CHAPTER 10
Conics, Parametric Equations, and Polar Coordinates

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Section 10.1 Conics and Calculus

1. $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right
Matches graph (h).

2. $x^2 = 8y$

Vertex: $(0, 0)$

$p = 2 > 0$

Opens upward
Matches graph (a).

3. $(x + 3)^2 = -2(y - 2)$

Vertex: $(-3, 2)$

$p = -\frac{1}{2} < 0$

Opens downward
Matches graph (e).

4. $\dfrac{(x - 2)^2}{16} + \dfrac{(y + 1)^2}{4} = 1$

Center: $(2, -1)$
Ellipse
Matches (b)

5. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Center: $(0, 0)$
Ellipse
Matches (f)

6. $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$

Circle radius 3.
Matches (g)

7. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$

Hyperbola

Center: $(0, 0)$

Vertical transverse axis
Matches (c)

8. $\dfrac{(x - 2)^2}{9} - \dfrac{y^2}{4} = 1$

Hyperbola

Center: $(-2, 0)$

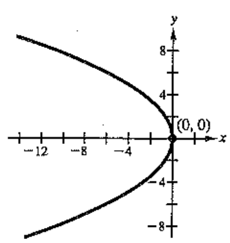
Horizontal transverse axis
Matches (d)

9. $y^2 = -6x = 4\left(-\frac{3}{2}\right)x$

Vertex: $(0, 0)$

Focus: $\left(-\frac{3}{2}, 0\right)$

Directrix: $x = \frac{3}{2}$



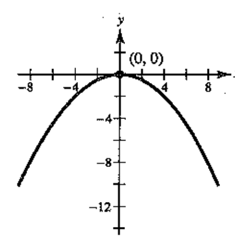
10. $x^2 + 8y = 0$

$x^2 = 4(-2)y$

Vertex: $(0, 0)$

Focus: $(0, -2)$

Directrix: $y = 2$



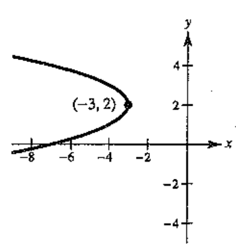
11. $(x + 3) + (y - 2)^2 = 0$

$(y - 2)^2 = 4\left(-\frac{1}{4}\right)(x + 3)$

Vertex: $(-3, 2)$

Focus: $(-3.25, 2)$

Directrix: $x = -2.75$



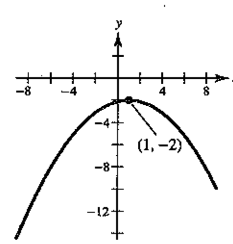
12. $(x - 1)^2 + 8(y + 2) = 0$

$(x - 1)^2 = 4(-2)(y + 2)$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$



13. $y^2 - 4y - 4x = 0$

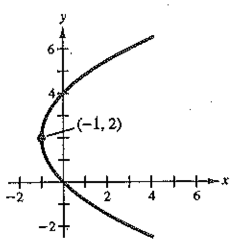
$y^2 - 4y + 4 = 4x + 4$

$(y - 2)^2 = 4(1)(x + 1)$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



14. $y^2 + 6y + 8x + 25 = 0$

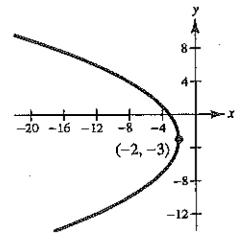
$y^2 + 6y + 9 = -8x - 25 + 9$

$(y + 3)^2 = 4(-2)(x + 2)$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



15. $x^2 + 4x + 4y - 4 = 0$

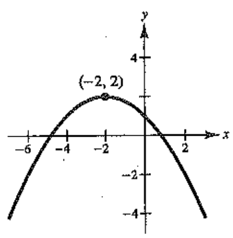
$x^2 + 4x + 4 = -4y + 4 + 4$

$(x + 2)^2 = 4(-1)(y - 2)$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



16. $y^2 + 4y + 8x - 12 = 0$

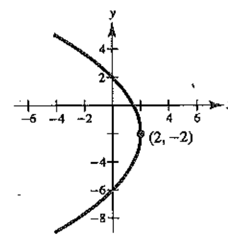
$y^2 + 4y + 4 = -8x + 12 + 4$

$(y + 2)^2 = 4(-2)(x - 2)$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$



17. $y^2 + x + y = 0$

$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$

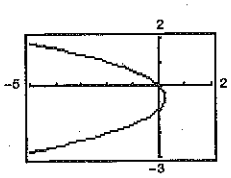
Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$

$y_1 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x}$

$y_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - x}$



18. $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$-6y = (x - 4)^2 - 10$

$-6y + 10 = (x - 4)^2$

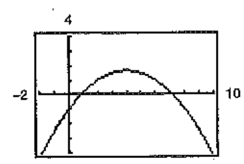
$(x - 4)^2 = -6\left(y - \frac{5}{3}\right)$

$(x - 4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$

Vertex: $\left(4, \frac{5}{3}\right)$

Focus: $\left(4, \frac{1}{6}\right)$

Directrix: $y = \frac{19}{6}$



19. $y^2 - 4x - 4 = 0$

$y^2 = 4x + 4$

$= 4(1)(x + 1)$

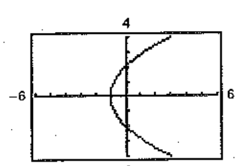
Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$

$y_1 = 2\sqrt{x + 1}$

$y_2 = -2\sqrt{x + 1}$



20. $x^2 - 2x + 8y + 9 = 0$

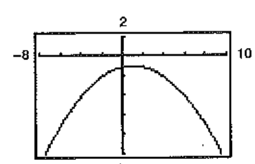
$x^2 - 2x + 1 = -8y - 9 + 1$

$(x - 1)^2 = 4(-2)(y + 1)$

Vertex: $(1, -1)$

Focus: $(1, -3)$

Directrix: $y = 1$



21. $(y - 2)^2 = 4(-2)(x - 3)$

$y^2 - 4y + 8x - 20 = 0$

22. $(x + 1)^2 = 4(-2)(y - 2)$

$x^2 + 2x + 8y - 15 = 0$

23. $(x - h)^2 = 4p(y - k)$

$x^2 = 4(6)(y - 4)$

$x^2 - 24y + 96 = 0$

24. Vertex: $(0, 2)$

$(y - 2)^2 = 4(2)(x - 0)$

$y^2 - 8x - 4y + 4 = 0$

25. $y = 4 - x^2$

$x^2 + y - 4 = 0$

26. $y = 4 - (x - 2)^2 = 4x - x^2$

$x^2 - 4x + y = 0$

27. Since the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, we obtain

$$3 = c,\ 4 = 9a + 3b + c,\ 11 = 16a + 4b + c.$$

Solving this system, we have $a = \frac{5}{3}$, $b = -\frac{14}{3}$, $c = 3$. Therefore,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

28. From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$(x - 4)^2 = 8(y - 0)$

$x^2 - 8x - 8y + 16 = 0$

29. $x^2 + 4y^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{1} = 1$

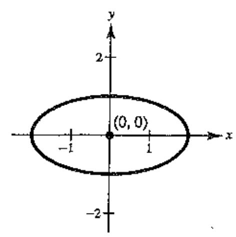
$a^2 = 4,\ b^2 = 1,\ c^2 = 3$

Center: $(0, 0)$

Foci: $(\pm\sqrt{3}, 0)$

Vertices: $(\pm 2, 0)$

$e = \frac{\sqrt{3}}{2}$



30. $5x^2 + 7y^2 = 70$

$\frac{x^2}{14} + \frac{y^2}{10} = 1$

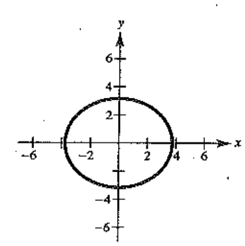
$a^2 = 14,\ b^2 = 10,\ c^2 = 4$

Center: $(0, 0)$

Foci: $(\pm 2, 0)$

Vertices: $(\pm\sqrt{14}, 0)$

$e = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$



31. $\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1$

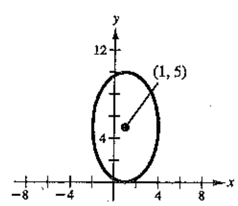
$a^2 = 25,\ b^2 = 9,\ c^2 = 16$

Center: $(1, 5)$

Foci: $(1, 9), (1, 1)$

Vertices: $(1, 10), (1, 0)$

$e = \frac{4}{5}$



32. $\frac{(x + 2)^2}{1} + \frac{(y + 4)^2}{1/4} = 1$

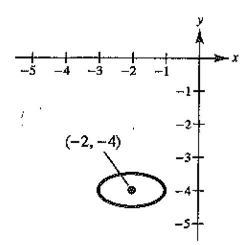
$a^2 = 1,\ b^2 = \frac{1}{4},\ c^2 = \frac{3}{4}$

Center: $(-2, -4)$

Foci: $\left(-2 \pm \frac{\sqrt{3}}{2}, -4\right)$

Vertices: $(-1, -4), (-3, -4)$

$e = \frac{\sqrt{3}}{2}$



33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$$
$$= 36$$

$$\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$$

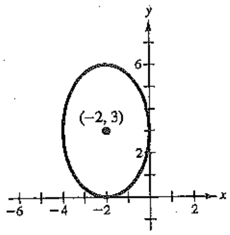
$a^2 = 9, b^2 = 4, c^2 = 5$

Center: $(-2, 3)$

Foci: $\left(-2, 3 \pm \sqrt{5}\right)$

Vertices: $(-2, 6), (-2, 0)$

$$e = \frac{\sqrt{5}}{3}$$



34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$$
$$= 10$$

$$\frac{(x - 2)^2}{(5/8)} + \frac{(y + 3)^2}{(2/5)} = 1$$

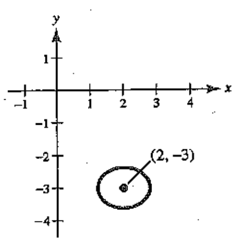
$a^2, \frac{5}{8}, b^2 = \frac{2}{5}, c^2 = a^2 - b^2 = \frac{9}{40}$

Center: $(2, -3)$

Foci: $\left(2 \pm \frac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \frac{\sqrt{10}}{4}, -3\right)$

$$e = \frac{c}{a} = \frac{3}{5}$$



35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$
$$= 60$$

$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$a^2 = 5, b^2 = 3, c^2 = 2$

Center: $\left(\frac{1}{2}, -1\right)$

Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$

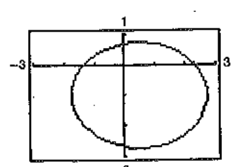
Solve for y:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$

$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = -43 + 16 + 36$$

$$= 9$$

$$\frac{[x + (2/3)]^2}{1/4} + \frac{(y - 2)^2}{1} = 1$$

$a^2 = 1,\ b^2 = \frac{1}{4},\ c^2 = \frac{3}{4}$

Center: $\left(-\frac{2}{3}, 2\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$

Vertices: $\left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$

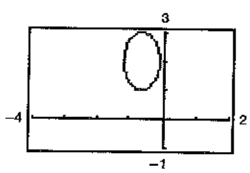
Solve for y:

$$9(y^2 - 4y + 4) = -36x^2 - 48x - 43 + 36$$

$$(y - 2)^2 = \frac{-(36x^2 + 48x + 7)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-(36x^2 + 48x + 7)}$$

(Graph each of these separately.)



37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$a^2 = 4,\ b^2 = 2,\ c^2 = 2$

Center: $\left(\frac{3}{2}, -1\right)$

Foci: $\left(\frac{3}{2} \pm \sqrt{2}, -1\right)$

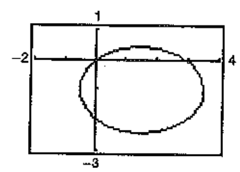
Vertices: $\left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$

Solve for y: $2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



38. $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

$$50x^2 + 25y^2 + 120x - 160y + 78 = 0$$

$$50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250$$

$$\frac{[x + (6/5)]^2}{5} + \frac{[y - (16/5)]^2}{10} = 1$$

$a^2 = 10,\ b^2 = 5,\ c^2 = 5$

Center: $\left(-\frac{6}{5}, \frac{16}{5}\right)$

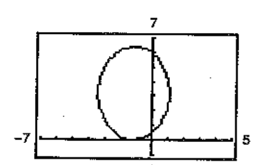
Foci: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$

Vertices: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$

Solve for y: $(y^2 - 6.4y + 10.24) = -2x^2 - 4.8x - 3.12 + 10.24$

$$(y - 3.2)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2}$$



(Graph each of these separately.)

39. Center: $(0, 0)$

Focus: $(2, 0)$

Vertex: $(3, 0)$

Horizontal major axis

$a = 3, c = 2 \Rightarrow b = \sqrt{5}$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

40. Vertices: $(0, 2), (4, 2)$

Eccentricity: $\frac{1}{2}$

Horizontal major axis

Center: $(2, 2)$

$a = 2, c = 1 \Rightarrow b = \sqrt{3}$

$$\frac{(x-2)^2}{4} + \frac{(y-2)^2}{3} = 1$$

41. Vertices: $(3, 1), (3, 9)$

Minor axis length: 6

Vertical major axis

Center: $(3, 5)$

$a = 4, b = 3$

$$\frac{(x-3)^2}{9} + \frac{(y-5)^2}{16} = 1$$

42 Foci: $(0, \pm 5)$

Major axis length: 14

Vertical major axis

Center: $(0, 0)$

$c = 5, a = 7 \Rightarrow b = \sqrt{24}$

$$\frac{x^2}{24} + \frac{y^2}{49} = 1$$

43. Center: $(0, 0)$

Horizontal major axis

Points on ellipse: $(3, 1), (4, 0)$

Since the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, we have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = 16/7$.

