{2y(x^2+1)^2}$

At  $(2, \frac{\sqrt{5}}{5})$ :  $y' = \frac{1+4-4}{[(2\sqrt{5})/5](4+1)^2} = \frac{1}{10\sqrt{5}}$

Tangent line:  $y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$

$10\sqrt{5}y - 10 = x - 2$

$x - 10\sqrt{5}y + 8 = 0$



53.  $x^2 + y^2 = 25$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$

At (4, 3):

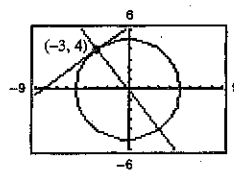
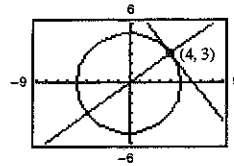
Tangent line:  $y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$

Normal line:  $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0$

At (-3, 4):

Tangent line:  $y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$

Normal line:  $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0$



54.  $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

At  $(0, 3)$ :

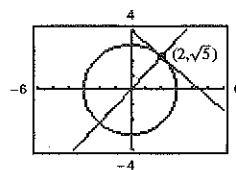
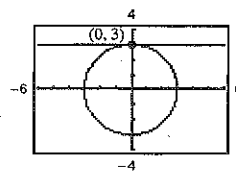
Tangent line:  $y = 3$

Normal line:  $x = 0$

At  $(2, \sqrt{5})$ :

Tangent line:  $y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2) \Rightarrow 2x + \sqrt{5}y - 9 = 0$

Normal line:  $y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2) \Rightarrow \sqrt{5}x - 2y = 0$



55.  $x^2 + y^2 = r^2$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let  $(x_0, y_0)$  be a point on the circle. If  $x_0 = 0$ , then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If  $x_0 \neq 0$ , then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

57.  $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$

Horizontal tangents occur when  $x = -4$ :

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

Horizontal tangents:  $(-4, 0), (-4, 10)$ Vertical tangents occur when  $y = 5$ :

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

Vertical tangents:  $(0, 5), (-8, 5)$ 

56.  $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal line at  $(1, 2)$  is  $y - 2 = -1(x - 1)$ ,  $y = 3 - x$ . The centers of the circles must be on the normal line and at a distance of 4 units from  $(1, 2)$ . Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

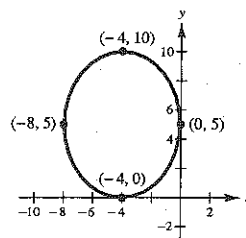
$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}$$

Centers of the circles:  $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$  and  $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

Equations:  $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$$



58.  $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when  $x = 1$ :

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \implies y = 0, -4$$

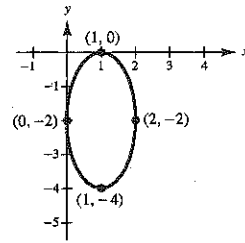
Horizontal tangents:  $(1, 0), (1, -4)$

Vertical tangents occur when  $y = -2$ :

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

$$4x^2 - 8x = 4x(x - 2) = 0 \implies x = 0, 2$$

Vertical tangents:  $(0, -2), (2, -2)$



59. Find the points of intersection by letting  $y^2 = 4x$  in the equation  $2x^2 + y^2 = 6$ .

$$2x^2 + 4x = 6 \quad \text{and} \quad (x + 3)(x - 1) = 0$$

The curves intersect at  $(1, \pm 2)$ .

Ellipse:

$$4x + 2yy' = 0$$

$$y' = -\frac{2x}{y}$$

Parabola:

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

At  $(1, 2)$ , the slopes are:

$$y' = -1$$

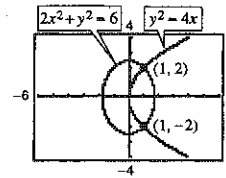
$$y' = 1$$

At  $(1, -2)$ , the slopes are:

$$y' = 1$$

$$y' = -1$$

Tangents are perpendicular.



60. Find the points of intersection by letting  $y^2 = x^3$  in the equation  $2x^2 + 3y^2 = 5$ .

$$2x^2 + 3x^3 = 5 \quad \text{and} \quad 3x^3 + 2x^2 - 5 = 0$$

Intersect when  $x = 1$ .

Points of intersection:  $(1, \pm 1)$

$$\underline{y^2 = x^3:}$$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y}$$

$$\underline{2x^2 + 3y^2 = 5:}$$

$$4x + 6yy' = 0$$

$$y' = -\frac{2x}{3y}$$

At  $(1, 1)$ , the slopes are:

$$y' = \frac{3}{2}$$

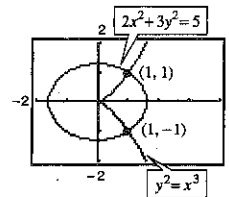
$$y' = -\frac{2}{3}$$

At  $(1, -1)$ , the slopes are:

$$y' = -\frac{3}{2}$$

$$y' = \frac{2}{3}$$

Tangents are perpendicular.



61.  $y = -x$  and  $x = \sin y$

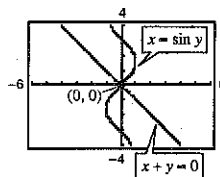
 Point of intersection:  $(0, 0)$ 

$$\begin{aligned} y = -x: & & x = \sin y: \\ y' = -1 & & 1 = y' \cos y \\ & & y' = \sec y \end{aligned}$$

 At  $(0, 0)$ , the slopes are:

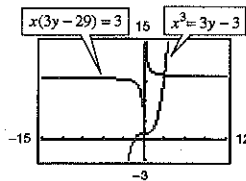
$$y' = -1 \quad y' = 1$$

Tangents are perpendicular.

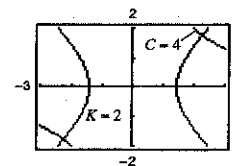
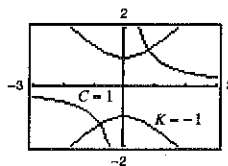


62. Rewriting each equation and differentiating:

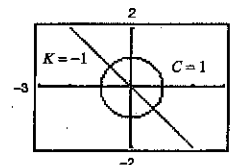
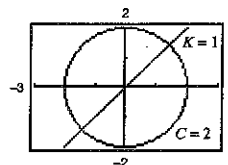
$$\begin{aligned} x^3 &= 3(y - 1) & x(3y - 29) &= 3 \\ y &= \frac{x^3}{3} + 1 & y &= \frac{1}{3}\left(\frac{3}{x} + 29\right) \\ y' &= x^2 & y' &= -\frac{1}{x^2} \end{aligned}$$


 For each value of  $x$ , the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

63.  $xy = C$        $x^2 - y^2 = K$   
 $xy' + y = 0$        $2x - 2yy' = 0$   
 $y' = -\frac{y}{x}$        $y' = \frac{x}{y}$


 At any point of intersection  $(x, y)$  the product of the slopes is  $(-y/x)(x/y) = -1$ . The curves are orthogonal.

64.  $x^2 + y^2 = C^2$        $y = Kx$   
 $2x + 2yy' = 0$        $y' = K$   
 $y' = -\frac{x}{y}$


 At the point of intersection  $(x, y)$ , the product of the slopes is  $(-x/y)(K) = (-x/Kx)(K) = -1$ . The curves are orthogonal.

65.  $2y^2 - 3x^4 = 0$

(a)  $4yy' - 12x^3 = 0$   
 $4yy' = 12x^3$   
 $y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$

(b)  $4y \frac{dy}{dt} - 12x^3 \frac{dx}{dt} = 0$   
 $y \frac{dy}{dt} = 3x^3 \frac{dx}{dt}$

66.  $x^2 - 3xy^2 + y^3 = 10$

(a)  $2x - 3y^2 - 6xyy' + 3y^2y' = 0$   
 $(-6xy + 3y^2)y' = 3y^2 - 2x$   
 $y' = \frac{3y^2 - 2x}{3y^2 - 6xy}$

(b)  $2x \frac{dx}{dt} - 3y^2 \frac{dx}{dt} - 6xy \frac{dy}{dt} + 3y^2 \frac{dy}{dt} = 0$   
 $(2x - 3y^2) \frac{dx}{dt} = (6xy - 3y^2) \frac{dy}{dt}$



67.  $\cos \pi y - 3 \sin \pi x = 1$

(a)  $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

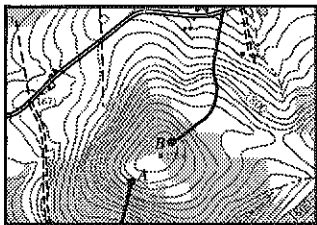
$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b)  $-\pi \sin(\pi y) \frac{dy}{dt} - 3\pi \cos(\pi x) \frac{dx}{dt} = 0$

$$-\sin(\pi y) \frac{dy}{dt} = 3 \cos(\pi x) \frac{dx}{dt}$$

69. A function is in explicit form if  $y$  is written as a function of  $x$ :  $y = f(x)$ . For example,  $y = x^3$ . An implicit equation is not in the form  $y = f(x)$ . For example,  $x^2 + y^2 = 5$ .

71.



Use starting point B.

73. (a)  $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm \sqrt{4x^2 - \frac{1}{4}x^4}$$

(b)  $y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$$36 = 16x^2 - x^4$$

$$x^4 - 16x^2 + 36 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

Note that  $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$ . Hence, there are four values of  $x$ :

$$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$$

To find the slope,  $2yy' = 8x - x^3 \Rightarrow y' = \frac{x(8 - x^2)}{2(3)}$ .

—CONTINUED—

68.  $4 \sin x \cos y = 1$

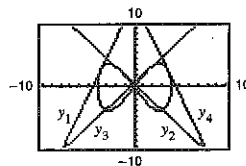
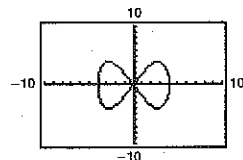
(a)  $4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

(b)  $4 \sin x(-\sin y) \frac{dy}{dt} + 4 \cos x \frac{dx}{dt} \cos y = 0$

$$\cos x \cos y \frac{dx}{dt} = \sin x \sin y \frac{dy}{dt}$$

70. Given an implicit equation, first differentiate both sides with respect to  $x$ . Collect all terms involving  $y'$  on the left, and all other terms to the right. Factor out  $y'$  on the left side. Finally, divide both sides by the left-hand factor that does not contain  $y'$ .

72. Highest wind speed near  $L$ 

## 73. —CONTINUED—

For  $x = -1 - \sqrt{7}$ ,  $y' = \frac{1}{3}(\sqrt{7} + 7)$ , and the line is

$$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23].$$

For  $x = 1 - \sqrt{7}$ ,  $y' = \frac{1}{3}(\sqrt{7} - 7)$ , and the line is

$$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}].$$

For  $x = -1 + \sqrt{7}$ ,  $y' = -\frac{1}{3}(\sqrt{7} - 7)$ , and the line is

$$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})].$$

For  $x = 1 + \sqrt{7}$ ,  $y' = -\frac{1}{3}(\sqrt{7} + 7)$ , and the line is

$$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)].$$

(c) Equating  $y_3$  and  $y_4$ :

$$\begin{aligned} -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 &= -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 \\ (\sqrt{7} - 7)(x + 1 - \sqrt{7}) &= (\sqrt{7} + 7)(x - 1 - \sqrt{7}) \\ \sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} &= \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7} \\ 16\sqrt{7} &= 14x \\ x &= \frac{8\sqrt{7}}{7} \end{aligned}$$

If  $x = \frac{8\sqrt{7}}{7}$ , then  $y = 5$  and the lines intersect at  $\left(\frac{8\sqrt{7}}{7}, 5\right)$ .