Therefore,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

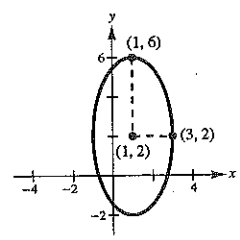
44. Center: $(1, 2)$

Vertical major axis

Points on ellipse: $(1, 6), (3, 2)$

From the sketch, we can see that $h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$



45. $\frac{y^2}{1} - \frac{x^2}{4} = 1$

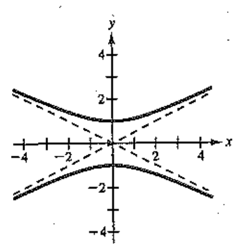
$a = 1, b = 2, c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $(0, \pm 1)$

Foci: $(0, \pm\sqrt{5})$

Asymptotes: $y = \pm\frac{1}{2}x$



46. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

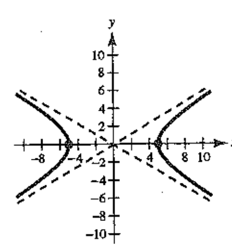
$a = 5, b = 3, c = \sqrt{a^2 + b^2} = \sqrt{34}$

Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

Foci: $(\pm\sqrt{34}, 0)$

Asymptotes: $y = \pm\frac{3}{5}x$



47. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

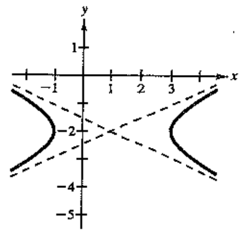
$a = 2, b = 1, c = \sqrt{5}$

Center: $(1, -2)$

Vertices: $(-1, -2), (3, -2)$

Foci: $\left(1 \pm \sqrt{5}, -2\right)$

Asymptotes: $y = -2 \pm \dfrac{1}{2}(x - 1)$



48. $\dfrac{(y+1)^2}{12^2} - \dfrac{(x-4)^2}{5^2} = 1$

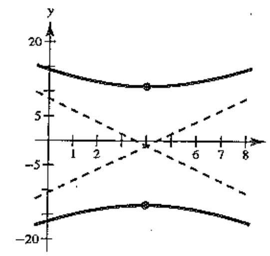
$a = 12, b = 5, c = \sqrt{a^2 + b^2} = 13$

Center: $(4, -1)$

Vertices: $(4, 11), (4, -13)$

Foci: $(4, -14), (4, 12)$

Asymptotes: $y = -1 \pm \dfrac{12}{5}(x - 4)$



49. $9x^2 - y^2 - 36x - 6y + 18 = 0$

$$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

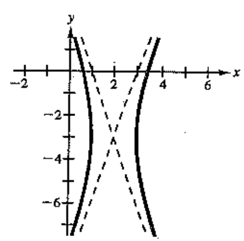
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $\left(2 \pm \sqrt{10}, -3\right)$

Asymptotes: $y = -3 \pm 3(x - 2)$



50. $y^2 - 9x^2 + 36x - 72 = 0$

$$y^2 - 9(x^2 - 4x + 4) = 72 - 36 = 36$$

$$\frac{y^2}{36} - \frac{(x-2)^2}{4} = 1$$

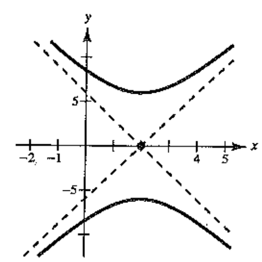
$a = 6, b = 2, c = \sqrt{a^2 + b^2} = 2\sqrt{10}$

Center: $(2, 0)$

Vertices: $(2, 6), (2, -6)$

Foci: $\left(2, 2\sqrt{10}\right), \left(2, -2\sqrt{10}\right)$

Asymptotes: $y = \pm 3(x - 2)$



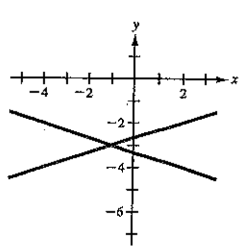
51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$$

$$(x + 1)^2 - 9(y + 3)^2 = 0$$

$$y + 3 = \pm\frac{1}{3}(x + 1)$$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.



52. $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x + 3)^2 - 4(y - 1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

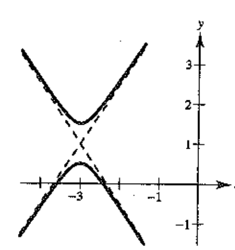
$a = \dfrac{1}{2}, b = \dfrac{1}{3}, c = \dfrac{\sqrt{13}}{6}$

Center: $(-3, 1)$

Vertices: $\left(-3, \dfrac{1}{2}\right), \left(-3, \dfrac{3}{2}\right)$

Foci: $\left(-3, 1 \pm \dfrac{1}{6}\sqrt{13}\right)$

Asymptotes: $y = 1 \pm \dfrac{3}{2}(x + 3)$



53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

$$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$$

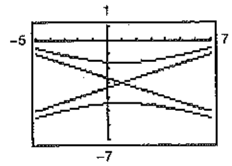
$$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $\left(1, -3 \pm \sqrt{2}\right)$

Foci: $\left(1, -3 \pm 2\sqrt{5}\right)$



Solve for y:

$$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$$

$$(y+3)^2 = \frac{x^2 - 2x + 19}{9}$$

$$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$$

(Graph each curve separately.)

54. $9x^2 - y^2 + 54x + 10y + 55 = 0$

$$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25$$
$$= 1$$

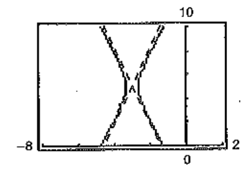
$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$a = \frac{1}{3}, b = 1, c = \frac{\sqrt{10}}{3}$

Center: $(-3, 5)$

Vertices: $\left(-3 \pm \frac{1}{3}, 5\right)$

Foci: $\left(-3 \pm \frac{\sqrt{10}}{3}, 5\right)$



Solve for y:

$$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$$

$$(y-5)^2 = 9x^2 + 54x + 80$$

$$y = 5 \pm \sqrt{9x^2 + 54x + 80}$$

(Graph each curve separately.)

55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

$$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$$

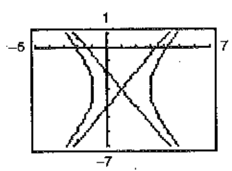
$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$$

$a = 2, b = \sqrt{6}, c = \sqrt{10}$

Center: $(1, -3)$

Vertices: $(-1, -3), (3, -3)$

Foci: $\left(1 \pm \sqrt{10}, -3\right)$



Solve for y:

$$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$$

$$(y+3)^2 = \frac{3x^2 - 6x - 9}{2}$$

$$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$$

(Graph each curve separately.)

56. $3y^2 - x^2 + 6x - 12y = 0$

$$3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$$

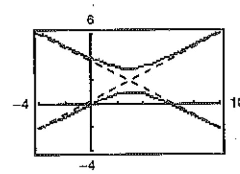
$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$a = 1, b = \sqrt{3}, c = 2$

Center: $(3, 2)$

Vertices: $(3, 1), (3, 3)$

Foci: $(3, 0), (3, 4)$



Solve for y:

$$3(y^2 - 4y + 4) = x^2 - 6x + 12$$

$$(y-2)^2 = \frac{x^2 - 6x + 12}{3}$$

$$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$$

(Graph each curve separately.)

57. Vertices: $(\pm 1, 0)$

Asymptotes: $y = \pm 3x$

Horizontal transverse axis

Center: $(0, 0)$

$a = 1, \pm\frac{b}{a} = \pm\frac{b}{1} = \pm 3 \implies b = 3$

Therefore, $\frac{x^2}{1} - \frac{y^2}{9} = 1$.

58. Vertices: $(0, \pm 3)$

Asymptotes: $y = \pm 3x$

Vertical transverse axis

$a = 3$

Slopes of asymptotes: $\pm\frac{a}{b} = \pm 3$

Thus, $b = 1$. Therefore,

$$\frac{y^2}{9} - \frac{x^2}{1} = 1.$$

59. Vertices: $(2, \pm 3)$

Point on graph: $(0, 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3$

Therefore, the equation is of the form

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, we have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

Therefore, the equation is $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{9/4} = 1.$

60. Vertices: $(2, \pm 3)$

Foci: $(2, \pm 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

Therefore, $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{16} = 1.$

61. Center: $(0, 0)$

Vertex: $(0, 2)$

Focus: $(0, 4)$

Vertical transverse axis

$a = 2, c = 4, b^2 = c^2 - a^2 = 12$

Therefore, $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1.$

62. Center: $(0, 0)$

Vertex: $(3, 0)$

Focus: $(5, 0)$

Horizontal transverse axis

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

Therefore, $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1.$

63. Vertices: $(0, 2), (6, 2)$

Asymptotes: $y = \dfrac{2}{3}x, y = 4 - \dfrac{2}{3}x$

Horizontal transverse axis

Center: $(3, 2)$

$a = 3$

Slopes of asymptotes: $\pm\dfrac{b}{a} = \pm\dfrac{2}{3}$

Thus, $b = 2$. Therefore,

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$

64. Focus: $(10, 0)$

Asymptotes: $y = \pm\dfrac{3}{4}x$

Horizontal transverse axis

Center: $(0, 0)$ since asymptotes intersect at the origin.

$c = 10$

Slopes of asymptotes: $\pm\dfrac{b}{a} = \pm\dfrac{3}{4}$ and $b = \dfrac{3}{4}a$

$c^2 = a^2 + b^2 = 100$

Solving these equations, we have $a^2 = 64$ and $b^2 = 36$. Therefore, the equation is

$$\frac{x^2}{64} - \frac{y^2}{36} = 1.$$

65. (a) $\dfrac{x^2}{9} - y^2 = 1, \dfrac{2x}{9} - 2yy' = 0, \dfrac{x}{9y} = y'$

At $x = 6$: $y = \pm\sqrt{3}, y' = \dfrac{\pm 6}{9\sqrt{3}} = \dfrac{\pm 2\sqrt{3}}{9}$

At $(6, \sqrt{3})$: $y - \sqrt{3} = \dfrac{2\sqrt{3}}{9}(x - 6)$

or $2x - 3\sqrt{3}y - 3 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \dfrac{-2\sqrt{3}}{9}(x - 6)$

or $2x + 3\sqrt{3}y - 3 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 9/(2\sqrt{3})$.

At $(6, \sqrt{3})$: $y - \sqrt{3} = -\dfrac{9}{2\sqrt{3}}(x - 6)$

or $9x + 2\sqrt{3}y - 60 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \dfrac{9}{2\sqrt{3}}(x - 6)$

or $9x - 2\sqrt{3}y - 60 = 0$

66. (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1,\ y^2 - 2x^2 = 4,\ 2yy' - 4x = 0,$

$$y' = \frac{4x}{2y} = \frac{2x}{y}$$

At $x = 4$: $y = \pm 6,\ y' = \frac{\pm 2(4)}{6} = \pm\frac{4}{3}$

At $(4, 6)$: $y - 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y - 34 = 0$

At $(4, -6)$: $y + 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y + 2 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 3/4$.

At $(4, 6)$: $y - 6 = -\frac{3}{4}(x - 4)$ or $3x + 4y - 36 = 0$

At $(4, -6)$: $y + 6 = \frac{3}{4}(x - 4)$ or $3x - 4y - 36 = 0$

67.
$$x^2 + 4y^2 - 6x + 16y + 21 = 0$$
$$(x^2 - 6x + 9) + 4(y^2 + 4y + 4) = -21 + 9 + 16$$
$$(x - 3)^2 + 4(y + 2)^2 = 4$$

Ellipse

68. $4x^2 - y^2 - 4x - 3 = 0$

$$4\left(x^2 - x + \tfrac{1}{4}\right) - y^2 = 3 + 1$$
$$4\left(x - \tfrac{1}{2}\right)^2 - y^2 = 4$$

Hyperbola

69. $y^2 - 4y - 4x = 0$

$$y^2 - 4y + 4 = 4x + 4$$
$$(y - 2)^2 = 4(x + 1)$$

Parabola

70. $25x^2 - 10x - 200y - 119 = 0$

$$25\left(x^2 - \tfrac{2}{5}x + \tfrac{1}{25}\right) = 200y + 119 + 1$$
$$25\left(x - \tfrac{1}{5}\right)^2 = 200(y + 1)$$

Parabola

71. $4x^2 + 4y^2 - 16y + 15 = 0$

$$4x^2 + 4(y^2 - 4y + 4) = -15 + 16$$
$$4x^2 + 4(y - 2)^2 = 1$$

Circle (Ellipse)

72.
$$y^2 - 4y = x + 5$$
$$y^2 - 4y + 4 = x + 5 + 4$$
$$(y - 2)^2 = x + 9$$

Parabola

73.
$$9x^2 + 9y^2 - 36x + 6y + 34 = 0$$
$$9(x^2 - 4x + 4) + 9\left(y^2 + \tfrac{2}{3}y + \tfrac{1}{9}\right) = -34 + 36 + 1$$
$$9(x - 2)^2 + 9\left(y + \tfrac{1}{3}\right)^2 = 3$$

Circle (Ellipse)

74.
$$2x(x - y) = y(3 - y - 2x)$$
$$2x^2 - 2xy = 3y - y^2 - 2xy$$
$$2x^2 + y^2 - 3y = 0$$
$$2x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Ellipse

75.
$$3(x - 1)^2 = 6 + 2(y + 1)^2$$
$$3(x - 1)^2 - 2(y + 1)^2 = 6$$
$$\frac{(x - 1)^2}{2} - \frac{(y + 1)^2}{3} = 1$$

Hyperbola

76.
$$9(x + 3)^2 = 36 - 4(y - 2)^2$$
$$9(x + 3)^2 + 4(y - 2)^2 = 36$$
$$\frac{(x + 3)^2}{4} + \frac{(y - 2)^2}{9} = 1$$

Ellipse

77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$

(c) See Theorem 10.2.

78. (a) An ellipse is the set of all points (x, y), the sum of whose distance from two distinct fixed points (foci) is constant.

(b) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ or $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distance fixed points (foci) is constant.