74.  $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at  $(x_0, y_0)$ :  $y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$

x-intercept:  $(x_0 + \sqrt{x_0\sqrt{y_0}}, 0)$

y-intercept:  $(0, y_0 + \sqrt{x_0\sqrt{y_0}})$

Sum of intercepts:

$$\begin{aligned} (x_0 + \sqrt{x_0\sqrt{y_0}}) + (y_0 + \sqrt{x_0\sqrt{y_0}}) &= x_0 + 2\sqrt{x_0\sqrt{y_0}} + y_0 \\ &= (\sqrt{x_0} + \sqrt{y_0})^2 \\ &= (\sqrt{c})^2 = c \end{aligned}$$

75.  $y = x^{p/q}$ ;  $p, q$  integers and  $q > 0$

$$y^q = x^p$$

$$qy^{q-1}y' = px^{p-1}$$

$$y' = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}y}{y^q}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^p} x^{p/q} = \frac{p}{q} x^{p/q-1}$$

Thus, if  $y = x^n$ ,  $n = p/q$ , then  $y' = nx^{n-1}$ .

76.  $x^2 + y^2 = 25$ , slope =  $\frac{3}{4}$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} = \frac{3}{4} \Rightarrow y = -\frac{4}{3}x$$

$$x^2 + \left(\frac{16}{9}x^2\right) = 25$$

$$\frac{25}{9}x^2 = 25$$

$$x = \pm 3$$

Points: (3, -4) and (-3, 4)

78.  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ , (4, 0)

$$\frac{2x}{4} + \frac{2yy'}{9} = 0$$

$$y' = -\frac{9x}{4y}$$

$$\frac{-9x}{4y} = \frac{y-0}{x-4}$$

$$-9x(x-4) = 4y^2$$

But,  $9x^2 + 4y^2 = 36 \Rightarrow 4y^2 = 36 - 9x^2$ . Hence,  $-9x^2 + 36x = 4y^2 = 36 - 9x^2 \Rightarrow x = 1$ .

Points on ellipse:  $\left(1, \pm\frac{3}{2}\sqrt{3}\right)$

$$\text{At } \left(1, \frac{3}{2}\sqrt{3}\right): y' = \frac{-9x}{4y} = \frac{-9}{4\left[\frac{3}{2}\sqrt{3}\right]} = -\frac{\sqrt{3}}{2}$$

$$\text{At } \left(1, -\frac{3}{2}\sqrt{3}\right): y' = \frac{\sqrt{3}}{2}$$

$$\text{Tangent lines: } y = -\frac{\sqrt{3}}{2}(x-4) = -\frac{\sqrt{3}}{2}x + 2\sqrt{3}$$

$$y = \frac{\sqrt{3}}{2}(x-4) = \frac{\sqrt{3}}{2}x - 2\sqrt{3}$$

79.  $x = y^2$

$$1 = 2yy'$$

$$y' = \frac{1}{2y}, \text{ slope of tangent line}$$

Consider the slope of the normal line joining  $(x_0, 0)$  and  $(x, y) = (y^2, y)$  on the parabola.

$$-2y = \frac{y-0}{y^2-x_0}$$

$$y^2 - x_0 = -\frac{1}{2}$$

$$y^2 = x_0 - \frac{1}{2}$$

77.  $y^4 = y^2 - x^2$

$$4y^3y' = 2yy' - 2x$$

$$2x = (2y - 4y^3)y'$$

$$y' = \frac{2x}{2y - 4y^3} = 0 \Rightarrow x = 0$$

Horizontal tangents at (0, 1) and (0, -1)

Note:  $y^4 - y^2 + x^2 = 0$

$$y^2 = \frac{1 \pm \sqrt{1 - 4x^2}}{2}$$

If you graph these four equations, you will see that these are horizontal tangents at  $(0, \pm 1)$ , but not at  $(0, 0)$ .

## 79. —CONTINUED—

(a) If  $x_0 = \frac{1}{4}$ , then  $y^2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ , which is impossible. Thus, the only normal line is the  $x$ -axis ( $y = 0$ ).

(b) If  $x_0 = \frac{1}{2}$ , then  $y^2 = 0 \Rightarrow y = 0$ . Same as part (a).

(c) If  $x_0 = 1$ , then  $y^2 = \frac{1}{2} = x$  and there are three normal lines:

The  $x$ -axis, the line joining  $(x_0, 0)$  and  $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ , and the line joining  $(x_0, 0)$  and  $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$

If two normals are perpendicular, then their slopes are  $-1$  and  $1$ . Thus,

$$-2y = -1 = \frac{y-0}{y^2-x_0} \Rightarrow y = \frac{1}{2} \quad \text{and} \quad \frac{1/2}{(1/4)-x_0} = -1 \Rightarrow \frac{1}{4} - x_0 = -\frac{1}{2} \Rightarrow x_0 = \frac{3}{4}$$

The perpendicular normal lines are  $y = -x + \frac{3}{4}$  and  $y = x - \frac{3}{4}$ .

80. (a)  $\frac{x^2}{32} + \frac{y^2}{8} = 1$

$$\frac{2x}{32} + \frac{2yy'}{8} = 0 \Rightarrow y' = \frac{-x}{4y}$$

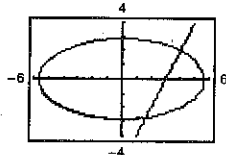
$$\text{At } (4, 2): y' = \frac{-4}{4(2)} = -\frac{1}{2}$$

Slope of normal line is 2.

$$y - 2 = 2(x - 4)$$

$$y = 2x - 6$$

(b)



(c)  $\frac{x^2}{32} + \frac{(2x-6)^2}{8} = 1$

$$x^2 + 4(4x^2 - 24x + 36) = 32$$

$$17x^2 - 96x + 112 = 0$$

$$(17x - 28)(x - 4) = 0 \Rightarrow x = 4, \frac{28}{17}$$

$$\text{Second point: } \left(\frac{28}{17}, -\frac{46}{17}\right)$$

## Section 2.6 Related Rates

1.  $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt}$$

(a) When  $x = 4$  and  $dx/dt = 3$ ,

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}$$

(b) When  $x = 25$  and  $dy/dt = 2$ ,

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20.$$

2.  $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6} \frac{dy}{dt}$$

(a) When  $x = 3$  and

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = [4(3) - 6](2) = 12.$$

(b) When  $x = 1$  and

$$\frac{dy}{dt} = 5, \frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}.$$

3.  $xy = 4$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right) \frac{dy}{dt}$$

(a) When  $x = 8, y = 1/2$ , and  $dx/dt = 10$ ,

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}$$

(b) When  $x = 1, y = 4$ , and  $dy/dt = -6$ ,

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}$$

5.  $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

(a) When  $x = -1$ ,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When  $x = 0$ ,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When  $x = 1$ ,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

7.  $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x \frac{dx}{dt}$$

(a) When  $x = -\pi/3$ ,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When  $x = -\pi/4$ ,

$$\frac{dy}{dt} = (\sqrt{2})^2(2) = 4 \text{ cm/sec.}$$

(c) When  $x = 0$ ,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

4.  $x^2 + y^2 = 25$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right) \frac{dy}{dt}$$

(a) When  $x = 3, y = 4$ , and  $dx/dt = 8$ ,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When  $x = 4, y = 3$ , and  $dy/dt = -2$ ,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}$$

6.  $y = \frac{1}{1+x^2}$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \left[ \frac{-2x}{(1+x^2)^2} \right] \frac{dx}{dt}$$

(a) When  $x = -2$ ,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When  $x = 0$ ,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When  $x = 2$ ,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = -\frac{8}{25} \text{ cm/sec.}$$

8.  $y = \sin x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \cos x \frac{dx}{dt}$$

(a) When  $x = \pi/6$ ,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{6}\right)(2) = \sqrt{3} \text{ cm/sec.}$$

(b) When  $x = \pi/4$ ,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{4}\right)(2) = \sqrt{2} \text{ cm/sec.}$$

(c) When  $x = \pi/3$ ,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{3}\right)(2) = 1 \text{ cm/sec.}$$

9. (a)  $\frac{dx}{dt}$  negative  $\Rightarrow \frac{dy}{dt}$  positive

(b)  $\frac{dy}{dt}$  positive  $\Rightarrow \frac{dx}{dt}$  negative

11. Yes,  $y$  changes at a constant rate.

$$\frac{dy}{dt} = a \cdot \frac{dx}{dt}$$

No, the rate  $dy/dt$  is a multiple of  $dx/dt$ .

13.  $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$$\frac{dx}{dt} = 2$$

$$\begin{aligned} \frac{dD}{dt} &= \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x) \frac{dx}{dt} \\ &= \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}} \frac{dx}{dt} \\ &= \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}} \end{aligned}$$

15.  $A = \pi r^2$

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(a) When  $r = 6$ ,  $\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2/\text{min}$ .

(b) When  $r = 24$ ,  $\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2/\text{min}$ .

17. (a)  $\sin \frac{\theta}{2} = \frac{(1/2)b}{s} \Rightarrow b = 2s \sin \frac{\theta}{2}$

$$\cos \frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s \cos \frac{\theta}{2}$$

$$\begin{aligned} A &= \frac{1}{2}bh = \frac{1}{2}\left(2s \sin \frac{\theta}{2}\right)\left(s \cos \frac{\theta}{2}\right) \\ &= \frac{s^2}{2}\left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right) = \frac{s^2}{2} \sin \theta \end{aligned}$$

(b)  $\frac{dA}{dt} = \frac{s^2}{2} \cos \theta \frac{d\theta}{dt}$  where  $\frac{d\theta}{dt} = \frac{1}{2} \text{ rad/min}$ .

When  $\theta = \frac{\pi}{6}$ ,  $\frac{dA}{dt} = \frac{s^2}{2} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$ .

When  $\theta = \frac{\pi}{3}$ ,  $\frac{dA}{dt} = \frac{s^2}{2} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{s^2}{8}$ .

10. (a)  $\frac{dx}{dt}$  negative  $\Rightarrow \frac{dy}{dt}$  negative

(b)  $\frac{dy}{dt}$  positive  $\Rightarrow \frac{dx}{dt}$  positive

12. Answers will vary. See page 149.

14.  $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$

$$\frac{dx}{dt} = 2$$

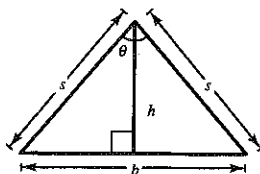
$$\begin{aligned} \frac{dD}{dt} &= \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2 \sin x \cos x) \frac{dx}{dt} \\ &= \frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}} \frac{dx}{dt} \\ &= \frac{2x + 2 \sin x \cos x}{\sqrt{x^2 + \sin^2 x}} \end{aligned}$$

16.  $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

If  $dr/dt$  is constant,  $dA/dt$  is not constant.