(b) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ or $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

(c) $y = k \pm \dfrac{b}{a}(x-h)$ or $y = k \pm \dfrac{a}{b}(x-h)$

80. $e = \dfrac{c}{a}, c = \sqrt{a^2 - b^2}, \quad 0 < e < 1$

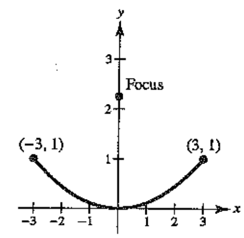
For $e \approx 0$, the ellipse is nearly circular.

For $e \approx 1$, the ellipse is elongated.

81. Assume that the vertex is at the origin.

$$x^2 = 4py$$
$$(3)^2 = 4p(1)$$
$$\frac{9}{4} = p$$

The pipe is located $\frac{9}{4}$ meters from the vertex.



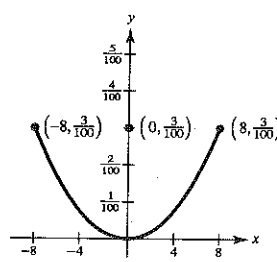
82. Assume that the vertex is at the origin.

(a) $x^2 = 4py$

$$8^2 = 4p\left(\frac{3}{100}\right)$$
$$\frac{1600}{3} = p$$
$$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$$

(b) The deflection is 1 cm when

$$y = \frac{2}{100} \Rightarrow x = \pm\sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$



83. $y = ax^2$

$y' = 2ax$

The equation of the tangent line is

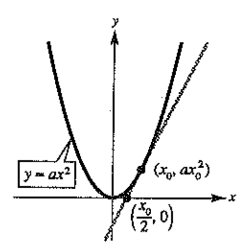
$$y - ax_0^2 = 2ax_0(x - x_0)$$

or $y = 2ax_0x - ax_0^2$.

Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$
$$ax_0^2 = 2ax_0x$$
$$x = \frac{x_0}{2}.$$

Therefore, $\left(\dfrac{x_0}{2}, 0\right)$ is the x-intercept.



84. (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, hence,

$$y' = \left(\frac{1}{2p}\right)x.$$

Therefore, for distinct tangent lines, the slopes are unequal and the lines intersect.

(b) $x^2 - 4x - 4y = 0$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At (0, 0), the slope is -1: $y = -x$. At (6, 3), the slope is 2: $y = 2x - 9$. Solving for x,

$$-x = 2x - 9$$

$$-3x = -9$$

$$x = 3$$

$$y = -3.$$

Point of intersection: $(3, -3)$

85. (a) Consider the parabola $x^2 = 4py$. Let m_0 be the slope of the one tangent line at (x_1, y_1) and therefore, $-1/m_0$ is the slope of the second at (x_2, y_2). Differentiating,

$2x = 4py'$ or $y' = \dfrac{x}{2p}$, and we have:

$$m_0 = \frac{1}{2p}x_1 \text{ or } x_1 = 2pm_0$$

$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \text{ or } x_2 = \frac{-2p}{m_0}.$$

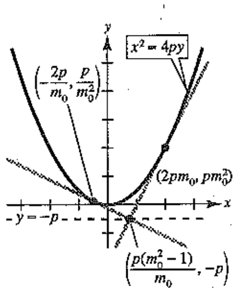
Substituting these values of x into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \quad \text{and}$$

$$\left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is

$\left(\dfrac{p(m_0^2 - 1)}{m_0}, -p\right)$ and is on the directrix, $y = -p$.



(b) $x^2 - 4x - 4y + 8 = 0$

$$(x - 2)^2 = 4(y - 1)$$

Vertex: (2, 1)

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $\dfrac{dy}{dx} = -2$. At $\left(3, \dfrac{5}{4}\right)$, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

Tangent line at $(-2, 5)$:

$$y - 5 = -2(x + 2) \implies 2x + y - 1 = 0.$$

Tangent line at $\left(3, \dfrac{5}{4}\right)$:

$$y - \frac{5}{4} = \frac{1}{2}(x - 3) \implies 2x - 4y - 1 = 0.$$

Since $m_1m_2 = (-2)\left(\dfrac{1}{2}\right) = -1$, the lines are perpendicular.

Point of intersection: $-2x + 1 = \dfrac{1}{2}x - \dfrac{1}{4}$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $\left(\dfrac{1}{2}, 0\right)$ lies on this line.

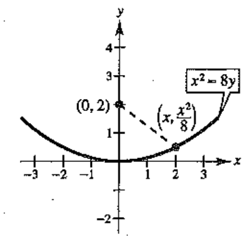
86. The focus of $x^2 = 8y = 4(2)y$ is $(0, 2)$. The distance from a point on the parabola, $(x, x^2/8)$, and the focus, $(0, 2)$, is

$$d = \sqrt{(x - 0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

Since d is minimized when d^2 is minimized, it is sufficient to minimize the function

$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$



$f'(x) = 0$ implies that $\frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \Rightarrow x = 0.$

This is a minimum by the First Derivative Test. Hence, the closest point to the focus is the vertex, $(0, 0)$.

87. $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At the point of tangency (x_1, y_1) on the mountain, $m = 1 - 2x_1$. Also, $m = \frac{y_1 - 1}{x_1 + 1}$.

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

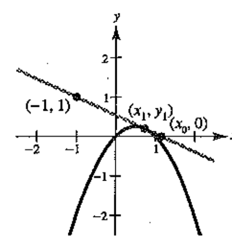
$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for x_1, we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2\left(-1 + \sqrt{3}\right) = 3 - 2\sqrt{3}$$

$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$



Thus, $-\frac{1}{x_0 + 1} = 3 - 2\sqrt{3}$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

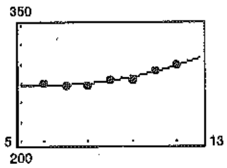
$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

$$\frac{2\sqrt{3}}{3} = x_0.$$

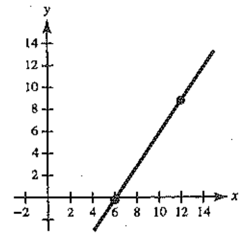
The closest the receiver can be to the hill is $\left(2\sqrt{3}/3\right) - 1 \approx 0.155$.

88. (a) $A = 0.75t^2 - 9.2t + 301$

(b)



(c) $\dfrac{dA}{dt} = 1.5t - 9.2, \quad 6 \le t \le 12$

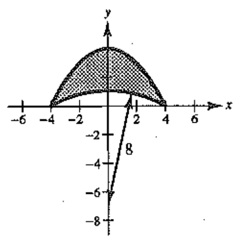


Women are spending more time watching TV each year.

89. Parabola

Vertex: $(0, 4)$

$$x^2 = 4p(y - 4)$$
$$4^2 = 4p(0 - 4)$$
$$p = -1$$
$$x^2 = -4(y - 4)$$
$$y = 4 - \frac{x^2}{4}$$



Circle

Center: $(0, k)$

Radius: 8

$$x^2 + (y - k)^2 = 64$$
$$4^2 + (0 - k)^2 = 64$$
$$k^2 = 48$$
$$k = -4\sqrt{3} \text{ (Center is on the negative } y\text{-axis.)}$$
$$x^2 + \left(y + 4\sqrt{3}\right)^2 = 64$$
$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

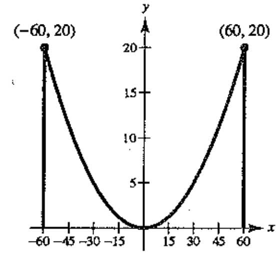
Since the y-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

$$A = 2\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) - \left(-4\sqrt{3} + \sqrt{64 - x^2}\right)\right] dx$$
$$= 2\left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2}\left(x\sqrt{64 - x^2} + 64 \arcsin \frac{x}{8}\right)\right]_0^4$$
$$= 2\left[16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin \frac{1}{2}\right]$$
$$= \frac{16\left(4 + 3\sqrt{3} - 2\pi\right)}{3} \approx 15.536 \text{ square feet}$$

90.
$$x = \frac{1}{4}y^2$$
$$x' = \frac{1}{2}y$$
$$1 + (x')^2 = 1 + \frac{y^2}{4}$$
$$s = \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)}\, dy = \frac{1}{2}\int_0^4 \sqrt{4 + y^2}\, dy$$
$$= \frac{1}{4}\left[y\sqrt{4 + y^2} + 4 \ln\left|y + \sqrt{4 + y^2}\right|\right]_0^4$$
$$= \frac{1}{4}\left[4\sqrt{20} + 4 \ln\left|4 + \sqrt{20}\right| - 4 \ln 2\right]$$
$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

91. (a) Assume that $y = ax^2$.

$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$



(b) $f(x) = \frac{1}{180}x^2, f'(x) = \frac{1}{90}x$

$$S = 2\int_0^{60} \sqrt{1 + \left(\frac{1}{90}x\right)^2}\,dx = \frac{2}{90}\int_0^{60} \sqrt{90^2 + x^2}\,dx$$

$$= \frac{2}{90}\frac{1}{2}\left[x\sqrt{90^2 + x^2} + 90^2 \ln\left|x + \sqrt{90^2 + x^2}\right|\right]_0^{60} \quad \text{(Formula 26)}$$

$$= \frac{1}{90}\left[60\sqrt{11{,}700} + 90^2 \ln\left(60 + \sqrt{11{,}700}\right) - 90^2 \ln 90\right]$$

$$= \frac{1}{90}\left[1800\sqrt{13} + 90^2 \ln\left(60 + 30\sqrt{13}\right) - 90^2 \ln 90\right]$$

$$= 20\sqrt{13} + 90 \ln\left(\frac{60 + 30\sqrt{13}}{90}\right)$$

$$= 10\left[2\sqrt{13} + 9 \ln\left(\frac{2 + \sqrt{13}}{3}\right)\right] \approx 128.4 \text{ m}$$

92. $x^2 = 20y$

$$y = \frac{x^2}{20}$$

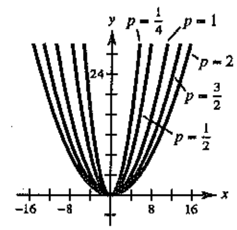
$$y' = \frac{x}{10}$$

$$S = 2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\,dx = 2\pi\int_0^r \frac{x\sqrt{100 + x^2}}{10}\,dx$$

$$= \left[\frac{\pi}{10} \cdot \frac{2}{3}(100 + x^2)^{3/2}\right]_0^r = \frac{\pi}{15}\left[(100 + r^2)^{3/2} - 1000\right]$$

93. $x^2 = 4py, p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As p increases, the graph becomes wider.



94. $A = 2\int_0^h \sqrt{4py}\,dy$

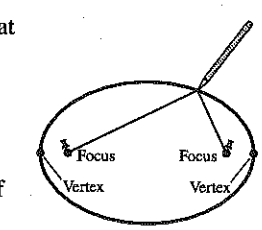
$$= 4\sqrt{p}\int_0^h y^{1/2}\,dy$$

$$= \left[4\sqrt{p}\left(\frac{2}{3}\right)y^{3/2}\right]_0^h$$

$$= \frac{8}{3}\sqrt{p}h^{3/2}$$

95. (a) At the vertices we notice that the string is horizontal and has a length of $2a$.

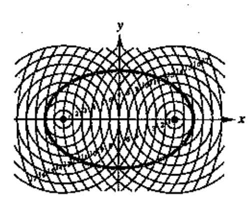
(b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.



96. $a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

97.



98. $e = \frac{c}{a}$

$$0.0167 = \frac{c}{149{,}598{,}000}$$

$$c \approx 2{,}498{,}286.6$$

Least distance: $a - c = 147{,}099{,}713.4$ km

Greatest distance: $a + c = 152{,}096{,}286.6$ km

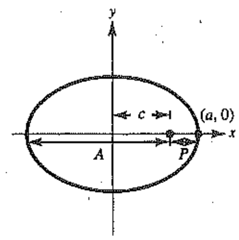
99. $e = \frac{c}{a}$

$$A + P = 2a$$

$$a = \frac{A + P}{2}$$

$$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$$

$$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$$



100. $e = \frac{A - P}{A + P}$

$$= \frac{(123{,}000 + 4000) - (119 + 4000)}{(123{,}000 + 4000) + (119 + 4000)}$$

$$= \frac{122{,}881}{131{,}119} \approx 0.9372$$

101. $e = \frac{A - P}{A + P} = \frac{35.29 - 0.59}{35.29 + 0.59} = 0.9671$

102. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As $e \to 0$, $1 - e^2 \to 1$ and we have

$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or the circle $x^2 + y^2 = a^2$.

103. $\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2x}{10^2y} = \frac{-x}{4y}$$

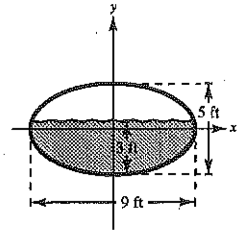
At $(-8, 3)$: $y' = \frac{8}{12} = \frac{2}{3}$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the y-axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

104. $\frac{x^2}{(4.5)^2} + \frac{y^2}{(2.5)^2} = 1$

$$x^2 = (4.5)^2\left[1 - \frac{y^2}{(2.5)^2}\right]$$

$$x = \pm\frac{9}{5}\sqrt{(2.5)^2 - y^2}$$



$V =$ (Area of bottom)(Length) + (Area of top)(Length)

$$V = \left[\frac{\pi(4.5)(2.5)}{2}\right](16) + 16\int_0^{0.5} 2\frac{9}{5}\sqrt{(2.5)^2 - y^2}\,dy$$ (Recall: Area of ellipse is πab.)

$$= 90\pi + \frac{144}{5}\cdot\left[y\sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin\frac{y}{2.5}\right]_0^{0.5} = 90\pi + \frac{144}{5}\left[0.5\sqrt{6} + (2.5)^2 \arcsin\frac{1}{5}\right] \approx 354.3 \text{ ft}^3$$

105. $16x^2 + 9y^2 + 96x + 36y + 36 = 0$

$$32x + 18yy' + 96 + 36y' = 0$$

$$y'(18y + 36) = -(32x + 96)$$

$$y' = \frac{-(32x + 96)}{18y + 36}$$

$y' = 0$ when $x = -3$. y' is undefined when $y = -2$.