$dA/dt$  depends on  $r$  and  $dr/dt$ .



(c) If  $\frac{d\theta}{dt}$  is constant,  $\frac{dA}{dt}$  is proportional to  $\cos \theta$ .

18.  $V = \frac{4}{3}\pi r^3$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(a) When  $r = 6$ ,

$$\frac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{min.}$$

When  $r = 24$ ,

$$\frac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3/\text{min.}$$

(b) If  $dr/dt$  is constant,  $dV/dt$  is proportional to  $r^2$ .

20.  $V = x^3$

$$\frac{dx}{dt} = 3$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

(a) When  $x = 1$ ,

$$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3/\text{sec.}$$

(b) When  $x = 10$ ,

$$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3/\text{sec.}$$

22.  $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

(a) When  $r = 6$ ,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min.}$$

(b) When  $r = 24$ ,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min.}$$

19.  $V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \left( \frac{dV}{dt} \right) = \frac{1}{4\pi r^2} (800)$$

(a) When  $r = 30$ ,

$$\frac{dr}{dt} = \frac{1}{4\pi(30)^2} (800) = \frac{2}{9\pi} \text{ cm/min.}$$

(b) When  $r = 60$ ,

$$\frac{dr}{dt} = \frac{1}{4\pi(60)^2} (800) = \frac{1}{18\pi} \text{ cm/min.}$$

21.  $s = 6x^2$

$$\frac{dx}{dt} = 3$$

$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

(a) When  $x = 1$ ,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec.}$$

(b) When  $x = 10$ ,

$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec.}$$

23.  $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left( \frac{9}{4}h^2 \right) h$  [since  $2r = 3h$ ]

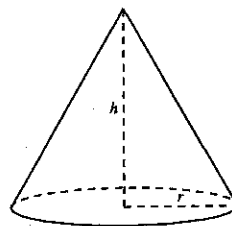
$$= \frac{3\pi}{4}h^3$$

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When  $h = 15$ ,

$$\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi} \text{ ft/min.}$$

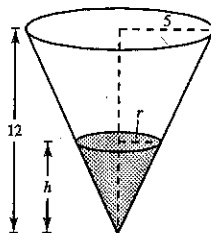


$$24. V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144} h^3 = \frac{25\pi}{3(144)} h^3 \quad \left(\text{By similar triangles, } \frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h.\right)$$

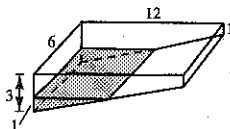
$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{25\pi}{144} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right) \frac{dV}{dt}$$

$$\text{When } h = 8, \frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi} \text{ ft/min.}$$



25.



$$(a) \text{ Total volume of pool} = \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$$

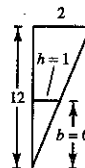
$$\text{Volume of 1 m of water} = \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3 \quad (\text{see similar triangle diagram})$$

$$\% \text{ pool filled} = \frac{18}{144}(100\%) = 12.5\%$$

(b) Since for  $0 \leq h \leq 2$ ,  $b = 6h$ , you have

$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h \frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

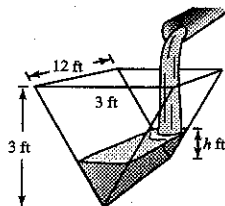


$$26. V = \frac{1}{2}bh(12) = 6bh = 6h^2 \quad (\text{since } b = h)$$

$$(a) \frac{dV}{dt} = 12h \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt}$$

$$\text{When } h = 1 \text{ and } \frac{dV}{dt} = 2, \frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6} \text{ ft/min.}$$

$$(b) \text{ If } \frac{dh}{dt} = \frac{3}{8} \text{ in./min} = \frac{1}{32} \text{ ft/min and } h = 2 \text{ feet, then } \frac{dV}{dt} = (12)(2)\left(\frac{1}{32}\right) = \frac{3}{4} \text{ ft}^3/\text{min.}$$



$$27. \quad x^2 + y^2 = 25^2$$

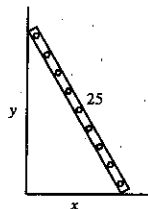
$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \quad \text{since } \frac{dx}{dt} = 2.$$

$$(a) \text{ When } x = 7, y = \sqrt{576} = 24, \frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12} \text{ ft/sec.}$$

$$\text{When } x = 15, y = \sqrt{400} = 20, \frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2} \text{ ft/sec.}$$

$$\text{When } x = 24, y = 7, \frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7} \text{ ft/sec.}$$



—CONTINUED—



## 27. —CONTINUED—

(b)  $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x \frac{dy}{dt} + y \frac{dx}{dt}\right)$$

From part (a) we have  $x = 7$ ,  $y = 24$ ,  $\frac{dx}{dt} = 2$ , and  $\frac{dy}{dt} = -\frac{7}{12}$ . Thus,

$$\frac{dA}{dt} = \frac{1}{2}\left[7\left(-\frac{7}{12}\right) + 24(2)\right] = \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec.}$$

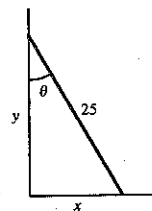
(c)  $\tan \theta = \frac{x}{y}$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left[ \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt} \right]$$

Using  $x = 7$ ,  $y = 24$ ,  $\frac{dx}{dt} = 2$ ,  $\frac{dy}{dt} = -\frac{7}{12}$  and  $\cos \theta = \frac{24}{25}$ , we have

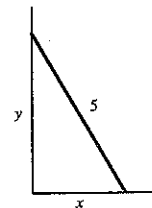
$$\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2 \left[ \frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right) \right] = \frac{1}{12} \text{ rad/sec.}$$



28.  $x^2 + y^2 = 25$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x} \quad \left(\text{since } \frac{dy}{dt} = 0.15\right)$$

When  $x = 2.5$ ,  $y = \sqrt{18.75}$ ,  $\frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5} \cdot 0.15 \approx -0.26 \text{ m/sec.}$ 

29. When  $y = 6$ ,  $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$ , and  $s = \sqrt{x^2 + (12 - y)^2} = \sqrt{108 + 36} = 12$ .

$$x^2 + (12 - y)^2 = s^2$$

$$2x \frac{dx}{dt} + 2(12 - y)(-1) \frac{dy}{dt} = 2s \frac{ds}{dt}$$

$$x \frac{dx}{dt} + (y - 12) \frac{dy}{dt} = s \frac{ds}{dt}$$

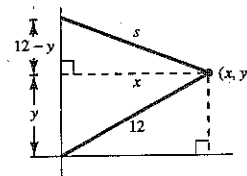
Also,  $x^2 + y^2 = 12^2$ .

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

Thus,  $x \frac{dx}{dt} + (y - 12) \left(-\frac{x}{y} \frac{dx}{dt}\right) = s \frac{ds}{dt}$

$$\frac{dx}{dt} \left[ x - x + \frac{12x}{y} \right] = s \frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})} (-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \left(\frac{-\sqrt{3}}{15}\right) = \frac{1}{5} \text{ m/sec (vertical)}$$



30. Let  $L$  be the length of the rope.

(a)  $L^2 = 144 + x^2$

$$2L \frac{dL}{dt} = 2x \frac{dx}{dt}$$

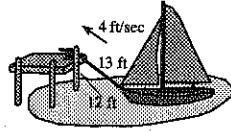
$$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x} \quad \left( \text{since } \frac{dL}{dt} = -4 \text{ ft/sec} \right)$$

When  $L = 13$ :

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec}$$

Speed of the boat increases as it approaches the dock.



(b) If  $\frac{dx}{dt} = -4$ , and  $L = 13$ :

$$\frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt} = \frac{5}{13}(-4) = -\frac{20}{13} \text{ ft/sec}$$

$$\frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt} = \frac{\sqrt{L^2 - 144}}{L}(-4)$$

$$\lim_{L \rightarrow 12^+} \frac{dL}{dt} = \lim_{L \rightarrow 12^+} \frac{-4}{L} \sqrt{L^2 - 144} = 0$$

31. (a)  $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

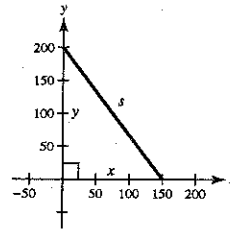
$$\frac{dy}{dt} = -600$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$

When  $x = 150$  and  $y = 200$ ,  $s = 250$  and  $\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750$  mph.

(b)  $t = \frac{250}{750} = \frac{1}{3} \text{ hr} = 20 \text{ min}$



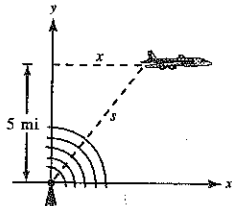
32.  $x^2 + y^2 = s^2$

$$2x \frac{dx}{dt} + 0 = 2s \frac{ds}{dt} \quad \left( \text{since } \frac{dy}{dt} = 0 \right)$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

When  $s = 10$ ,  $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$ ,

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(240) = \frac{480}{\sqrt{3}} = 160\sqrt{3} \approx 277.13 \text{ mph.}$$



33.  $s^2 = 90^2 + x^2$

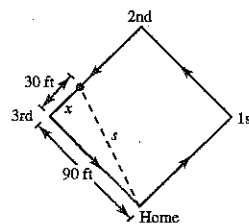
$$x = 30$$

$$\frac{dx}{dt} = -28$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt}$$

When  $x = 30$ ,  $s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$ ,

$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$



34.  $s^2 = 90^2 + x^2$

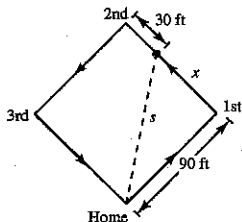
$$x = 60$$

$$\frac{dx}{dt} = 28$$

$$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When  $x = 60$ ,  $s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$ ,

$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$



36. (a)

$$\frac{20}{6} = \frac{y}{y-x}$$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$

$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7} \frac{dx}{dt} = \frac{10}{7}(-5) = -\frac{50}{7} \text{ ft/sec}$$

(b) 
$$\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$$

37.  $x(t) = \frac{1}{2} \sin \frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period:  $\frac{2\pi}{\pi/6} = 12$  seconds

(b) When  $x = \frac{1}{2}$ ,  $y = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$  m.

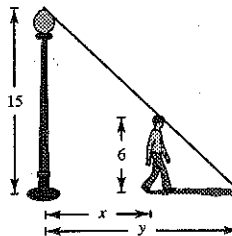
Lowest point:  $\left(0, \frac{\sqrt{3}}{2}\right)$

35. (a)  $\frac{15}{6} = \frac{y}{y-x} \Rightarrow 15y - 15x = 6y$

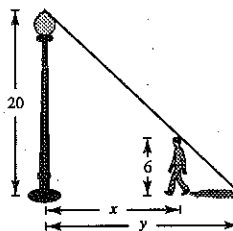
$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$



(b) 
$$\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3} \text{ ft/sec}$$



(c) When  $x = \frac{1}{4}$ ,  $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$  and  $t = 1$ :

$$\frac{dx}{dt} = \frac{1}{2} \left(\frac{\pi}{6}\right) \cos \frac{\pi t}{6} = \frac{\pi}{12} \cos \frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

Thus, 
$$\frac{dy}{dt} = -\frac{1/4}{\sqrt{15}/4} \cdot \frac{\pi}{12} \cos\left(\frac{\pi}{6}\right)$$

$$= -\frac{\pi}{\sqrt{15}} \left(\frac{1}{12}\right) \frac{\sqrt{3}}{2} = -\frac{\pi}{24} \frac{1}{\sqrt{5}} = -\frac{\sqrt{5}\pi}{120}$$

$$\text{Speed} = \left| \frac{-\sqrt{5}\pi}{120} \right| = \frac{\sqrt{5}\pi}{120} \text{ m/sec}$$

$$38. x(t) = \frac{3}{5} \sin \pi t, x^2 + y^2 = 1$$

$$(a) \text{ Period: } \frac{2\pi}{\pi} = 2 \text{ seconds}$$

$$(b) \text{ When } x = \frac{3}{5}, y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5} \text{ m.}$$

$$\text{Lowest point: } \left(0, \frac{4}{5}\right)$$

$$(c) \text{ When } x = \frac{3}{10}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \text{ and } \frac{3}{10} = \frac{3}{5} \sin \pi t \Rightarrow \sin \pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5} \pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

$$\begin{aligned} \text{Thus, } \frac{dy}{dt} &= \frac{-3/10}{\sqrt{15}/4} \cdot \frac{3}{5} \pi \cos\left(\frac{\pi}{6}\right) \\ &= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125} \end{aligned}$$

$$\text{Speed} = \left| \frac{-9\sqrt{5}\pi}{125} \right| \approx 0.5058 \text{ m/sec}$$

39. Since the evaporation rate is proportional to the surface area,  $dV/dt = k(4\pi r^2)$ . However, since  $V = (4/3)\pi r^3$ , we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\text{Therefore, } k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}$$

$$41. \quad pV^{1.3} = k$$

$$1.3pV^{0.3} \frac{dV}{dt} + V^{1.3} \frac{dp}{dt} = 0$$

$$V^{0.3} \left( 1.3p \frac{dV}{dt} + V \frac{dp}{dt} \right) = 0$$

$$1.3p \frac{dV}{dt} = -V \frac{dp}{dt}$$

$$40. \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt}$$

$$\text{When } R_1 = 50 \text{ and } R_2 = 75:$$

$$R = 30$$

$$\frac{dR}{dt} = (30)^2 \left[ \frac{1}{(50)^2} (1) + \frac{1}{(75)^2} (1.5) \right] = 0.6 \text{ ohms/sec}$$

$$42. \quad rg \tan \theta = v^2$$

$$32r \tan \theta = v^2, \quad r \text{ is a constant.}$$

$$32r \sec^2 \theta \frac{d\theta}{dt} = 2v \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v} \sec^2 \theta \frac{d\theta}{dt}$$

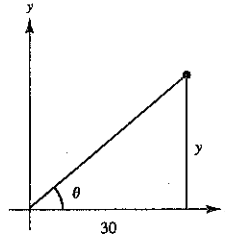
$$\text{Likewise, } \frac{d\theta}{dt} = \frac{v}{16r} \cos^2 \theta \frac{dv}{dt}$$

43.  $\tan \theta = \frac{y}{30}$

$$\frac{dy}{dt} = 3 \text{ m/sec}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{30} \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{30} \cos^2 \theta \cdot \frac{dy}{dt}$$



When  $y = 30$ ,  $\theta = \frac{\pi}{4}$ , and  $\cos \theta = \frac{\sqrt{2}}{2}$ . Thus,  $\frac{d\theta}{dt} = \frac{1}{30} \left(\frac{1}{2}\right)(3) = \frac{1}{20} \text{ rad/sec}$ .

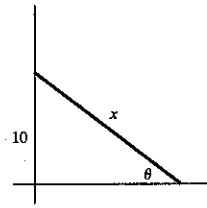
44.  $\sin \theta = \frac{10}{x}$

$$\frac{dx}{dt} = (-1) \text{ ft/sec}$$

$$\cos \theta \left(\frac{d\theta}{dt}\right) = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2} \frac{dx}{dt} (\sec \theta)$$

$$= \frac{-10}{25^2} (-1) \frac{25}{\sqrt{25^2 - 10^2}} = \frac{10}{25} \frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}} = \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$



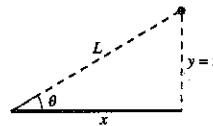
45.  $\tan \theta = \frac{y}{x}$ ,  $y = 5$

$$\frac{dx}{dt} = -600 \text{ mi/hr}$$

$$(\sec^2 \theta) \frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left(-\frac{5}{x^2}\right) \frac{dx}{dt} = \frac{x^2}{L^2} \left(-\frac{5}{x^2}\right) \frac{dx}{dt}$$

$$= \left(-\frac{5^2}{L^2}\right) \left(\frac{1}{5}\right) \frac{dx}{dt} = (-\sin^2 \theta) \left(\frac{1}{5}\right) (-600) = 120 \sin^2 \theta$$


 (a) When  $\theta = 30^\circ$ ,

$$\frac{d\theta}{dt} = \frac{120}{4}$$

$$= 30 \text{ rad/hr} = \frac{1}{2} \text{ rad/min.}$$

 (b) When  $\theta = 60^\circ$ ,

$$\frac{d\theta}{dt} = 120 \left(\frac{3}{4}\right)$$

$$= 90 \text{ rad/hr} = \frac{3}{2} \text{ rad/min.}$$

 (c) When  $\theta = 75^\circ$ ,

$$\frac{d\theta}{dt} = 120 \sin^2 75^\circ$$

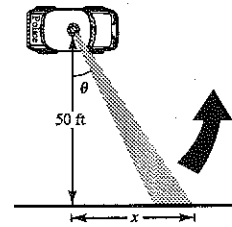
$$\approx 111.96 \text{ rad/hr} \approx 1.87 \text{ rad/min.}$$

46.  $\tan \theta = \frac{x}{50}$

$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2 \theta \left(\frac{d\theta}{dt}\right) = \frac{1}{50} \left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = 50 \sec^2 \theta \left(\frac{d\theta}{dt}\right)$$



(a) When  $\theta = 30^\circ$ ,  $\frac{dx}{dt} = \frac{200\pi}{3} \text{ ft/sec}$ . (b) When  $\theta = 60^\circ$ ,  $\frac{dx}{dt} = 200\pi \text{ ft/sec}$ . (c) When  $\theta = 70^\circ$ ,  $\frac{dx}{dt} \approx 427.43\pi \text{ ft/sec}$ .

$$47. \frac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$$

$$(a) \quad \cos \theta = \frac{x}{30}$$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{30} \frac{dx}{dt}$$

$$\frac{dx}{dt} = -30 \sin \theta \frac{d\theta}{dt}$$

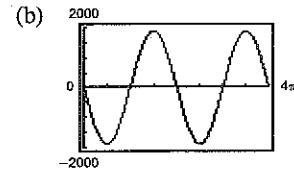
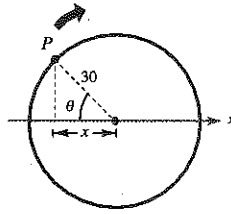
$$= -30 \sin \theta (20\pi)$$

$$= -600\pi \sin \theta$$

(c)  $|dx/dt| = |-600\pi \sin \theta|$  is greatest when

$$|\sin \theta| = 1 \Rightarrow \theta = \frac{\pi}{2} + n\pi \text{ (or } 90^\circ + n \cdot 180^\circ\text{)}.$$

$|dx/dt|$  is least when  $\theta = n\pi$  (or  $n \cdot 180^\circ$ ).



(d) For  $\theta = 30^\circ$ ,

$$\frac{dx}{dt} = -600\pi \sin(30^\circ)$$

$$= -600\pi \frac{1}{2} = -300\pi \text{ cm/sec.}$$

For  $\theta = 60^\circ$ ,

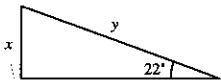
$$\frac{dx}{dt} = -600\pi \sin(60^\circ)$$

$$= -600\pi \frac{\sqrt{3}}{2} = -300\sqrt{3}\pi \text{ cm/sec.}$$

$$48. \sin 22^\circ = \frac{x}{y}$$

$$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 22^\circ)(240) \approx 89.9056 \text{ mi/hr}$$



$$49. \tan \theta = \frac{x}{50} \Rightarrow x = 50 \tan \theta$$

$$\frac{dx}{dt} = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$2 = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25} \cos^2 \theta, \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

50. (a)  $dy/dt = 3(dx/dt)$  means that  $y$  changes three times as fast as  $x$  changes.

(b)  $y$  changes slowly when  $x \approx 0$  or  $x \approx L$ .  $y$  changes more rapidly when  $x$  is near the middle of the interval.

$$51. x^2 + y^2 = 25; \text{ acceleration of the top of the ladder} = \frac{d^2y}{dt^2}$$

$$\text{First derivative: } 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

$$\text{Second derivative: } x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[ -x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When  $x = 7$ ,  $y = 24$ ,  $\frac{dy}{dt} = -\frac{7}{12}$ , and  $\frac{dx}{dt} = 2$  (see Exercise 27). Since  $\frac{dx}{dt}$  is constant,  $\frac{d^2x}{dt^2} = 0$ .

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[ -7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[ -4 - \frac{49}{144} \right] = \frac{1}{24} \left[ -\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

52.  $L^2 = 144 + x^2$ ; acceleration of the boat  $= \frac{d^2x}{dt^2}$

First derivative:  $2L \frac{dL}{dt} = 2x \frac{dx}{dt}$

$$L \frac{dL}{dt} = x \frac{dx}{dt}$$

Second derivative:  $L \frac{d^2L}{dt^2} + \frac{dL}{dt} \cdot \frac{dL}{dt} = x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right) \left[ L \frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2 \right]$$

When  $L = 13$ ,  $x = 5$ ,  $\frac{dx}{dt} = -10.4$ , and  $\frac{dL}{dt} = -4$  (see Exercise 30). Since  $\frac{dL}{dt}$  is constant,  $\frac{d^2L}{dt^2} = 0$ .

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{1}{5} [13(0) + (-4)^2 - (-10.4)^2] \\ &= \frac{1}{5} [16 - 108.16] = \frac{1}{5} [-92.16] = -18.432 \text{ ft/sec}^2 \end{aligned}$$

53. (a)  $m(s) = 0.3754s^3 - 18.780s^2 + 313.23s - 1707.8$

(b)  $\frac{dm}{dt} = (1.1262s^2 - 37.560s + 313.23) \frac{ds}{dt}$

If  $t = 10$  and  $\frac{ds}{dt} = 0.75$ , then  $s = 17.8$  and  $\frac{dm}{dt} \approx 1.1154$  million/year.