At $x = -3$, $y = 2$ or -6.

Endpoints of major axis: $(-3, 2), (-3, -6)$

At $y = -2$, $x = 0$ or -6.

Endpoints of minor axis: $(0, -2), (-6, -2)$

Note: Equation of ellipse is $\dfrac{(x+3)^2}{9} + \dfrac{(y+2)^2}{16} = 1$

106. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$18x + 8yy' + 36 - 24y' = 0$$

$$(8y - 24)y' = -(18x + 36)$$

$$y' = \frac{-(18x + 36)}{8y - 24}$$

$y' = 0$ when $x = -2$. y' undefined when $y = 3$.
At $x = -2$, $y = 0$ or 6.

Endpoints of major axis: $(-2, 0), (-2, 6)$

At $y = 3$, $x = 0$ or -4.

Endpoints of minor axis: $(0, 3), (-4, 3)$

Note: Equation of ellipse is $\dfrac{(x+2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$

107. (a) $A = 4\displaystyle\int_0^2 \frac{1}{2}\sqrt{4 - x^2}\,dx = \left[x\sqrt{4 - x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 2\pi$ [or, $A = \pi ab = \pi(2)(1) = 2\pi$]

(b) **Disk:** $V = 2\pi\displaystyle\int_0^2 \frac{1}{4}(4 - x^2)\,dx = \frac{1}{2}\pi\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{8\pi}{3}$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi)\int_0^2 y\left(\frac{\sqrt{16 - 3x^2}}{4y}\right)dx = \pi\int_0^2 \sqrt{16 - 3x^2}\,dx$$

$$= \frac{\pi}{2\sqrt{3}}\left[\sqrt{3}x\sqrt{16 - 3x^2} + 16\arcsin\left(\frac{\sqrt{3}x}{4}\right)\right]_0^2$$

$$= \frac{2\pi}{9}\left(9 + 4\sqrt{3}\pi\right) \approx 21.48$$

(c) **Shell:** $V = 2\pi\displaystyle\int_0^2 x\sqrt{4 - x^2}\,dx = -\pi\int_0^2 -2x(4 - x^2)^{1/2}\,dx = -\frac{2\pi}{3}\left[(4 - x^2)^{3/2}\right]_0^2 = \frac{16\pi}{3}$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi)\int_0^1 2\sqrt{1 - y^2}\,\frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}\,dy = 8\pi\int_0^1 \sqrt{1 + 3y^2}\,dy$$

$$= \frac{8\pi}{2\sqrt{3}}\left[\sqrt{3}y\sqrt{1 + 3y^2} + \ln\left|\sqrt{3}y + \sqrt{1 + 3y^2}\right|\right]_0^1$$

$$= \frac{4\pi}{3}\left|6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right| \approx 34.69$$

108. (a) $A = 4\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\, dx = \frac{3}{2}\left[x\sqrt{16 - x^2} + 16 \arcsin \frac{x}{4}\right]_0^4 = 12\pi$

(b) **Disk:** $V = 2\pi\int_0^4 \frac{9}{16}(16 - x^2)\, dx = \frac{9\pi}{8}\left[\left(16x - \frac{1}{3}x^3\right)\right]_0^4 = 48\pi$

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16(16 - x^2)}}$$

$$S = 2(2\pi)\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\sqrt{\frac{16(16 - x^2) + 9x^2}{16(16 - x^2)}}\, dx = 4\pi\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,\frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}}\, dx = \frac{3\pi}{4}\int_0^4 \sqrt{256 - 7x^2}\, dx$$

$$= \frac{3\pi}{8\sqrt{7}}\left[\sqrt{7}x\sqrt{256 - 7x^2} + 256 \arcsin \frac{\sqrt{7}x}{16}\right]_0^4 = \frac{3\pi}{8\sqrt{7}}\left(48\sqrt{7} + 256 \arcsin \frac{\sqrt{7}}{4}\right) \approx 138.93$$

(c) **Shell:** $V = 4\pi\int_0^4 x\left[\frac{3}{4}\sqrt{16 - x^2}\right] dx = 3\pi\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16 - x^2)^{3/2}\right]_0^4 = 64\pi$

$$x = \frac{4}{3}\sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$S = 2(2\pi)\int_0^3 \frac{4}{3}\sqrt{9 - y^2}\sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}}\, dy$$

$$= 4\pi\int_0^3 \frac{4}{9}\sqrt{81 + 7y^2}\, dy$$

$$= \frac{16}{9}\left(\frac{\pi}{2\sqrt{7}}\right)\left[\sqrt{7}y\sqrt{81 + 7y^2} + 81 \ln\left|\sqrt{7}y + \sqrt{81 + 7y^2}\right|\right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}}\, 3\sqrt{7}(12) + 81 \ln\left(3\sqrt{7} + 12\right) - 81 \ln 9 \approx 168.53$$

109. From Example 5,

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, we have

$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$

$$C = 4(7)\int_0^{\pi/2} \sqrt{1 - \frac{24}{49}\sin^2 \theta}\, d\theta$$

$$\approx 28(1.3558) \approx 37.9614$$

110. (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0$

$y' = -\dfrac{xb^2}{ya^2}$

At P, $y' = -\dfrac{b^2}{a^2} \cdot \dfrac{x_0}{y_0} = m.$

(b) Slope of line through $(-c, 0)$ and (x_0, y_0): $m_1 = \dfrac{y_0}{x_0 + c}$

Slope of line through $(c, 0)$ and (x_0, y_0): $m_2 = \dfrac{y_0}{x_0 - c}$

(c) $$\tan\alpha = \frac{m_2 - m}{1 + m_2 m} = \frac{\dfrac{y_0}{x_0 - c} - \left(-\dfrac{b^2x_0}{a^2y_0}\right)}{1 + \left(\dfrac{y_0}{x_0 - c}\right)\left(-\dfrac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0^2 + b^2x_0(x_0 - c)}{a^2y_0(x_0 - c) - b^2x_0y_0}$$

$$= \frac{a^2y_0^2 + b^2x_0^2 - b^2x_0c}{x_0y_0(a^2 - b^2) - a^2y_0c} = \frac{a^2b^2 - b^2x_0c}{x_0y_0c^2 - a^2y_0c} = \frac{b^2(a^2 - x_0c)}{y_0c(x_0c - a^2)} = -\frac{b^2}{y_0c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0c}\right) = -\arctan\left(\frac{b^2}{y_0c}\right)$$

$$\tan\beta = \frac{m_1 - m}{1 + m_1 m} = \frac{\dfrac{y_0}{x_0 + c} - \left(-\dfrac{b^2x_0}{a^2y_0}\right)}{1 + \left(\dfrac{y_0}{x_0 + c}\right)\left(-\dfrac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0^2 + b^2x_0(x_0 + c)}{a^2y_0(x_0 + c) - b^2x_0y_0}$$

$$= \frac{a^2y_0^2 + b^2x_0^2 + b^2x_0c}{a^2x_0y_0 + a^2cy_0 - b^2x_0y_0} = \frac{a^2b^2 + b^2x_0c}{x_0y_0(a^2 - b^2) + a^2cy_0} = \frac{b^2(a^2 + x_0c)}{y_0c(x_0c + a^2)} = \frac{b^2}{y_0c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0c}\right)$$

Since $|\alpha| = |\beta|$, the tangent line to an ellipse at a point P makes equal angles with the lines through P and the foci.

111. Area circle $= \pi r^2 = 100\pi$

Area ellipse $= \pi ab = \pi a(10)$

$2(100\pi) = 10\pi a \Rightarrow a = 20$

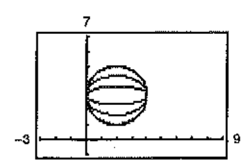
Hence, the length of the major axis is $2a = 40$.

112. (a) $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. Hence,

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1.$$

(b) $\dfrac{(x - 2)^2}{4} + \dfrac{(y - 3)^2}{4(1 - e^2)} = 1$



(c) As e approaches 0, the ellipse approaches a circle.

113. The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).

Center: $(6, 2)$

$c = 4,\ 2a = 6,\ b^2 = c^2 - a^2 = 7$

Therefore, the equation is

$$\frac{(x - 6)^2}{9} - \frac{(y - 2)^2}{7} = 1.$$

114. The transverse axis is vertical since $(-3, 0)$ and $(-3, 3)$ are the foci.

Center: $\left(-3, \dfrac{3}{2}\right)$

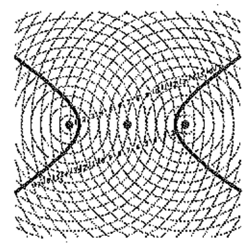
$c = \dfrac{3}{2},\ 2a = 2,\ b^2 = c^2 - a^2 = \dfrac{5}{4}$

Therefore, the equation is

$$\frac{[y - (3/2)]^2}{1} - \frac{(x + 3)^2}{5/4} = 1.$$

115. $2a = 10 \implies a = 5$

$c = 6 \implies b = \sqrt{11}$



116. Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

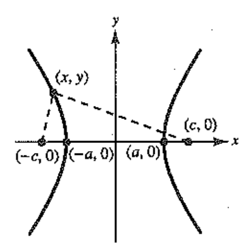
Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is

$$(a + c) - (c - a) = 2a.$$

Now, for any point (x, y) on the hyperbola, the difference of the distances between (x, y) and the two foci must also be $2a$.

$$\sqrt{(x - c)^2 + (y - 0)^2} - \sqrt{(x + c)^2 + (y - 0)^2} = 2a$$
$$\sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2}$$
$$(x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$
$$-4xc - 4a^2 = 4a\sqrt{(x + c)^2 + y^2}$$
$$-(xc + a^2) = a\sqrt{(x + c)^2 + y^2}$$
$$x^2c^2 + 2a^2cx + a^4 = a^2[x^2 + 2cx + c^2 + y^2]$$
$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$
$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

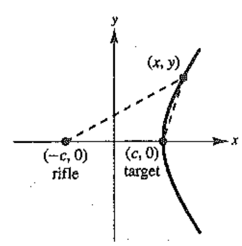


Since $a^2 + b^2 = c^2$, we have $(x^2/a^2) - (y^2/b^2) = 1$.

117. Time for sound of bullet hitting target to reach (x, y): $\dfrac{2c}{v_m} + \dfrac{\sqrt{(x - c)^2 + y^2}}{v_s}$

Time for sound of rifle to reach (x, y): $\dfrac{\sqrt{(x + c)^2 + y^2}}{v_s}$

Since the times are the same, we have: $\dfrac{2c}{v_m} + \dfrac{\sqrt{(x - c)^2 + y^2}}{v_s} = \dfrac{\sqrt{(x + c)^2 + y^2}}{v_s}$



$$\frac{4c^2}{v_m^2} + \frac{4c}{v_m v_s}\sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{v_s^2} = \frac{(x + c)^2 + y^2}{v_s^2}$$
$$\sqrt{(x - c)^2 + y^2} = \frac{v_m^2 x - v_s^2 c}{v_s v_m}$$
$$\left(1 - \frac{v_m^2}{v_s^2}\right)x^2 + y^2 = \left(\frac{v_s^2}{v_m^2} - 1\right)c^2$$
$$\frac{x^2}{c^2v_s^2/v_m^2} - \frac{y^2}{c^2(v_m^2 - v_s^2)/v_m^2} = 1$$

118. $c = 150,\ 2a = 0.001(186{,}000),\ a = 93,$

$b = \sqrt{150^2 - 93^2} = \sqrt{13{,}851}$

$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

When $y = 75$, we have

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right)$$

$$x \approx 110.3 \text{ miles.}$$

119. The point (x, y) lies on the line between $(0, 10)$ and $(10, 0)$. Thus, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, we have:

$$\frac{x^2}{36} - \frac{(10 - x)^2}{64} = 1$$

$$16x^2 - 9(10 - x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)}$$

$$= \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for x we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and}$$

$$y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

120. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$

$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$

$$a^2b^2 = b^2x_0x - a^2y_0y$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

121. $$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \implies \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2}. \text{ Let } c^2 = a^2 - b^2.$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \implies \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \implies 2 = x^2\left(\frac{1}{a^2} + \frac{1}{a^2 - b^2}\right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \implies x = \pm\frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2}\left(\frac{2a^2c^2}{2a^2 - b^2}\right) \implies \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \implies y = \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right), \left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \implies \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \implies y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \implies \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \implies y'_h = \frac{b^2x}{2c^2y}$$

—CONTINUED—

121. —CONTINUED—

At $\left(\dfrac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \dfrac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, the slopes of the tangent lines are:

$$y'_e = \frac{-b^2\left(\dfrac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2a^2\left(\dfrac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2\left(\dfrac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2c^2\left(\dfrac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = \frac{a}{c}.$$

Since the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

122. $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (Assume $A \neq 0$ and $C \neq 0$; see (b) below)

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\dfrac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\dfrac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

(a) If $A = C$, we have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

(b) If $C = 0$, we have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If $A = 0$, we have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

(c) If $AC < 0$, we have

$$\frac{\left[x + \left(\dfrac{D}{2A}\right)\right]^2}{\left|\dfrac{R}{A}\right|} + \frac{\left[y + \left(\dfrac{E}{2C}\right)\right]^2}{\left|\dfrac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

(d) If $AC < 0$, we have

$$\frac{\left[x + \left(\dfrac{D}{2A}\right)\right]^2}{\left|\dfrac{R}{A}\right|} - \frac{\left[y + \left(\dfrac{E}{2C}\right)\right]^2}{\left|\dfrac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

123. False. See the definition of a parabola.

124. True

125. True

126. False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines:

$y + 1 = \pm(x - 1)$

127. True

128. True

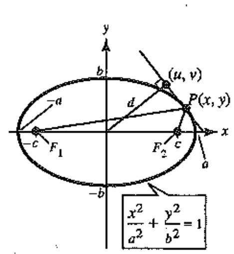
129. Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equation of the ellipse with $a > b > 0$. Let $(\pm c, 0)$ be the foci,

$c^2 = a^2 - b^2$. Let (u, v) be a point on the tangent line at $P(x, y)$, as indicated in the figure.

$$x^2b^2 + y^2a^2 = a^2b^2$$

$$2xb^2 + 2yy'a^2 = 0$$

$$y' = -\frac{b^2x}{a^2y} \quad \text{Slope at } P(x, y)$$



—CONTINUED—

129. —CONTINUED—

Now, $\dfrac{y-v}{x-u} = -\dfrac{b^2x}{a^2y}$

$$y^2a^2 - a^2vy = -b^2x^2 + b^2xu$$
$$y^2a^2 + x^2b^2 = a^2vy + b^2ux$$
$$a^2b^2 = a^2vy + b^2ux$$

Since there is a right angle at (u, v),

$$\frac{v}{u} = \frac{a^2y}{b^2x}$$
$$vb^2x = a^2uy.$$

We have two equations:

$$a^2vy + b^2ux = a^2b^2$$
$$a^2uy - b^2vx = 0.$$

Multiplying the first by v and the second by u, and adding,

$$a^2v^2y + a^2u^2y = a^2b^2v$$
$$y[u^2 + v^2] = b^2v$$
$$yd^2 = b^2v$$
$$v = \frac{yd^2}{b^2}.$$

Similarly, $u = \dfrac{xd^2}{a^2}$.