54.  $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles:  $\frac{20}{x} = \frac{y}{x-12}$

$$20x - 240 = xy$$

When  $y = 15.1$ :  $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

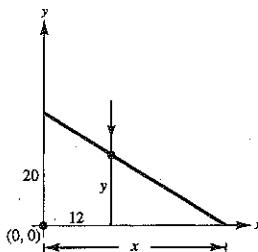
$$x = \frac{240}{4.9}$$

$$20x - 240 = xy$$

$$20 \frac{dx}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y} \frac{dy}{dt}$$

At  $t = 1$ ,  $\frac{dx}{dt} = \frac{240/4.9}{20 - 15.1} (-9.8) \approx -97.96$  m/sec.



## Review Exercises for Chapter 2

1.  $f(x) = x^2 - 2x + 3$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2
 \end{aligned}$$

2.  $f(x) = \sqrt{x} + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

3.  $f(x) = \frac{x + 1}{x - 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x + \Delta x + 1}{x + \Delta x - 1} - \frac{x + 1}{x - 1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1)(x - 1) - (x + \Delta x - 1)(x + 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + x\Delta x + x - x - \Delta x - 1) - (x^2 + x\Delta x - x + x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} = \lim_{\Delta x \rightarrow 0} \frac{-2}{(x + \Delta x - 1)(x - 1)} = \frac{-2}{(x - 1)^2}
 \end{aligned}$$

4.  $f(x) = \frac{2}{x}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{2}{x + \Delta x} - \frac{2}{x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x - (2x + 2\Delta x)}{\Delta x(x + \Delta x)x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x(x + \Delta x)x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2}{(x + \Delta x)x} = \frac{-2}{x^2}
 \end{aligned}$$

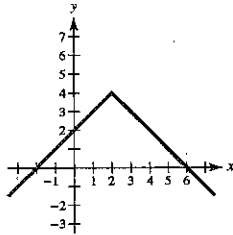


5.  $f$  is differentiable for all  $x \neq -1$ .

7.  $f(x) = 4 - |x - 2|$

(a) Continuous at  $x = 2$

(b) Not differentiable at  $x = 2$  because of the sharp turn in the graph.



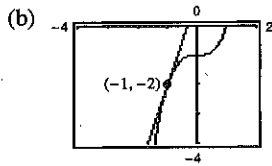
9. Using the limit definition, you obtain  $g'(x) = \frac{4}{3}x - \frac{1}{6}$ . At  $x = -1$ ,

$$g'(-1) = -\frac{4}{3} - \frac{1}{6} = -\frac{3}{2}.$$

11. (a) Using the limit definition,  $f'(x) = 3x^2$ . At  $x = -1$ ,  $f'(-1) = 3$ . The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1.$$



$$\begin{aligned} 13. \quad g'(2) &= \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^2(x - 1) - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x^2 + x + 2) = 8 \end{aligned}$$

15.  $y = 25$   
 $y' = 0$

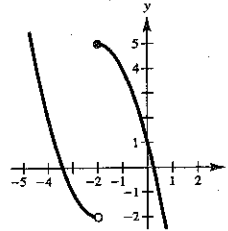
16.  $y = -12$   
 $y' = 0$

6.  $f$  is differentiable for all  $x \neq -3$ .

8.  $f(x) = \begin{cases} x^2 + 4x + 2, & \text{if } x < -2 \\ 1 - 4x - x^2, & \text{if } x \geq -2 \end{cases}$

(a) Nonremovable discontinuity at  $x = -2$ .

(b) Not differentiable at  $x = -2$  because the function is discontinuous there.



10. Using the limit definition, you obtain  $h'(x) = \frac{3}{8} - 4x$ . At  $x = -2$ ,

$$h'(-2) = \frac{3}{8} - 4(-2) = \frac{67}{8}.$$

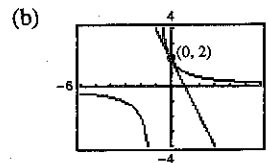
12. (a) Using the limit definition,

$$f'(x) = \frac{-2}{(x + 1)^2}.$$

At  $x = 0$ ,  $f'(0) = -2$ . The tangent line is

$$y - 2 = -2(x - 0)$$

$$y = -2x + 2.$$



$$\begin{aligned} 14. \quad f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\frac{1}{x+1} - \frac{1}{3}}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{3 - x - 1}{(x - 2)(x + 1)3} \\ &= \lim_{x \rightarrow 2} \frac{-1}{(x + 1)3} = -\frac{1}{9} \end{aligned}$$

17.  $f(x) = x^8$   
 $f'(x) = 8x^7$

18.  $g(x) = x^{12}$   
 $g'(x) = 12x^{11}$

19.  $h(t) = 3t^4$   
 $h'(t) = 12t^3$

20.  $f(t) = -8t^5$   
 $f'(t) = -40t^4$

21.  $f(x) = x^3 - 3x^2$   
 $f'(x) = 3x^2 - 6x$   
 $= 3x(x - 2)$

22.  $g(s) = 4s^4 - 5s^2$   
 $g'(s) = 16s^3 - 10s$

23.  $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$   
 $h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

24.  $f(x) = x^{1/2} - x^{-1/2}$   
 $f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{x+1}{2x^{3/2}}$

25.  $g(t) = \frac{2}{3}t^{-2}$   
 $g'(t) = \frac{-4}{3}t^{-3} = \frac{-4}{3t^3}$

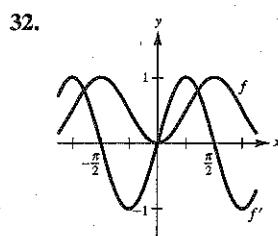
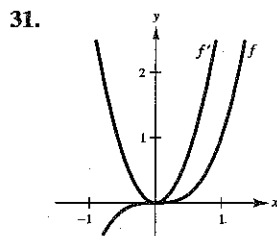
26.  $h(x) = \frac{2}{9}x^{-2}$   
 $h'(x) = \frac{-4}{9}x^{-3} = \frac{-4}{9x^3}$

27.  $f(\theta) = 2\theta - 3\sin\theta$   
 $f'(\theta) = 2 - 3\cos\theta$

28.  $g(\alpha) = 4\cos\alpha + 6$   
 $g'(\alpha) = -4\sin\alpha$

29.  $f(\theta) = 3\cos\theta - \frac{\sin\theta}{4}$   
 $f'(\theta) = -3\sin\theta - \frac{\cos\theta}{4}$

30.  $g(\alpha) = \frac{5\sin\alpha}{3} - 2\alpha$   
 $g'(\alpha) = \frac{5\cos\alpha}{3} - 2$



33.  $F = 200\sqrt{T}$   
 $F'(t) = \frac{100}{\sqrt{T}}$   
 (a) When  $T = 4$ ,  
 $F'(4) = 50$  vibrations/sec/lb.  
 (b) When  $T = 9$ ,  
 $F'(9) = 33\frac{1}{3}$  vibrations/sec/lb.

34.  $s = -16t^2 + s_0$

First ball:

$$-16t^2 + 100 = 0$$

$$t = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5 \text{ seconds to hit ground}$$

Second ball:

$$-16t^2 + 75 = 0$$

$$t^2 = \sqrt{\frac{75}{16}} = \frac{5\sqrt{3}}{4} \approx 2.165 \text{ seconds to hit ground}$$

Since the second ball was released one second after the first ball, the first ball will hit the ground first. The second ball will hit the ground  $3.165 - 2.5 = 0.665$  second later.

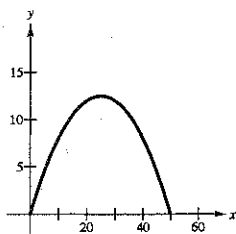
35.  $s(t) = -16t^2 + s_0$   
 $s(9.2) = -16(9.2)^2 + s_0 = 0$   
 $s_0 = 1354.24$

The building is approximately 1354 feet high (or 415 m).

36.  $s(t) = -16t^2 + 14,400 = 0$   
 $16t^2 = 14,400$   
 $t = 30 \text{ sec}$

Since  $600 \text{ mph} = \frac{1}{6} \text{ mi/sec}$ , in 30 seconds the bomb will move horizontally  $(\frac{1}{6})(30) = 5$  miles.

37. (a)



Total horizontal distance: 50

(b)  $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

(c) Ball reaches maximum height when  $x = 25$ .

(d)  $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

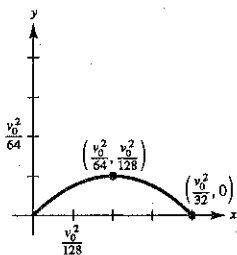
$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e)  $y'(25) = 0$

38.



$$(a) \quad y = x - \frac{32}{v_0^2}x^2 = x\left(1 - \frac{32}{v_0^2}x\right)$$

$$= 0 \text{ if } x = 0 \text{ or } x = \frac{v_0^2}{32}$$

Projectile strikes the ground when  $x = v_0^2/32$ .Projectile reaches its maximum height at  $x = v_0^2/64$  (one-half the distance).

(c)  $y = x - \frac{32}{v_0^2}x^2 = x\left(1 - \frac{32}{v_0^2}x\right) = 0$

when  $x = 0$  and  $x = v_0^2/32$ . Therefore, the range is  $x = v_0^2/32$ . When the initial velocity is doubled the range is

$$x = \frac{(2v_0)^2}{32} = \frac{4v_0^2}{32}$$

or four times the initial range. From part (a), the maximum height occurs when  $x = v_0^2/64$ . The maximum height is

$$y\left(\frac{v_0^2}{64}\right) = \frac{v_0^2}{64} - \frac{32}{v_0^2}\left(\frac{v_0^2}{64}\right)^2 = \frac{v_0^2}{64} - \frac{v_0^2}{128} = \frac{v_0^2}{128}$$

If the initial velocity is doubled, the maximum height is

$$y\left[\frac{(2v_0)^2}{64}\right] = \frac{(2v_0)^2}{128} = 4\left(\frac{v_0^2}{128}\right)$$

or four times the original maximum height.

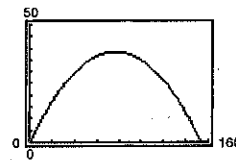
(b)  $y' = 1 - \frac{64}{v_0^2}x$

When  $x = \frac{v_0^2}{64}$ ,  $y' = 1 - \frac{64}{v_0^2}\left(\frac{v_0^2}{64}\right) = 0$ .

(d)  $v_0 = 70$  ft/sec

Range:  $x = \frac{v_0^2}{32} = \frac{(70)^2}{32} = 153.125$  ft

Maximum height:  $y = \frac{v_0^2}{128} = \frac{(70)^2}{128} \approx 38.28$  ft



$$39. x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$$

$$(a) v(t) = x'(t) = 2t - 3$$

$$(c) v(t) = 0 \text{ for } t = \frac{3}{2}$$

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

$$(b) v(t) < 0 \text{ for } t < \frac{3}{2}$$

$$(d) x(t) = 0 \text{ for } t = 1, 2$$

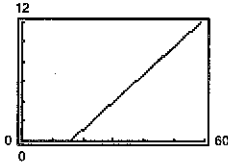
$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

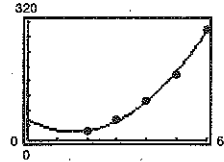
$$40. (a) y = 0.14x^2 - 4.43x + 58.4$$

(c)



(e) As the speed increases, the stopping distance increases at an increasing rate.