From the figure, $u = d\cos\theta$ and $v = d\sin\theta$. Thus, $\cos\theta = \dfrac{xd}{a^2}$ and $\sin\theta = \dfrac{yd}{b^2}$.

$$\cos^2\theta + \sin^2\theta = \frac{x^2d^2}{a^4} + \frac{y^2d^2}{b^4} = 1$$
$$x^2b^4d^2 + y^2a^4d^2 = a^4b^4$$
$$d^2 = \frac{a^4b^4}{x^2b^4 + y^2a^4}$$

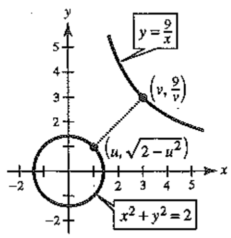
Let $r_1 = PF_1$ and $r_2 = PF_2$, $r_1 + r_2 = 2a$.

$$r_1r_2 = \frac{1}{2}[(r_1 + r_2)^2 - r_1^2 - r_2^2]$$
$$= \frac{1}{2}[4a^2 - (x + c)^2 - y^2 - (x - c)^2 - y^2]$$
$$= 2a^2 - x^2 - y^2 - c^2$$
$$= a^2 + b^2 - x^2 - y^2$$

Finally, $d^2r_1r_2 = \dfrac{a^4b^4}{x^2b^4 + y^2a^4} \cdot [a^2 + b^2 - x^2 - y^2]$

$$= \frac{a^4b^4}{b^2(b^2x^2) + a^2(a^2y^2)} \cdot [a^2 + b^2 - x^2 - y^2]$$
$$= \frac{a^4b^4}{b^2(a^2b^2 - a^2y^2) + a^2(a^2b^2 - b^2x^2)} \cdot [a^2 + b^2 - x^2 - y^2]$$
$$= \frac{a^4b^4}{a^2b^2[a^2 + b^2 - x^2 - y^2]} \cdot [a^2 + b^2 - x^2 - y^2]$$
$$= a^2b^2, \quad \text{a constant!}$$

130. Consider circle $x^2 + y^2 = 2$ and hyperbola $y = \dfrac{9}{x}$.



Let $\left(u, \sqrt{2 - u^2}\right)$ and $\left(v, \dfrac{9}{v}\right)$ be points on the circle and hyperbola, respectively. We need to minimize the distance between these 2 points:

$$(\text{Distance})^2 = f(u, v) = (u - v)^2 + \left(\sqrt{2 + u^2} - \frac{9}{v}\right)^2.$$

The tangent lines at $(1, 1)$ and $(3, 3)$ are both perpendicular to $y = x$, and hence parallel.

The minimum value is $(3 - 1)^2 + (3 - 1)^2 = 8$.

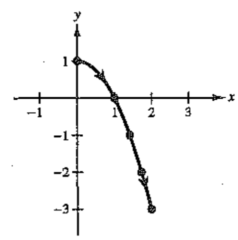
Section 10.2 Plane Curves and Parametric Equations

1. $x = \sqrt{t},\ y = 1 - t$

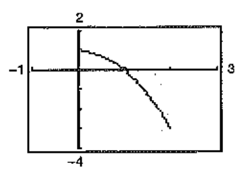
(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3

(b)



(c)



(d) $x^2 = t$

$y = 1 - x^2, x \geq 0$

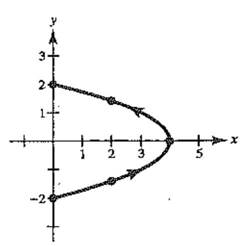
2. $x = 4\cos^2\theta \qquad y = 2\sin\theta$

$0 \leq x \leq 4 \qquad -2 \leq y \leq 2$

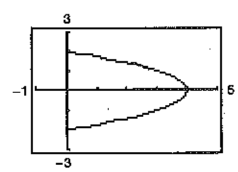
(a)

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	0	2	4	2	0
y	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2

(b)



(c)



(d) $\dfrac{x}{4} = \cos^2\theta$

$\dfrac{y^2}{4} = \sin^2\theta$

$\dfrac{x}{4} + \dfrac{y^2}{4} = 1$

$x = 4 - y^2, -2 \leq y \leq 2$

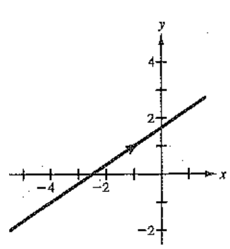
(e) The graph would be oriented in the opposite direction.

3. $x = 3t - 1$

$y = 2t + 1$

$y = 2\left(\dfrac{x + 1}{3}\right) + 1$

$2x - 3y + 5 = 0$

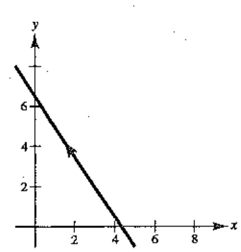


4. $x = 3 - 2t$

$y = 2 + 3t$

$y = 2 + 3\left(\dfrac{3 - x}{2}\right)$

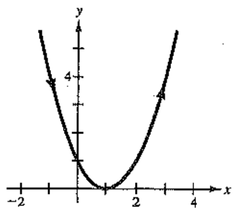
$2y + 3x - 13 = 0$



5. $x = t + 1$

$y = t^2$

$y = (x - 1)^2$



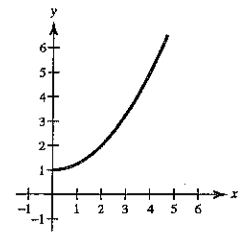
6. $x = 2t^2$

$y = t^4 + 1$

$y = \left(\frac{x}{2}\right)^2 + 1 = \frac{x^2}{4} + 1, x \geq 0$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.

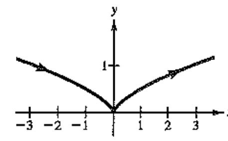


7. $x = t^3$

$y = \frac{1}{2}t^2$

$x = t^3$ implies $t = x^{1/3}$

$y = \frac{1}{2}x^{2/3}$



8. $x = t^2 + t, \; y = t^2 - t$

Subtracting the second equation from the first, we have

$x - y = 2t \quad \text{or} \quad t = \frac{x - y}{2}.$

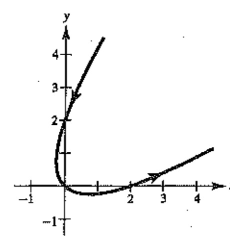
$y = \frac{(x - y)^2}{4} - \frac{x - y}{2}$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2

Since the discriminant is

$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$

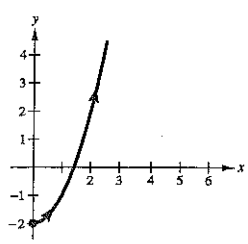
the graph is a rotated parabola.



9. $x = \sqrt{t}, t \geq 0$

$y = t - 2$

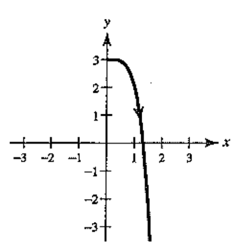
$y = x^2 - 2, x \geq 0$



10. $x = \sqrt[4]{t}, t \geq 0$

$y = 3 - t$

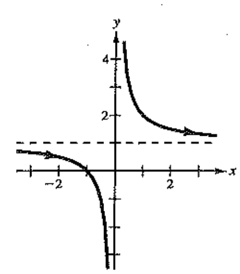
$y = 3 - x^4, x \geq 0$



11. $x = t - 1$

$y = \frac{t}{t - 1}$

$y = \frac{x + 1}{x}$

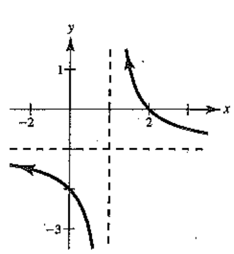


12. $x = 1 + \frac{1}{t}$

$y = t - 1$

$x = 1 + \frac{1}{t}$ implies $t = \frac{1}{x - 1}$

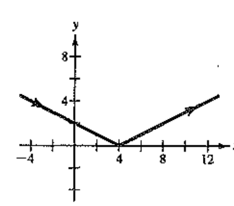
$y = \frac{1}{x - 1} - 1$



13. $x = 2t$

$y = |t - 2|$

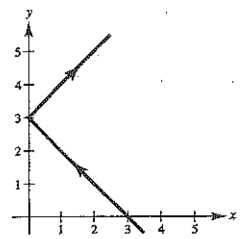
$y = \left|\frac{x}{2} - 2\right| = \frac{|x - 4|}{2}$



14. $x = |t - 1|$

$y = t + 2$

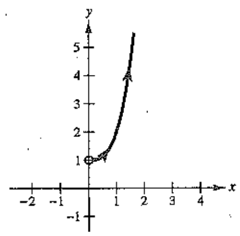
$x = |(y - 2) - 1| = |y - 3|$



15. $x = e^t, x > 0$

$y = e^{3t} + 1$

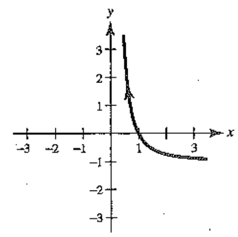
$y = x^3 + 1, x > 0$



16. $x = e^{-t}, x > 0$

$y = e^{2t} - 1$

$y = x^{-2} - 1 = \frac{1}{x^2} - 1, x > 0$



17. $x = \sec\theta$

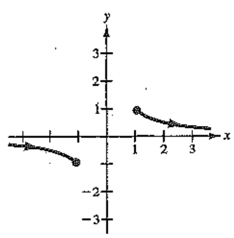
$y = \cos\theta$

$0 \le \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \le \pi$

$xy = 1$

$y = \frac{1}{x}$

$|x| \ge 1, |y| \le 1$



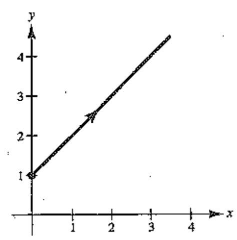
18. $x = \tan^2\theta$

$y = \sec^2\theta$

$\sec^2\theta = \tan^2\theta + 1$

$y = x + 1$

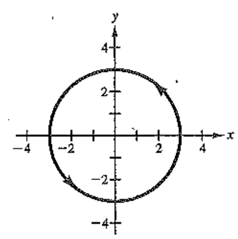
$x \ge 0$



19. $x = 3\cos\theta,\ y = 3\sin\theta$

Squaring both equations and adding, we have

$x^2 + y^2 = 9.$

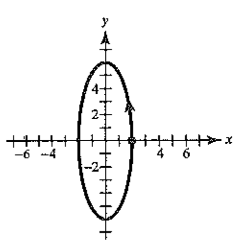


20. $x = 2\cos\theta$

$y = 6\sin\theta$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{6}\right)^2 = \cos^2 + \sin^2\theta = 1$

$\frac{x^2}{4} + \frac{y^2}{36} = 1$ ellipse



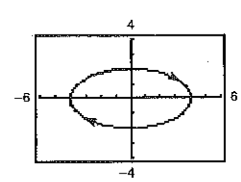
21. $x = 4\sin 2\theta$

$y = 2\cos 2\theta$

$\frac{x^2}{16} = \sin^2 2\theta$

$\frac{y^2}{4} = \cos^2 2\theta$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$



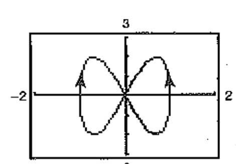
22. $x = \cos\theta$

$y = 2\sin 2\theta$

$y = 4\sin\theta\cos\theta$

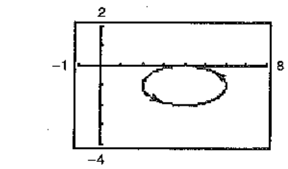
$1 - x^2 = \sin^2\theta$

$y = \pm 4x\sqrt{1 - x^2}$



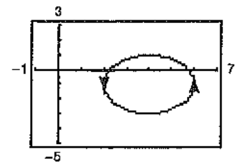
23.

$$x = 4 + 2\cos\theta$$
$$y = -1 + \sin\theta$$
$$\frac{(x-4)^2}{4} = \cos^2\theta$$
$$\frac{(y+1)^2}{1} = \sin^2\theta$$
$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$$



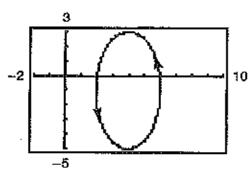
24.

$$x = 4 + 2\cos\theta$$
$$y = -1 + 2\sin\theta$$
$$(x-4)^2 = 4\cos^2\theta$$
$$(y+1)^2 = 4\sin^2\theta$$
$$(x-4)^2 + (y+1)^2 = 4$$



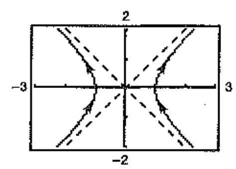
25.