(b)



(d) If  $x = 65$ ,  $y \approx 362$  feet.

$$41. f(x) = (3x^2 + 7)(x^2 - 2x + 3)$$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

$$42. g(x) = (x^3 - 3x)(x + 2)$$

$$\begin{aligned} g'(x) &= (x^3 - 3x)(1) + (x + 2)(3x^2 - 3) \\ &= x^3 - 3x + 3x^3 + 6x^2 - 3x - 6 \\ &= 4x^3 + 6x^2 - 6x - 6 \end{aligned}$$

$$43. h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

$$44. f(t) = t^3 \cos t$$

$$\begin{aligned} f'(t) &= t^3(-\sin t) + \cos t(3t^2) \\ &= -t^3 \sin t + 3t^2 \cos t \end{aligned}$$

$$45. f(x) = \frac{x^2 + x - 1}{x^2 - 1}$$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

$$46. f(x) = \frac{6x - 5}{x^2 + 1}$$

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1)(6) - (6x - 5)(2x)}{(x^2 + 1)^2} \\ &= \frac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2} \end{aligned}$$

$$47. f(x) = (4 - 3x^2)^{-1}$$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

$$48. f(x) = 9(3x^2 - 2x)^{-1}$$

$$f'(x) = -9(3x^2 - 2x)^{-2}(6x - 2) = \frac{18(1 - 3x)}{(3x^2 - 2x)^2}$$

$$49. y = \frac{x^2}{\cos x}$$

$$y' = \frac{\cos x(2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

$$50. y = \frac{\sin x}{x^2}$$

$$y' = \frac{(x^2) \cos x - (\sin x)(2x)}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}$$

$$51. y = 3x^2 \sec x$$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

$$52. y = 2x - x^2 \tan x$$

$$y' = 2 - x^2 \sec^2 x - 2x \tan x$$

53.  $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

55.  $f(x) = \frac{2x^3 - 1}{x^2} = 2x - x^{-2}, (1, 1)$

$$f'(x) = 2 + 2x^{-3}$$

$$f'(1) = 4$$

Tangent line:  $y - 1 = 4(x - 1)$

$$y = 4x - 3$$

57.  $f(x) = -x \tan x, (0, 0)$

$$f'(x) = -x \sec^2 x - \tan x$$

$$f'(0) = 0$$

Tangent line:  $y - 0 = 0(x - 0)$

$$y = 0$$

59.  $v(t) = 36 - t^2, 0 \leq t \leq 6$

$$a(t) = v'(t) = -2t$$

$$v(4) = 36 - 16 = 20 \text{ m/sec}$$

$$a(4) = -8 \text{ m/sec}^2$$

60.  $v(t) = \frac{90t}{4t + 10}$

$$a(t) = \frac{(4t + 10)90 - 90t(4)}{(4t + 10)^2}$$

$$= \frac{900}{(4t + 10)^2} = \frac{225}{(2t + 5)^2}$$

(a)  $v(1) = \frac{90}{14} \approx 6.43 \text{ ft/sec}$

$$a(1) = \frac{225}{49} \approx 4.59 \text{ ft/sec}^2$$

(b)  $v(5) = \frac{90(5)}{30} = 15 \text{ ft/sec}$

$$a(5) = \frac{225}{15^2} = 1 \text{ ft/sec}^2$$

(c)  $v(10) = \frac{90(10)}{50} = 18 \text{ ft/sec}$

$$a(10) = \frac{225}{25^2} = 0.36 \text{ ft/sec}^2$$

61.  $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

62.  $f(x) = 12x^{1/4}$

$$f'(x) = 3x^{-3/4}$$

$$f''(x) = \frac{-9}{4}x^{-7/4} = \frac{-9}{4x^{7/4}}$$

63.  $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) \\ = 6 \sec^2 \theta \tan \theta$$

54.  $g(x) = 3x \sin x + x^2 \cos x$

$$g'(x) = 3x \cos x + 3 \sin x - x^2 \sin x + 2x \cos x \\ = 5x \cos x + (3 - x^2) \sin x$$

56.  $f(x) = \frac{x+1}{x-1}, \left(\frac{1}{2}, -3\right)$

$$f'(x) = \frac{(x-1) - (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$f'\left(\frac{1}{2}\right) = \frac{-2}{(1/4)} = -8$$

Tangent line:  $y + 3 = -8\left(x - \frac{1}{2}\right)$

$$y = -8x + 1$$

58.  $f(x) = \frac{1 + \sin x}{1 - \sin x}, (\pi, 1)$

$$f'(x) = \frac{(1 - \sin x) \cos x - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2}$$

$$f'(\pi) = \frac{-1 - 1}{1} = -2$$

Tangent line:  $y - 1 = -2(x - \pi)$

$$y = -2x + 2\pi + 1$$

64.  $h(t) = 4 \sin t - 5 \cos t$

$h'(t) = 4 \cos t + 5 \sin t$

$h''(t) = -4 \sin t + 5 \cos t$

66.  $y = \frac{(10 - \cos x)}{x}$

$xy + \cos x = 10$

$xy' + y - \sin x = 0$

$xy' + y = \sin x$

68.  $f(x) = \left(x^2 + \frac{1}{x}\right)^5$

$f'(x) = 5\left(x^2 + \frac{1}{x}\right)^4\left(2x - \frac{1}{x^2}\right)$

70.  $h(\theta) = \frac{\theta}{(1 - \theta)^3}$

$h'(\theta) = \frac{(1 - \theta)^3 - \theta[3(1 - \theta)^2(-1)]}{(1 - \theta)^6}$

$= \frac{(1 - \theta)^2(1 - \theta + 3\theta)}{(1 - \theta)^6} = \frac{2\theta + 1}{(1 - \theta)^4}$

72.  $y = 1 - \cos 2x + 2 \cos^2 x$

$y' = 2 \sin 2x - 4 \cos x \sin x$

$= 2[2 \sin x \cos x] - 4 \sin x \cos x$

$= 0$

74.  $y = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5}$

$y' = \sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x)$

$= \sec^5 x \tan x(\sec^2 x - 1)$

$= \sec^5 x \tan^3 x$

76.  $f(x) = \frac{3x}{\sqrt{x^2 + 1}}$

$f'(x) = \frac{3(x^2 + 1)^{1/2} - 3x \cdot \frac{1}{2}(x^2 + 1)^{-1/2}(2x)}{x^2 + 1}$

$= \frac{3(x^2 + 1) - 3x^2}{(x^2 + 1)^{3/2}} = \frac{3}{(x^2 + 1)^{3/2}}$

65.  $y = 2 \sin x + 3 \cos x$

$y' = 2 \cos x - 3 \sin x$

$y'' = -2 \sin x - 3 \cos x$

$y'' + y = -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x)$

$= 0$

67.  $h(x) = \left(\frac{x - 3}{x^2 + 1}\right)^2$

$h'(x) = 2\left(\frac{x - 3}{x^2 + 1}\right)\left(\frac{(x^2 + 1)(1) - (x - 3)(2x)}{(x^2 + 1)^2}\right)$

$= \frac{2(x - 3)(-x^2 + 6x + 1)}{(x^2 + 1)^3}$

69.  $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$

$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$

$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$

71.  $y = 3 \cos(3x + 1)$

$y' = -3 \sin(3x + 1)(3)$

$y' = -9 \sin(3x + 1)$

73.  $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$y' = \frac{1}{2} - \frac{1}{4} \cos 2x(2) = \frac{1}{2}(1 - \cos 2x) = \sin^2 x$

75.  $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$y' = \sin^{1/2} x \cos x - \sin^{5/2} x \cos x$

$= (\cos x)\sqrt{\sin x}(1 - \sin^2 x)$

$= (\cos^3 x)\sqrt{\sin x}$

77.  $y = \frac{\sin \pi x}{x + 2}$

$y' = \frac{(x + 2)\pi \cos \pi x - \sin \pi x}{(x + 2)^2}$

$$78. \quad y = \frac{\cos(x-1)}{x-1}$$

$$y' = \frac{-(x-1)\sin(x-1) - \cos(x-1)(1)}{(x-1)^2}$$

$$= -\frac{1}{(x-1)^2}[(x-1)\sin(x-1) + \cos(x-1)]$$

$$80. \quad f(x) = \sqrt[3]{x^2 - 1}$$

$$f'(x) = \frac{1}{3}(x^2 - 1)^{-2/3}(2x) = \frac{2x}{3(x^2 - 1)^{2/3}}$$

$$f'(3) = \frac{2(3)}{3(4)} = \frac{1}{2}$$

$$82. \quad y = \csc 3x + \cot 3x$$

$$y' = -3 \csc 3x \cot 3x - 3 \csc^2 3x$$

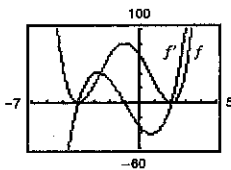
$$y'\left(\frac{\pi}{6}\right) = 0 - 3 = -3$$

$$84. \quad f(x) = [(x-2)(x+4)]^2 = (x^2 + 2x - 8)^2$$

$$f'(x) = 4(x^3 + 3x^2 - 6x - 8)$$

$$= 4(x-2)(x+1)(x+4)$$

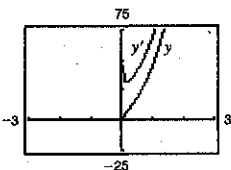
The zeros of  $f'$  correspond to the points on the graph of  $f$  where the tangent line is horizontal.



$$86. \quad y = \sqrt{3x}(x+2)^3$$

$$y' = \frac{3(x+2)^2(7x+2)}{2\sqrt{3x}}$$

$y'$  does not equal zero for any  $x$  in the domain. The graph has no horizontal tangent lines.



$$79. \quad f(x) = \sqrt{1-x^3}$$

$$f'(x) = \frac{1}{2}(1-x^3)^{-1/2}(-3x^2) = \frac{-3x^2}{2\sqrt{1-x^3}}$$

$$f'(-2) = \frac{-12}{2(3)} = -2$$

$$81. \quad y = \frac{1}{2} \csc 2x$$

$$y' = -\csc 2x \cot 2x$$

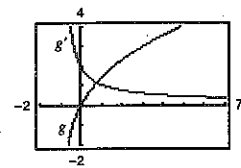
$$y'\left(\frac{\pi}{4}\right) = 0$$

$$83. \quad g(x) = 2x(x+1)^{-1/2}$$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

$g'$  does not equal zero for any value of  $x$  in the domain.

The graph of  $g$  has no horizontal tangent lines.



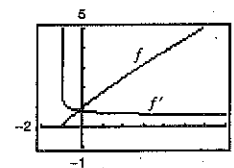
$$85. \quad f(t) = \sqrt{t+1} \sqrt[3]{t+1}$$

$$f(t) = (t+1)^{1/2}(t+1)^{1/3} = (t+1)^{5/6}$$

$$f'(t) = \frac{5}{6(t+1)^{1/6}}$$

$f'$  does not equal zero for any  $x$  in the domain.