$$x = 4 + 2\cos\theta$$
$$y = -1 + 4\sin\theta$$
$$\frac{(x-4)^2}{4} = \cos^2\theta$$
$$\frac{(y+1)^2}{16} = \sin^2\theta$$
$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1$$



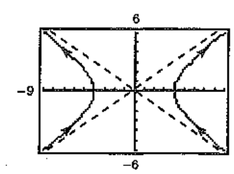
26.

$$x = \sec\theta$$
$$y = \tan\theta$$
$$x^2 = \sec^2\theta$$
$$y^2 = \tan^2\theta$$
$$x^2 - y^2 = 1$$



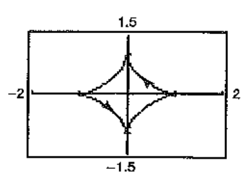
27.

$$x = 4\sec\theta$$
$$y = 3\tan\theta$$
$$\frac{x^2}{16} = \sec^2\theta$$
$$\frac{y^2}{9} = \tan^2\theta$$
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



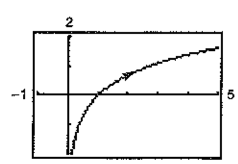
28.

$$x = \cos^3\theta$$
$$y = \sin^3\theta$$
$$x^{2/3} = \cos^2\theta$$
$$y^{2/3} = \sin^2\theta$$
$$x^{2/3} + y^{2/3} = 1$$



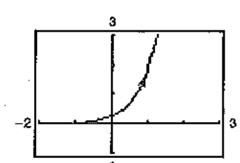
29. $x = t^3$

$$y = 3\ln t$$
$$y = 3\ln\sqrt[3]{x} = \ln x$$



30. $x = \ln 2t$

$$y = t^2$$
$$t = \frac{e^x}{2}$$
$$y = \frac{e^{2x}}{r} = \frac{1}{4}e^{2x}$$



31. $x = e^{-t}$

$y = e^{3t}$

$e^t = \dfrac{1}{x}$

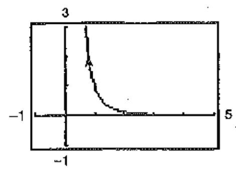
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \dfrac{1}{x}$

$y = \dfrac{1}{x^3}$

$x > 0$

$y > 0$



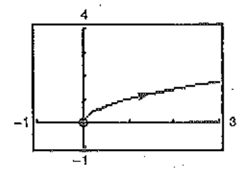
32. $x = e^{2t}$

$y = e^t$

$y^2 = x$

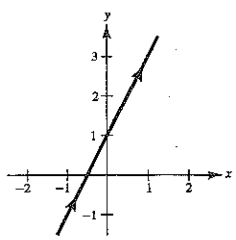
$y > 0$

$y = \sqrt{x}, x > 0$



33. By eliminating the parameters in (a) – (d), we get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

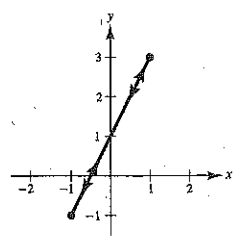
(a) $x = t,\; y = 2t + 1$



(b) $x = \cos\theta$ $\qquad y = 2\cos\theta + 1$

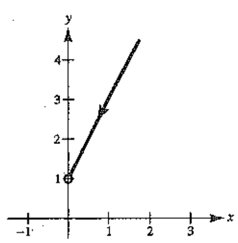
$-1 \le x \le 1 \qquad -1 \le y \le 3$

$\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$



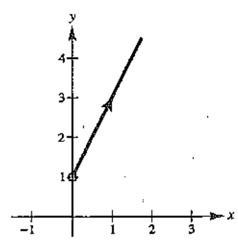
(c) $x = e^{-t}$ $\qquad y = 2e^{-t} + 1$

$x > 0 \qquad y > 1$



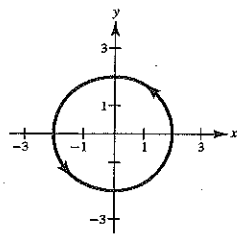
(d) $x = e^t$ $\qquad y = 2e^t + 1$

$x > 0 \qquad y > 1$



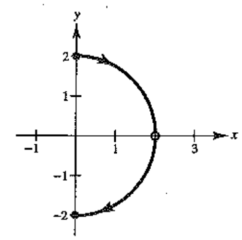
34. By eliminating the parameters in (a) – (d), we get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

(a) $x = 2\cos\theta,\ y = 2\sin\theta$



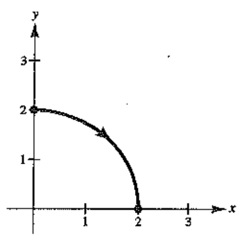
(b) $x = \dfrac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \dfrac{1}{t^2}}$ $\qquad y = \dfrac{1}{t}$

$x \geq 0,\ x \neq 2$ $\qquad y \neq 0$



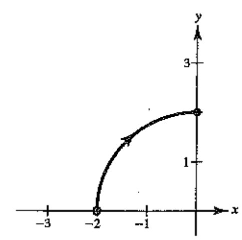
(c) $x = \sqrt{t}$ $\qquad y = \sqrt{4 - t}$

$x \geq 0$ $\qquad y \geq 0$



(d) $x = -\sqrt{4 - e^{2t}}$ $\qquad y = e^t$

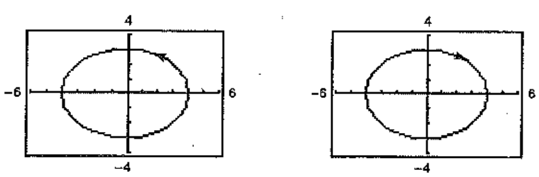
$-2 < x \leq 0$ $\qquad y > 0$



35. The curves are identical on $0 < \theta < \pi$. They are both smooth. Represent $y = 2(1 - x^2)$

36. The orientations are reversed. The graphs are the same. They are both smooth.

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Many answers possible. For example, $x = 1 + t$, $y = 1 + 2t$, and $x = 1 - t$, $x = 1 - 2t$.

38. The set of points (x, y) corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

39.

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$\frac{x - x_1}{x_2 - x_1} = t$$
$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - y_1 = m(x - x_1)$$

40.

$$x = h + r\cos\theta$$
$$y = k + r\sin\theta$$
$$\cos\theta = \frac{x - h}{r}$$
$$\sin\theta = \frac{y - k}{r}$$
$$\cos^2\theta + \sin^2\theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$
$$(x - h)^2 + (y - k)^2 = r^2$$

41.
$$x = h + a\cos\theta$$
$$y = k + b\sin\theta$$
$$\frac{x-h}{a} = \cos\theta$$
$$\frac{y-k}{b} = \sin\theta$$
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

42.
$$x = h + a\sec\theta$$
$$y = k + b\tan\theta$$
$$\frac{x-h}{a} = \sec\theta$$
$$\frac{y-k}{b} = \tan\theta$$
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

43. From Exercise 39 we have

$x = 5t$

$y = -2t.$

Solution not unique

44. From Exercise 39 we have

$x = 1 + 4t$

$y = 4 - 6t.$

Solution not unique

45. From Exercise 40 we have

$x = 2 + 4\cos\theta$

$y = 1 + 4\sin\theta.$

Solution not unique

46. From Exercise 40 we have

$x = -3 + 3\cos\theta$

$y = 1 + 3\sin\theta.$

Solution not unique

47. From Exercise 41 we have

$a = 5, c = 4 \implies b = 3$

$x = 5\cos\theta$

$y = 3\sin\theta.$

Center: $(0, 0)$
Solution not unique

48. From Exercise 41 we have

$a = 5, c = 3 \implies b = 4$

$x = 4 + 5\cos$

$y = 2 + 4\sin\theta.$

Center: $(4, 2)$
Solution not unique

49. From Exercise 42 we have

$a = 4, c = 5 \implies b = 3$

$x = 4\sec\theta$

$y = 3\tan\theta.$

Center: $(0, 0)$
Solution not unique

50. From Exercise 42 we have

$a = 1, c = 2 \implies b = \sqrt{3}$

$x = \sqrt{3}\tan\theta$

$y = \sec\theta.$

Center: $(0, 0)$
Solution not unique
The transverse axis is vertical, therefore, x and y are interchanged.

51. $y = 3x - 2$

Example

$x = t, \quad y = 3t - 2$

$x = t - 3, \quad y = 3t - 11$

52. $y = \dfrac{2}{x-1}$

Example

$x = t, y = \dfrac{2}{t-1}$

$x = -t, y = \dfrac{2}{-t-1}$

53. $y = x^3$

Example

$x = t, \quad y = t^3$

$x = \sqrt[3]{t}, \quad y = t$

$x = \tan t, \quad y = \tan^3 t$

54. $y = x^2$

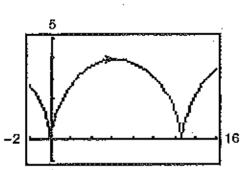
Example

$x = t, \quad y = t^2$

$x = t^3, \quad y = t^6$

55. $x = 2(\theta - \sin\theta)$

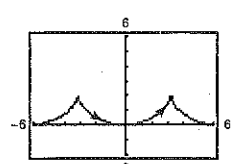
$y = 2(1 - \cos\theta)$



Not smooth at $\theta = 2n\pi$

56. $x = \theta + \sin\theta$

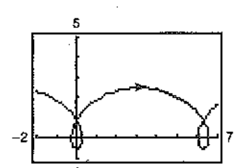
$y = 1 - \cos\theta$



Not smooth at $x = (2n - 1)\pi$

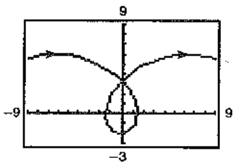
57. $x = \theta - \frac{3}{2}\sin\theta$

$y = 1 - \frac{3}{2}\cos\theta$



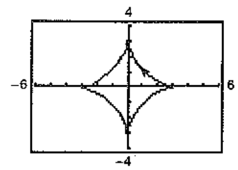
58. $x = 2\theta - 4\sin\theta$

$y = 2 - 4\cos\theta$



59. $x = 3\cos^3\theta$

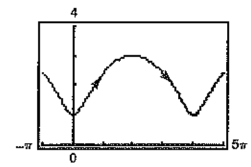
$y = 3\sin^3\theta$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

60. $x = 2\theta - \sin\theta$

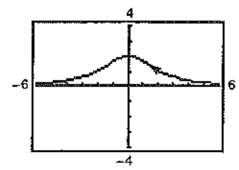
$y = 2 - \cos\theta$



Smooth everywhere

61. $x = 2\cot\theta$

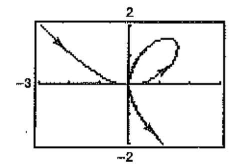
$y = 2\sin^2\theta$



Smooth everywhere

62. $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$



Smooth everywhere

63. See definition on page 709.

64. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t, plot (x, y). As t increases, the curve is traced out in a specific direction called the orientation of the curve.

65. A plane curve C, represented by $x = f(t)$, $y = g(t)$, is smooth if f' and g' are continuous and not simultaneously 0. See page 714.

66. (a) Matches (iv) because $(0, 2)$ is on the graph.

(b) Matches (v) because $(1, 0)$ is on the graph.

(c) Matches (ii) because $-1 \le x \le 0$ and $1 \le y \le 3$.

(d) Matches (iii) because $(4, 0)$ is on the graph.

(e) Matches (vi) because undefined at $\theta = 0$.

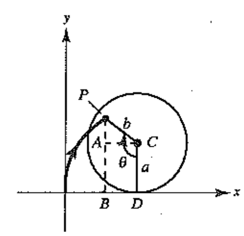
(f) Matches (i) because $x = (y - 2)^2 - 1$ for all y.

67. When the circle has rolled θ radians, we know that the center is at $(a\theta, a)$.

$$\sin\theta = \sin(180^\circ - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \quad \text{or} \quad |BD| = b\sin\theta$$

$$\cos\theta = -\cos(180^\circ - \theta) = \frac{|AP|}{-b} \quad \text{or} \quad |AP| = -b\cos\theta$$

Therefore, $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$.



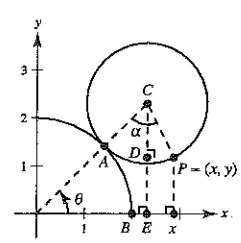
68. Let the circle of radius 1 be centered at C. A is the point of tangency on the line OC. $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle θ changes $\widehat{AB} = \widehat{AP}$, $\widehat{AB} = 2\theta$ and $\widehat{AP} = \alpha \implies \alpha = 2\theta$. Form the right triangle $\triangle CDP$. The angle $OCE = (\pi/2) - \theta$ and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3\sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3\cos\theta - \cos 3\theta$$

$$y = EC - CD = 3\sin\theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3\sin\theta - \sin 3\theta$$

Hence, $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$.



69. False

$x = t^2 \implies x \geq 0$

$y = t^2 \implies y \geq 0$

The graph of the parametric equations is only a portion of the line $y = x$.

70. False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and y is not a function of x.

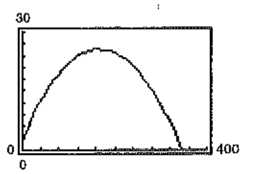
71. (a) $100 \text{ mi/hr} = \dfrac{(100)(5280)}{3600} = \dfrac{440}{3} \text{ ft/sec}$

$$x = (v_0 \cos\theta)t = \left(\frac{440}{3}\cos\theta\right)t$$

$$y = h + (v_0 \sin\theta)t - 16t^2$$

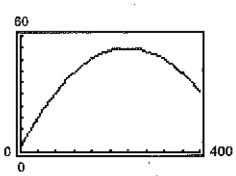
$$= 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$$

(b)



It is not a home run—when $x = 400$, $y < 10$.

(c)



Yes, it's a home run when $x = 400$, $y > 10$.

(d) We need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3}\cos\theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440\cos\theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{1200}{440\cos\theta}\right) - 16\left(\frac{1200}{440\cos\theta}\right)^2$$

$$7 = 400\tan\theta - 16\left(\frac{120}{44}\right)^2\sec^2\theta$$

$$= 400\tan\theta - 16\left(\frac{120}{44}\right)^2(\tan^2\theta + 1).$$

We now solve the quadratic for $\tan\theta$:

$$16\left(\frac{120}{44}\right)^2\tan^2\theta - 400\tan\theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0$$

$\tan\theta \approx 0.35185 \implies \theta \approx 19.4°$

72. (a) $x = (v_0 \cos\theta)t$

$$y = h + (v_0 \sin\theta)t - 16t^2$$

$$t = \frac{x}{v_0\cos\theta} \implies y = h + (v_0\sin\theta)\frac{x}{v_0\cos\theta} - 16\left(\frac{x}{v_0\cos\theta}\right)^2$$

$$y = h + (\tan\theta)x - \frac{16\sec^2\theta}{v_0^2}x^2$$

(b) $y = 5 + x - 0.005x^2 = h + (\tan\theta)x - \dfrac{16\sec^2\theta}{v_0^2}x^2$

$h = 5$, $\tan\theta = 1 \implies \theta = \dfrac{\pi}{4}$, and

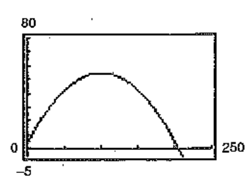
$$0.005 = \frac{16\sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \implies v_0 = 80.$$

Hence, $x = (80\cos(45°))t$

$y = 5 + (80\sin(45°))t - 16t^2.$

(c)



(d) Maximum height: $y = 55$ (at $x = 100$)

Range: 204.88

Section 10.3 Parametric Equations and Calculus

1. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-4}{2t} = \dfrac{-2}{t}$

2. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

3. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-2\cos\theta\sin\theta}{2\sin\theta\cos\theta} = -1$

$\left[\text{Note: } x + y = 1 \Rightarrow y = 1 - x \text{ and } \dfrac{dy}{d\theta} = -1\right]$

4. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(-1/2)e^{-\theta/2}}{2e^{\theta}} = -\dfrac{1}{4}e^{-3\theta/2} = \dfrac{-1}{4e^{3\theta/2}}$

5. $x = 2t,\ y = 3t - 1$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3}{2}$

$\dfrac{d^2y}{dx^2} = 0$ Line

6. $x = \sqrt{t},\ y = 3t - 1$

$\dfrac{dy}{dx} = \dfrac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6$ when $t = 1$.

$\dfrac{d^2y}{dx^2} = \dfrac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6$ concave upwards

7. $x = t + 1,\ y = t^2 + 3t$

$\dfrac{dy}{dx} = \dfrac{2t+3}{1} = 1$ when $t = -1$.

$\dfrac{d^2y}{dx^2} = 2$ concave upwards

8. $x = t^2 + 3t + 2, y = 2t$

$\dfrac{dy}{dx} = \dfrac{2}{2t+3} = \dfrac{2}{3}$ when $t = 0$.

$\dfrac{d^2y}{dx^2} = \dfrac{-2(2)/(2t+3)^2}{2t+3} = \dfrac{-4}{(2t+3)^3} = \dfrac{-4}{27}$ when $t = 0$.

concave downward

9. $x = 2\cos\theta,\ y = 2\sin\theta$

$\dfrac{dy}{dx} = \dfrac{2\cos\theta}{-2\sin\theta} = -\cot\theta = -1$ when $\theta = \dfrac{\pi}{4}$.

$\dfrac{d^2y}{dx^2} = \dfrac{\csc^2\theta}{-2\sin\theta} = \dfrac{-\csc^3\theta}{2} = -\sqrt{2}$ when $\theta = \dfrac{\pi}{4}$.

concave downward

10. $x = \cos\theta,\ y = 3\sin\theta$

$\dfrac{dy}{dx} = \dfrac{3\cos\theta}{-\sin\theta} = -3\cot\theta \cdot \dfrac{dy}{dx}$ is undefined when $\theta = 0$.

$\dfrac{d^2y}{dx^2} = \dfrac{3\csc^2\theta}{-\sin\theta} = \dfrac{-3}{\sin^3\theta} \cdot \dfrac{d^2y}{dx^2}$ is undefined when $\theta = 0$.

11. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$

$\dfrac{dy}{dx} = \dfrac{2\sec^2\theta}{\sec\theta\tan\theta}$

$= \dfrac{2\sec\theta}{\tan\theta} = 2\csc\theta = 4$ when $\theta = \dfrac{\pi}{6}$.

$\dfrac{d^2y}{dx^2} = \dfrac{d\left[\dfrac{dy}{dx}\right]}{\dfrac{dx}{d\theta}} = \dfrac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta}$

$= -2\cot^3\theta = -6\sqrt{3}$ when $\theta = \dfrac{\pi}{6}$.

concave downward

12. $x = \sqrt{t},\ y = \sqrt{t-1}$

$\dfrac{dy}{dx} = \dfrac{1/(2\sqrt{t-1})}{1/(2\sqrt{t})}$

$= \dfrac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2}$ when $t = 2$.

$\dfrac{d^2y}{dx^2} = \dfrac{\left[\sqrt{t-1}/(2\sqrt{t}) - \sqrt{t}(1/2\sqrt{t-1})\right]/(t-1)}{1/(2\sqrt{t})}$

$= \dfrac{-1}{(t-1)^{3/2}} = -1$ when $t = 2$.

concave downward

13. $x = \cos^3\theta,\ y = \sin^3\theta$

$$\frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}$$

$$= -\tan\theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta} = \frac{1}{3\cos^4\theta\sin\theta}$$

$$= \frac{\sec^4\theta\csc\theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

14. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{[(1-\cos\theta)\cos\theta - \sin^2\theta]}{(1-\cos\theta)^2}}{(1-\cos\theta)}$$

$$= \frac{-1}{(1-\cos\theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

concave downward

15. $x = 2\cot\theta,\ y = 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} = -2\sin^3\theta\cos\theta$$

At $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $\theta = \frac{2\pi}{3}$, and $\frac{dy}{dx} = \frac{3\sqrt{3}}{8}$.

Tangent line: $y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$

$$3\sqrt{3}x - 8y + 18 = 0$$

At $(0, 2)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y - 2 = 0$

At $\left(2\sqrt{3}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{6}$, and $\frac{dy}{dx} = -\frac{\sqrt{3}}{8}$.

Tangent line: $y - \frac{1}{2} = -\frac{\sqrt{3}}{8}(x - 2\sqrt{3})$

$$\sqrt{3}x + 8y - 10 = 0$$

16. $x = 2 - 3\cos\theta,\ y = 3 + 2\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$$

At $(-1, 3)$, $\theta = 0$, and $\frac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y = 5$

At $\left(\frac{4+3\sqrt{3}}{2}, 2\right)$, $\theta = \frac{7\pi}{6}$, and $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$.

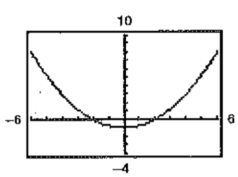
Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3}\left(x - \frac{4+3\sqrt{3}}{2}\right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

17. $x = 2t,\ y = t^2 - 1,\ t = 2$

(a)



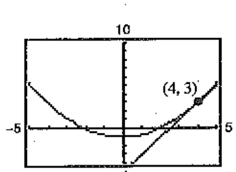
(b) At $t = 2$, $(x, y) = (4, 3)$, and

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 4,\ \frac{dy}{dx} = 2.$$

(c) $\frac{dy}{dx} = 2$. At $(4, 3)$, $y - 3 = 2(x - 4)$

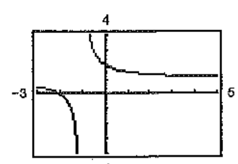
$$y = 2x - 5.$$

(d)



18. $x = t - 1,\ y = \frac{1}{t} + 1,\ t = 1$

(a)



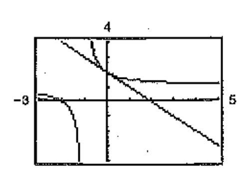
(b) At $t = 1$, $(x, y) = (0, 2)$, and

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1,\ \frac{dy}{dx} = -1.$$

(c) $\frac{dy}{dx} = -1$. At $(0, 2)$, $y - 2 = -1(x - 0)$

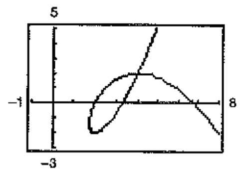
$$y = -x + 2.$$

(d)



19. $x = t^2 - t + 2,\ y = t^3 - 3t,\ t = -1$

(a)



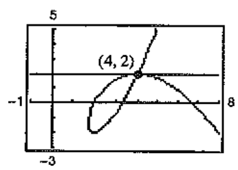
(b) At $t = -1$, $(x, y) = (4, 2)$, and

$$\frac{dx}{dt} = -3,\ \frac{dy}{dt} = 0,\ \frac{dy}{dx} = 0.$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$

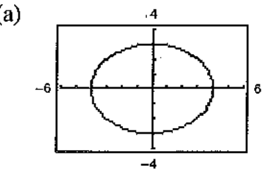
$$y = 2$$

(d)



20. $x = 4\cos\theta,\ y = 3\sin\theta,\ \theta = \frac{3\pi}{4}$

(a)



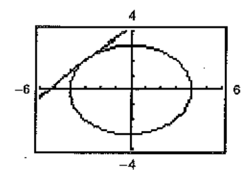
(b) At $\theta = \frac{3\pi}{4}$, $(x, y) = \left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, and

$$\frac{dx}{dt} = -2\sqrt{2},\ \frac{dy}{dt} = -\frac{3\sqrt{2}}{2},\ \frac{dy}{dx} = \frac{3}{4}.$$

(c) $\frac{dy}{dx} = \frac{3}{4}$. At $\left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, $y - \frac{3}{\sqrt{2}} = \frac{3}{4}\left(x + \frac{4}{\sqrt{2}}\right)$

$$y = \frac{3}{4}x + 3\sqrt{2}.$$

(d)



21. $x = 2\sin 2t, y = 3\sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$$

At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = \frac{-3}{4}x$. Tangent Line

22. $x = 2 - \pi\cos t, y = 2t - \pi\sin t$ crosses itself at a point on the x-axis: $(2, 0)$. The corresponding t-values are $t = \pm\pi/2$.

$$\frac{dy}{dt} = 2 - \pi\cos t,\ \frac{dx}{dt} = \pi\sin t,\ \frac{dy}{dx} = \frac{2 - \pi\cos t}{\pi\sin t}$$

At $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{2}{\pi}$.

Tangent line: $y - 0 = \frac{2}{\pi}(x - 2)$

$$y = \frac{2}{\pi}x - \frac{4}{\pi}$$

At $t = -\frac{\pi}{2}$: $\frac{dy}{dx} = -\frac{2}{\pi}$.

Tangent line: $y - 0 = -\frac{2}{\pi}(x - 2)$

$$y = -\frac{2}{\pi}x + \frac{4}{\pi}$$

23. $x = t^2 - t, y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1$, $\frac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2$, $\frac{dy}{dt} = \frac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$. Tangent Line

24. $x = t^3 - 6t$, $y = t^2$ crosses itself at $(0, 6)$. The corresponding t-values are $t = \pm\sqrt{6}$.

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$$

At $t = \sqrt{6}$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$.

Tangent line: $y - 6 = \frac{\sqrt{6}}{6}(x - 0)$

$$y = \frac{\sqrt{6}}{6}x + 6$$

At $t = -\sqrt{6}$, $\frac{dy}{dx} = -\frac{2\sqrt{6}}{12} = -\frac{\sqrt{6}}{6}$.

Tangent line: $y = -\frac{\sqrt{6}}{6}x + 6$

25. $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta\sin\theta = 0$ when $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Points: $(-1, [2n - 1]\pi)$, $(1, 2n\pi)$ where n is an integer. Points shown: $(1, 0)$, $(-1, \pi)$, $(1, -2\pi)$

Vertical tangents: $\frac{dx}{d\theta} = \theta\cos\theta = 0$ when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

Note: $\theta = 0$ corresponds to the cusp at $(x, y) = (1, 0)$.

$$\frac{dy}{dx} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta = 0 \text{ at } \theta = 0$$

Points: $\left(\frac{(-1)^{n+1}(2n - 1)\pi}{2}, (-1)^{n+1}\right)$ Points shown: $\left(\frac{\pi}{2}, 1\right)$, $\left(-\frac{3\pi}{2}, -1\right)$, $\left(\frac{5\pi}{2}, 1\right)$

26. $x = 2\theta$, $y = 2(1 - \cos\theta)$

Horizontal tangents: $\frac{dy}{d\theta} = 2\sin\theta = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

Points: $(4n\pi, 0)$, $(2[2n - 1]\pi, 4)$ where n is an integer

Points shown: $(0, 0)$, $(2\pi, 4)$, $(4\pi, 0)$

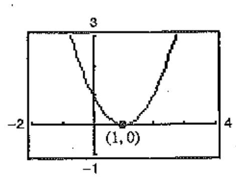
Vertical tangents: $\frac{dx}{d\theta} = 2 \neq 0$; none

27. $x = 1 - t$, $y = t^2$

Horizontal tangents: $\frac{dy}{dt} = 2t = 0$ when $t = 0$

Point: $(1, 0)$

Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none



28. $x = t + 1$, $y = t^2 + 3t$

Horizontal tangents: $\frac{dy}{dt} = 2t + 3 = 0$ when $t = -\frac{3}{2}$

Point: $\left(-\frac{1}{2}, -\frac{9}{4}\right)$

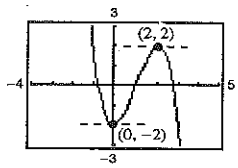
Vertical tangents: $\frac{dx}{dt} = 1 \neq 0$; none

29. $x = 1 - t,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(0, -2), (2, 2)$

Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$; none



30. $x = t^2 - t + 2,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 2t - 1 = 0$ when $t = \dfrac{1}{2}$.