The graph of  $f$  has no horizontal tangent lines.



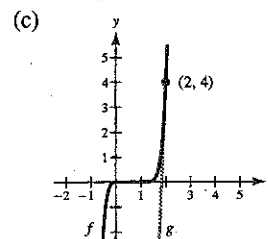
$$87. \quad f(t) = t^2(t-1)^5, \quad (2, 4)$$

$$(a) \quad f'(t) = t(t-1)^4(7t-2)$$

$$f'(2) = 24$$

$$(b) \quad y - 4 = 24(t - 2)$$

$$y = 24t - 44$$



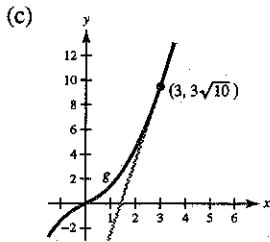
88.  $g(x) = x\sqrt{x^2 + 1}, (3, 3\sqrt{10})$

(a)  $g'(x) = \sqrt{x^2 + 1} + \frac{x^2}{\sqrt{x^2 + 1}}$

$$g'(3) = \frac{19\sqrt{10}}{10}$$

(b)  $y - 3\sqrt{10} = \frac{19\sqrt{10}}{10}(x - 3)$

$$y = \frac{19\sqrt{10}}{10}x - \frac{27\sqrt{10}}{10}$$



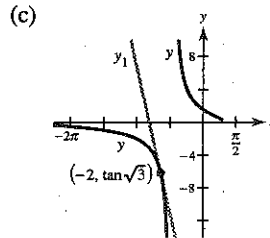
89.  $y = \tan \sqrt{1-x}, (-2, \tan \sqrt{3})$

(a)  $y' = \frac{-1}{2\sqrt{1-x} \cos^2 \sqrt{1-x}}$

$$y'(-2) = \frac{-\sqrt{3}}{6 \cos^2 \sqrt{3}} \approx -11.1983$$

(b)  $y - \tan \sqrt{3} = \frac{-\sqrt{3}}{6 \cos^2 \sqrt{3}}(x + 2)$

$$y = \frac{-\sqrt{3}}{6 \cos^2 \sqrt{3}}x + \tan \sqrt{3} - \frac{\sqrt{3}}{3 \cos^2 \sqrt{3}}$$



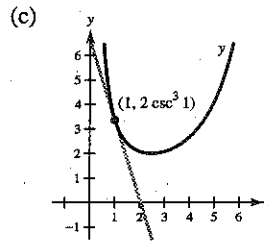
90.  $y = 2 \csc^3(\sqrt{x}) = \frac{2}{\sin^3(\sqrt{x})}, (1, 2 \csc^3(1))$

(a)  $y' = \frac{-3 \cos(\sqrt{x})}{\sqrt{x} \sin^4(\sqrt{x})}$

$$y'(1) = \frac{-3 \cos(1)}{\sin^4(1)}$$

(b)  $y - 2 \csc^3(1) = \frac{-3 \cos(1)}{\sin^4(1)}(x - 1)$

$$y = \frac{-3 \cos(1)}{\sin^4(1)}x + \frac{2}{\sin^3(1)} + \frac{3 \cos(1)}{\sin^4(1)}$$



91.  $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

92.  $y = x^{-1} + \tan x$

$$y' = -x^{-2} + \sec^2 x$$

$$y'' = 2x^{-3} + 2 \sec x(\sec x \tan x)$$

$$= \frac{2}{x^3} + 2 \sec^2 x \tan x$$

93.  $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$f''(x) = -2 \csc x(-\csc x \cdot \cot x)$$

$$= 2 \csc^2 x \cot x$$

94.  $y = \sin^2 x$

$$y' = 2 \sin x \cos x = \sin 2x$$

$$y'' = 2 \cos 2x$$

95.  $f(t) = \frac{t}{(1-t)^2}$

$$f'(t) = \frac{t+1}{(1-t)^3}$$

$$f''(t) = \frac{2(t+2)}{(1-t)^4}$$

96.  $g(x) = \frac{6x-5}{x^2+1}$

$$g'(x) = \frac{2(-3x^2+5x+3)}{(x^2+1)^2}$$

$$g''(x) = \frac{2(6x^3-15x^2-18x+5)}{(x^2+1)^3}$$

97.  $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

98.  $h(x) = x\sqrt{x^2-1}$

$$h'(x) = \frac{2x^2-1}{\sqrt{x^2-1}}$$

$$h''(x) = \frac{x(2x^2-3)}{(x^2-1)^{3/2}}$$



$$99. T = \frac{700}{t^2 + 4t + 10}$$

$$T = 700(t^2 + 4t + 10)^{-1}$$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When  $t = 1$ ,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(c) When  $t = 5$ ,

$$T' = \frac{-1400(5 + 2)}{(25 + 20 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(b) When  $t = 3$ ,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(d) When  $t = 10$ ,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

$$100. v = \sqrt{2gh} = \sqrt{2(32)h} = 8\sqrt{h}$$

$$\frac{dv}{dh} = \frac{4}{\sqrt{h}}$$

(a) When  $h = 9$ ,  $\frac{dv}{dh} = \frac{4}{3}$  ft/sec.

(b) When  $h = 4$ ,  $\frac{dv}{dh} = 2$  ft/sec.

$$101. x^2 + 3xy + y^3 = 10$$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

$$102. x^2 + 9y^2 - 4x + 3y = 0$$

$$2x + 18yy' - 4 + 3y' = 0$$

$$3(6y + 1)y' = 4 - 2x$$

$$y' = \frac{4 - 2x}{3(6y + 1)}$$

$$103. y\sqrt{x} - x\sqrt{y} = 16$$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

$$104. y^2 = x^3 - x^2y + xy - y^2$$

$$0 = 3x^2 - x^2y' + xy' - 2y^2$$

$$0 = 3x^2 - x^2y' - 2xy + xy' + y - 4yy'$$

$$(x^2 - x + 4y)y' = 3x^2 - 2xy + y$$

$$y' = \frac{3x^2 - 2xy + y}{x^2 - x + 4y}$$

$$105. x \sin y = y \cos x$$

$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$

$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

106.  $\cos(x + y) = x$

$$-(1 + y') \sin(x + y) = 1$$

$$-y' \sin(x + y) = 1 + \sin(x + y)$$

$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)}$$

$$= -\csc(x + y) - 1$$

108.  $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

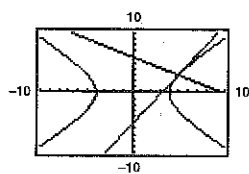
At (5, 3):  $y' = \frac{5}{3}$

Tangent line:  $y - 3 = \frac{5}{3}(x - 5)$

$$5x - 3y - 16 = 0$$

Normal line:  $y - 3 = -\frac{3}{5}(x - 5)$

$$3x + 5y - 30 = 0$$



107.  $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

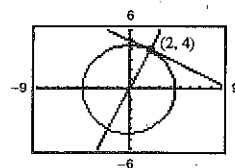
At (2, 4):  $y' = -\frac{1}{2}$

Tangent line:  $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line:  $y - 4 = 2(x - 2)$

$$2x - y = 0$$



109.  $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When  $x = \frac{1}{2}$ ,  $\frac{dx}{dt} = 2\sqrt{2}$  units/sec.

(b) When  $x = 1$ ,  $\frac{dx}{dt} = 4$  units/sec.

(c) When  $x = 4$ ,  $\frac{dx}{dt} = 8$  units/sec.

110. Surface area =  $A = 6x^2$ ,  $x$  = length of edge

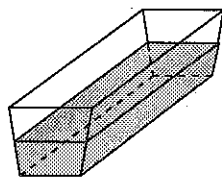
$$\frac{dx}{dt} = 5$$

$$\frac{dA}{dt} = 12x \frac{dx}{dt} = 12(4.5)(5) = 270 \text{ cm}^2/\text{sec}$$

111.  $\frac{s}{h} = \frac{1}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$


 Width of water at depth  $h$ :

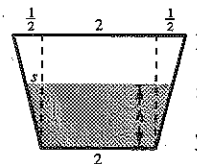
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When  $h = 1$ ,  $\frac{dh}{dt} = \frac{2}{25}$  m/min.



112.  $\tan \theta = x$

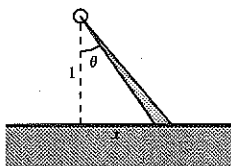
$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$

$$\sec^2 \theta \left( \frac{d\theta}{dt} \right) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = (\tan^2 \theta + 1)(6\pi) = 6\pi(x^2 + 1)$$

When  $x = \frac{1}{2}$ ,

$$\frac{dx}{dt} = 6\pi \left( \frac{1}{4} + 1 \right) = \frac{15\pi}{2} \text{ km/min} = 450\pi \text{ km/hr.}$$



113.  $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

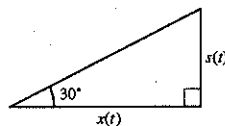
$$4.9t^2 = 25$$

$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30 = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}} \approx -38.34 \text{ m/sec}$$



### Problem Solving for Chapter 2

1. (a)  $x^2 + (y - r)^2 = r^2$ , Circle

$$x^2 = y, \text{ Parabola}$$

Substituting:

$$(y - r)^2 = r^2 - y$$

$$y^2 - 2ry + r^2 = r^2 - y$$

$$y^2 - 2ry + y = 0$$

$$y(y - 2r + 1) = 0$$

Since you want only one solution, let  $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$ . Graph  $y = x^2$  and  $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$ .(b) Let  $(x, y)$  be a point of tangency:

$$x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \frac{x}{b - y}, \text{ Circle}$$

$$y = x^2 \Rightarrow y' = 2x, \text{ Parabola}$$

Equating:

$$2x = \frac{x}{b - y}$$

$$2(b - y) = 1$$

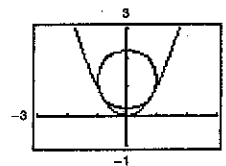
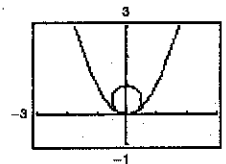
$$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$$

Also,  $x^2 + (y - b)^2 = 1$  and  $y = x^2$  imply:

$$y + (y - b)^2 = 1 \Rightarrow y + \left[ y - \left( y + \frac{1}{2} \right) \right]^2 = 1 \Rightarrow y + \frac{1}{4} = 1 \Rightarrow y = \frac{3}{4} \text{ and } b = \frac{5}{4}$$

Center:  $\left( 0, \frac{5}{4} \right)$

Graph  $y = x^2$  and  $x^2 + (y - \frac{5}{4})^2 = 1$ .





## 4. —CONTINUED—

- (d) Let  $(a, a^2)$ ,  $a \neq 0$ , be a point on the parabola  $y = x^2$ . Tangent line at  $(a, a^2)$  is  $y = 2a(x - a) + a^2$ .  
Normal line at  $(a, a^2)$  is  $y = -(1/2a)(x - a) + a^2$ . To find points of intersection, solve:

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm \left(a + \frac{1}{4a}\right)$$

$$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a \quad (\text{Point of tangency})$$

$$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$$

The normal line intersects a second time at  $x = -\frac{2a^2 + 1}{2a}$ .

5. Let
- $p(x) = Ax^3 + Bx^2 + Cx + D$

$$p'(x) = 3Ax^2 + 2Bx + C.$$

$$\text{At } (1, 1): \quad A + B + C + D = 1 \quad \text{Equation 1}$$

$$3A + 2B + C = 14 \quad \text{Equation 2}$$

$$\text{At } (-1, -3): \quad -A + B - C + D = -3 \quad \text{Equation 3}$$

$$3A - 2B + C = -2 \quad \text{Equation 4}$$

$$\text{Adding Equations 1 and 3: } 2B + 2D = -2$$

$$\text{Subtracting Equations 1 and 3: } 2A + 2C = 4$$

$$\text{Adding Equations 2 and 4: } 6A + 2C = 12$$

$$\text{Subtracting Equations 2 and 4: } 4B = 16$$

Hence,  $B = 4$  and  $D = \frac{1}{2}(-2 - 2B) = -5$ . Subtracting

$2A + 2C = 4$  and  $6A + 2C = 12$ , you obtain

$4A = 8 \Rightarrow A = 2$ . Finally,  $C = \frac{1}{2}(4 - 2A) = 0$ . Thus,

$$p(x) = 2x^3 + 4x^2 - 5.$$

- 6.
- $f(x) = a + b \cos cx$

$$f'(x) = -bc \sin cx$$

$$\text{At } (0, 1): \quad a + b = 1 \quad \text{Equation 1}$$

$$\text{At } \left(\frac{\pi}{4}, \frac{3}{2}\right): \quad a + b \cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \quad \text{Equation 2}$$

$$-bc \sin\left(\frac{c\pi}{4}\right) = 1 \quad \text{Equation 3}$$

From Equation 1,  $a = 1 - b$ . Equation 2 becomes

$$(1 - b) + b \cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b \cos\frac{c\pi}{4} = \frac{1}{2}$$

From Equation 3,  $b = \frac{-1}{c \sin(c\pi/4)}$ . Thus:

$$\frac{1}{c \sin(c\pi/4)} + \frac{-1}{c \sin(c\pi/4)} \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$$

$$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2} c \sin\left(\frac{c\pi}{4}\right)$$

Graphing the equation

$$g(c) = \frac{1}{2} c \sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1,$$

you see that many values of  $c$  will work. One answer:

$$c = 2, b = -\frac{1}{2}, a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2} \cos 2x$$

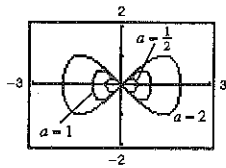
7. (a)  $x^4 = a^2x^2 - a^2y^2$

$$a^2y^2 = a^2x^2 - x^4$$

$$y = \frac{\pm\sqrt{a^2x^2 - x^4}}{a}$$

Graph:  $y_1 = \frac{\sqrt{a^2x^2 - x^4}}{a}$  and  $y_2 = -\frac{\sqrt{a^2x^2 - x^4}}{a}$

(b)



$(\pm a, 0)$  are the  $x$ -intercepts, along with  $(0, 0)$ .

(c) Differentiating implicitly:

$$4x^3 = 2a^2x - 2a^2yy'$$

$$y' = \frac{2a^2x - 4x^3}{2a^2y}$$

$$= \frac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{\pm a}{\sqrt{2}}$$

$$\left(\frac{a^2}{2}\right)^2 = a^2\left(\frac{a^2}{2}\right) - a^2y^2$$

$$\frac{a^4}{4} = \frac{a^4}{2} - a^2y^2$$

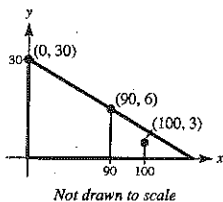
$$a^2y^2 = \frac{a^4}{4}$$

$$y^2 = \frac{a^2}{4}$$

$$y = \pm \frac{a}{2}$$

Four points:  $\left(\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(\frac{a}{\sqrt{2}}, -\frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{a}{2}\right)$

9. (a)



Line determined by  $(0, 30)$  and  $(90, 6)$ :

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0)$$

$$= -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When  $x = 100$ :

$$y = -\frac{4}{15}(100) + 30 = \frac{10}{3} > 3 \Rightarrow \text{Shadow determined by man}$$

8. (a)  $b^2y^2 = x^3(a - x); a, b > 0$

$$y^2 = \frac{x^3(a - x)}{b^2}$$

Graph  $y_1 = \frac{\sqrt{x^3(a - x)}}{b}$  and  $y_2 = -\frac{\sqrt{x^3(a - x)}}{b}$

(b)  $a$  determines the  $x$ -intercept on the right:  $(a, 0)$ .  
 $b$  affects the height.

(c) Differentiating implicitly:

$$2b^2yy' = 3x^2(a - x) - x^3 = 3ax^2 - 4x^3$$

$$y' = \frac{(3ax^2 - 4x^3)}{2b^2y} = 0$$

$$\Rightarrow 3ax^2 = 4x^3$$

$$3a = 4x$$

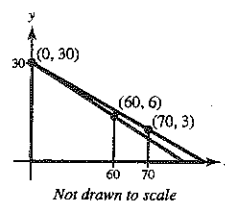
$$x = \frac{3a}{4}$$

$$b^2y^2 = \left(\frac{3a}{4}\right)^3 \left(a - \frac{3a}{4}\right) = \frac{27a^3}{64} \left(\frac{1}{4}a\right)$$

$$y^2 = \frac{27a^4}{256b^2} \Rightarrow y = \pm \frac{3\sqrt{3}a^2}{16b}$$

Two points:  $\left(\frac{3a}{4}, \frac{3\sqrt{3}a^2}{16b}\right), \left(\frac{3a}{4}, -\frac{3\sqrt{3}a^2}{16b}\right)$

(b)



Line determined by  $(0, 30)$  and  $(60, 6)$ :

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When  $x = 70$ :

$$y = -\frac{2}{5}(70) + 30$$

$$= 2 < 3 \Rightarrow \text{Shadow determined by child}$$

—CONTINUED—

## 9. —CONTINUED—

(c) Need  $(0, 30)$ ,  $(d, 6)$ ,  $(d + 10, 3)$  collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Rightarrow \frac{24}{d} = \frac{3}{10} \Rightarrow d = 80 \text{ feet}$$

(d) Let  $y$  be the distance from the base of the street light to the tip of the shadow. We know that  $dx/dt = -5$ .For  $x > 80$ , the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4} \frac{dx}{dt} = \frac{-25}{4}$$

For  $x < 80$ , the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9} \frac{dx}{dt} = \frac{-50}{9}$$

Therefore:

$$\frac{dy}{dt} = \begin{cases} -\frac{25}{4}, & x > 80 \\ -\frac{50}{9}, & 0 < x < 80 \end{cases}$$

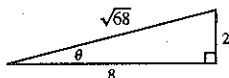
 $dy/dt$  is not continuous at  $x = 80$ .

$$10. (a) y = x^{1/3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^{-2/3} \frac{dx}{dt}$$

$$1 = \frac{1}{3}(8)^{-2/3} \frac{dx}{dt}$$

$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

$$(c) \tan \theta = \frac{y}{x} \Rightarrow \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{x(dy/dt) - y(dx/dt)}{x^2}$$

From the triangle,  $\sec \theta = \sqrt{68}/8$ . Hence

$$\frac{d\theta}{dt} = \frac{8(1) - 2(12)}{64(68/64)} = \frac{-16}{68} = \frac{-4}{17} \text{ rad/sec.}$$

$$\begin{aligned} 11. L'(x) &= \lim_{\Delta x \rightarrow 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x)}{\Delta x} \end{aligned}$$

$$\text{Also, } L'(0) = \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x) - L(0)}{\Delta x}$$

But,  $L(0) = 0$  because

$$L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0.$$

Thus,  $L'(x) = L'(0)$  for all  $x$ . The graph of  $L$  is a line through the origin of slope  $L'(0)$ .

$$\begin{aligned} (b) D &= \sqrt{x^2 + y^2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2} \left( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) \\ &= \frac{x(dx/dt) + y(dy/dt)}{\sqrt{x^2 + y^2}} \\ &= \frac{8(12) + 2(1)}{\sqrt{64 + 4}} \\ &= \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec} \end{aligned}$$

$$\begin{aligned} 12. E'(x) &= \lim_{\Delta x \rightarrow 0} \frac{E(x + \Delta x) - E(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{E(x)E(\Delta x) - E(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} E(x) \left( \frac{E(\Delta x) - 1}{\Delta x} \right) \\ &= E(x) \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - 1}{\Delta x} \end{aligned}$$

$$\text{But, } E'(0) = \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - E(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - 1}{\Delta x} = 1.$$

Thus,  $E'(x) = E(x)E'(0) = E(x)$  exists for all  $x$ .For example:  $E(x) = e^x$ .

13. (a)

$z$ (degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$	0.0174524	0.0174533	0.0174533

$$\begin{aligned}
 \text{(c) } \frac{d}{dz}(\sin z) &= \lim_{\Delta z \rightarrow 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z} \\
 &= \lim_{\Delta z \rightarrow 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z} \\
 &= \lim_{\Delta z \rightarrow 0} \left[ \sin z \left( \frac{\cos \Delta z - 1}{\Delta z} \right) \right] + \lim_{\Delta z \rightarrow 0} \left[ \cos z \left( \frac{\sin \Delta z}{\Delta z} \right) \right] \\
 &= (\sin z)(0) + (\cos z) \left( \frac{\pi}{180} \right) = \frac{\pi}{180} \cos z
 \end{aligned}$$

14. (a)  $v(t) = -\frac{27}{5}t + 27$  ft/sec

$$a(t) = -\frac{27}{5} \text{ ft/sec}^2$$

(b)  $v(t) = -\frac{27}{5}t + 27 = 0 \Rightarrow \frac{27}{5}t = 27 \Rightarrow t = 5$  seconds

$$S(5) = -\frac{27}{10}(5)^2 + 27(5) + 6 = 73.5 \text{ feet}$$

(c) The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

(b)  $\lim_{z \rightarrow 0} \frac{\sin z}{z} \approx 0.0174533$

In fact,  $\lim_{z \rightarrow 0} \frac{\sin z}{z} = \frac{\pi}{180}$ .

(d)  $S(90) = \sin\left(\frac{\pi}{180} \cdot 90\right) = \sin \frac{\pi}{2} = 1$

$$C(180) = \cos\left(\frac{\pi}{180} \cdot 180\right) = -1$$

$$\frac{d}{dz}S(z) = \frac{d}{dz} \sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180}C(z)$$

(e) The formulas for the derivatives are more complicated in degrees.

15.  $j(t) = a'(t)$

(a)  $j(t)$  is the rate of change of acceleration.

(b)  $s(t) = -8.25t^2 + 66t$

$$v(t) = -16.5t + 66$$

$$a(t) = -16.5$$

$$a'(t) = j(t) = 0$$

The acceleration is constant, so  $j(t) = 0$ .

(c)  $a$  is position.

$b$  is acceleration.

$c$  is jerk.

$d$  is velocity.