Point: $\left(\dfrac{7}{4}, -\dfrac{11}{8}\right)$

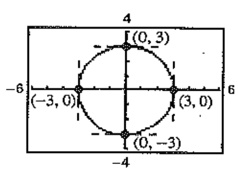
31. $x = 3\cos\theta, y = 3\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 3\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points: $(0, 3), (0, -3)$

Vertical tangents: $\dfrac{dx}{d\theta} = -3\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(3, 0), (-3, 0)$



32. $x = \cos\theta, y = 2\sin 2\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 4\cos 2\theta = 0$ when

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Points: $\left(\dfrac{\sqrt{2}}{2}, 2\right), \left(-\dfrac{\sqrt{2}}{2}, -2\right), \left(-\dfrac{\sqrt{2}}{2}, 2\right), \left(\dfrac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\dfrac{dx}{d\theta} = -\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$

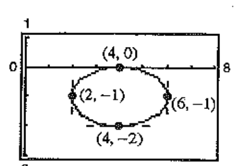
33. $x = 4 + 2\cos\theta,\ y = -1 + \sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points: $(4, 0), (4, -2)$

Vertical tangents: $\dfrac{dx}{d\theta} = -2\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(6, -1), (2, -1)$



34. $x = 4\cos^2\theta,\ y = 2\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Since $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -8\cos\theta\sin\theta = 0$ when

$\theta = 0, \pi$.

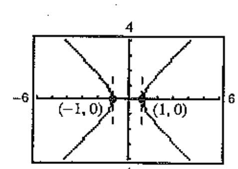
Point: $(4, 0)$

35. $x = \sec\theta,\ y = \tan\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \sec^2\theta \neq 0$; none

Vertical tangents: $\dfrac{dx}{d\theta} = \sec\theta\tan\theta = 0$ when $x = 0, \pi$.

Points: $(1, 0), (-1, 0)$



36. $x = \cos^2\theta,\ y = \cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = -\sin\theta = 0$ when $x = 0, \pi$.

Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -2\cos\theta\sin\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

(Exclude $0, \pi$.)

Point: $(0, 0)$

37. $x = t^2, y = t^3 - t$

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$$

$$\frac{d^2y}{dx^2} = \left[\frac{\dfrac{2t(6t) - (3t^2 - 1)2}{4t^2}}{2t}\right] = \frac{6t^2 + 2}{8t^3} = \frac{1 + 3t^2}{4t^3}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

38. $x = 2 + t^2, y = t^2 + t^3$

$$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

39. $x = 2t + \ln t, y = 2t - \ln t, t > 0$

$$\frac{dy}{dx} = \frac{2 - (1/t)}{2 + (1/t)} = \frac{2t - 1}{2t + 1}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(2t + 1)2 - (2t - 1)2}{(2t + 1)^2}\right]\Big/\left(2 + \frac{1}{t}\right)$$

$$= \frac{4}{(2t + 1)^2} \cdot \frac{t}{2t + 1}$$

$$= \frac{4t}{(2t + 1)^3}$$

Because $t > 0, \dfrac{d^2y}{dx^2} > 0$

Concave upward for $t > 0$

40. $x = t^2, y = \ln t, t > 0$

$$\frac{dy}{dx} = \frac{1/t}{2t} = \frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1/t^3}{2t} = -\frac{1}{2t^4}$$

Because $t > 0, \dfrac{d^2y}{dx^2} < 0$

Concave downward for $t > 0$

41. $x = \sin t, y = \cos t, 0 < t < \pi$

$$\frac{dy}{dx} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{\cos t} = -\frac{1}{\cos^3 t}$$

Concave upward on $\pi/2 < t < \pi$

Concave downward on $0 < t < \pi/2$

42. $x = 2\cos t, y = \sin t, 0 < t < 2\pi$

$$\frac{dy}{dx} = \frac{-\cos t}{2\sin t} = -\frac{1}{2}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{(1/2)\csc^2 t}{-2\sin t} = \frac{-1}{4\sin^3 t}$$

Concave upward on $\pi < t < 2\pi$

Concave downward on $0 < t < \pi$

43. $x = 2t - t^2, y = 2t^{3/2}, 1 \le t \le 2$

$$\frac{dx}{dt} = 2 - 2t, \frac{dy}{dt} = 3t^{1/2}$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{(2 - 2t)^2 + (3t^{1/2})^2}\, dt$$

$$= \int_1^2 \sqrt{4 - 8t + 4t^2 + 9t}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + t + 4}\, dt$$

44. $x = \ln t, y = t + 1, 1 \le t \le 6$

$$\frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = 1$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_1^6 \sqrt{\frac{1}{t^2} + 1}\,dt$$

45. $x = e^t + 2, y = 2t + 1, -2 \le t \le 2$

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = 2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{-2}^2 \sqrt{e^{2t} + 4}\,dt$$

46. $x = t + \sin t, y = t - \cos t, 0 \le t \le \pi$

$$\frac{dx}{dt} = 1 + \cos t, \frac{dy}{dt} = 1 + \sin t$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^\pi \sqrt{(1 + \cos t)^2 + (1 + \sin t)^2}\,dt$$

$$= \int_0^\pi \sqrt{3 + 2\cos t + 2\sin t}\,dt$$

47. $x = t^2, y = 2t, 0 \le t \le 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$s = 2\int_0^2 \sqrt{t^2 + 1}\,dt$$

$$= \left[t\sqrt{t^2 + 1} + \ln\left|t + \sqrt{t^2 + 1}\right|\right]_0^2$$

$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

48. $x = t^2 + 1, y = 4t^3 + 3, -1 \le t \le 0$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4$$

$$s = \int_{-1}^0 \sqrt{4t^2 + 144t^4}\,dt = \int_{-1}^0 -2t\sqrt{1 + 36t^2}\,dt$$

$$= \left[\frac{-(1 + 36t^2)^{3/2}}{54}\right]_{-1}^0 = \frac{-1}{54}(1 - 37^{3/2}) \approx 4.149$$

49. $x = e^{-t}\cos t, y = e^{-t}\sin t, 0 \le t \le \frac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t), \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{2e^{-2t}}\,dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\,dt$$

$$= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$$

50. $x = \arcsin t, y = \ln\sqrt{1 - t^2}, 0 \le t \le \frac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1 - t^2)^2}}\,dt = \int_0^{1/2} \frac{1}{1 - t^2}\,dt$$

$$= \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2}$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$

51. $x = \sqrt{t},\ y = 3t - 1,\ \frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 3$

$$s = \int_0^1 \sqrt{\frac{1}{4t} + 9}\,dt = \frac{1}{2}\int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}}\,dt$$

$$= \frac{1}{6}\int_0^6 \sqrt{1 + u^2}\,du$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{1 + u^2} + u\right) + u\sqrt{1 + u^2}\right]_0^6$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$$

$$u = 6\sqrt{t},\ du = \frac{3}{\sqrt{t}}\,dt$$

52. $x = t,\ y = \dfrac{t^5}{10} + \dfrac{1}{6t^3},\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = \dfrac{t^4}{2} - \dfrac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\, dt =$$
$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\, dt$$
$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right) dt$$
$$= \left[\frac{t^5}{10} - \frac{1}{6t^3}\right]_1^2 = \frac{779}{240}$$

53. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$
$$s = 4\int_0^{\pi/2} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$
$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$
$$= 6a\int_0^{\pi/2} \sin 2\theta\, d\theta = \Big[-3a\cos 2\theta\Big]_0^{\pi/2} = 6a$$

54. $x = a\cos\theta,\ y = a\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = a\cos\theta$

$$S = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\, d\theta$$
$$= 4a\int_0^{\pi/2} d\theta = \Big[4a\theta\Big]_0^{\pi/2} = 2\pi a$$

55. $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),$

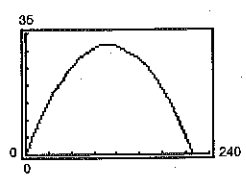
$$\frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$
$$s = 2\int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$
$$= 2\sqrt{2}a\int_0^{\pi} \sqrt{1 - \cos\theta}\, d\theta$$
$$= 2\sqrt{2}a\int_0^{\pi} \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\, d\theta$$
$$= \Big[-4\sqrt{2}a\sqrt{1 + \cos\theta}\Big]_0^{\pi} = 8a$$

56. $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta,\ \dfrac{dx}{d\theta} = \theta\cos\theta$

$$\frac{dy}{d\theta} = \theta\sin\theta$$
$$S = \int_0^{2\pi} \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\, d\theta$$
$$= \int_0^{2\pi} \theta\, d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi} = 2\pi^2$$

57. $x = (90\cos 30°)t,\ y = (90\sin 30°)t - 16t^2$

(a)



(b) Range: 219.2 ft, $\left(t = \dfrac{45}{16}\right)$

(c) $\dfrac{dx}{dt} = 90\cos 30°,\ \dfrac{dy}{dt} = 90\sin 30° - 32t$

$y = 0$ for $t = \dfrac{45}{16}$.

$$s = \int_0^{45/16} \sqrt{(90\cos 30°)^2 + (90\sin 30° - 32t)^2}\, dt$$
$$\approx 230.8 \text{ ft}$$

58. $y = 0 \Rightarrow (90 \sin \theta)t = 16t^2 \Rightarrow t = 0, \dfrac{90}{16} \sin \theta$

$$x = (90 \cos \theta)t = (90 \cos \theta)\frac{90}{16} \sin \theta$$

$$= \frac{90^2}{16} \sin \theta \cos \theta = \frac{90^2}{32} \sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32} 2 \cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

By the First Derivative Test, $\theta = \frac{\pi}{4}(45°)$ maximizes the range ($x = 253.125$ feet).

To maximize the arc length, we have $\dfrac{dx}{dt} = 90 \cos \theta$, $\dfrac{dy}{dt} = 90 \sin \theta - 32t$.

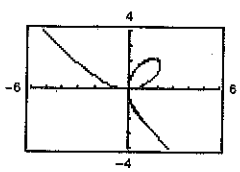
$$s = \int_0^{(90/16)\sin \theta} \sqrt{(90 \cos \theta)^2 + (90 \sin \theta - 32t)^2}\, dt$$

$$= \frac{2025}{8} \sin \theta + \frac{2025}{16} \cos^2 \theta \ln\left[\frac{1 + \sin \theta}{1 - \sin \theta}\right]$$

Using a graphing utility, we see that s is a maximum of approximately 303.67 feet at $\theta \approx 0.9855(56.5°)$.

59. $x = \dfrac{4t}{1 + t^3}$, $y = \dfrac{4t^2}{1 + t^3}$

(a) $x^3 + y^3 = 4xy$



(b) $\dfrac{dy}{dt} = \dfrac{(1 + t^3)(8t) - 4t^2(3t^2)}{(1 + t^3)^2}$

$= \dfrac{4t(2 - t^3)}{(1 + t^3)^2} = 0$ when $t = 0$ or $t = \sqrt[3]{2}$.

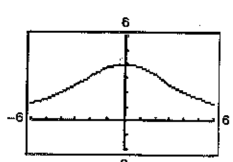
Points: $(0, 0), \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

(c) $$s = 2\int_0^1 \sqrt{\left[\frac{4(1 - 2t^3)}{(1 + t^3)^2}\right]^2 + \left[\frac{4t(2 - t^3)}{(1 + t^3)^2}\right]^2}\, dt = 2\int_0^1 \sqrt{\frac{16}{(1 + t^3)^4}[t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1]}\, dt$$

$$= 8\int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1 + t^3)^2}\, dt \approx 6.557$$

60. $x = 4 \cot \theta = \dfrac{4}{\tan \theta}$, $y = 4 \sin^2 \theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

(a)



(b) $\dfrac{dy}{d\theta} = 8 \sin \theta \cdot \cos \theta$

$\dfrac{dx}{d\theta} = -4 \csc^2 \theta$

$\dfrac{dy}{d\theta} = 0$ for $\theta = 0, \pm\dfrac{\pi}{2}$

Horizontal tangent at $(x, y) = (0, 4)\left(\theta = \pm\dfrac{\pi}{2}\right)$

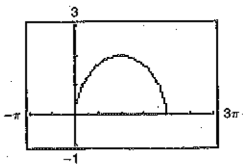
(Function is not defined at $\theta = 0$)

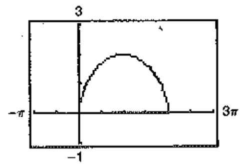
(c) Arc length over $\dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}$: 4.5183

61. (a) $x = t - \sin t$ $\quad$ $x = 2t - \sin(2t)$

$y = 1 - \cos t$ $\quad$ $y = 1 - \cos(2t)$

$0 \le t \le 2\pi$ $\quad$ $0 \le t \le \pi$





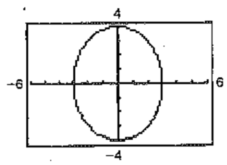
(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$

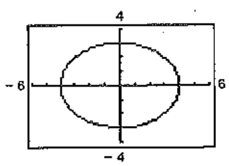
$y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is $t = 4\pi$.

62. (a) First particle: $x = 3\cos t,\ y = 4\sin t, 0 \le t \le 2\pi$



Second particle: $x = 4\sin t,\ y = 3\cos t,\ 0 \le t \le 2\pi$



(b) There are 4 points of intersection.

(c) Suppose at time t that

$3\cos t = 4\sin t$ and $4\sin t = 3\cos t$

$\tan t = \frac{3}{4}$ and $\tan t = \frac{3}{4}$

Yes, the particles are at the same place at the same time for $\tan t = \frac{3}{4}$. $t \approx 0.6435, 3.7851$. The intersection points are $(2.4, 2.4)$ and $(-2.4, -2.4)$.

(d) The curves intersect twice, but not at the same time.

63. $x = 4t,\ \dfrac{dx}{dt} = 4$

$y = t + 1,\ \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^2 (t+1)\sqrt{4^2 + 1^2}\,dt$$

$$= 2\pi\int_0^2 \sqrt{17}(t+1)\,dt = 8\pi\sqrt{17}$$

$$\approx 103.6249$$

64. $x = \dfrac{1}{4}t^2,\ \dfrac{dx}{dt} = \dfrac{1}{2}t$

$y = t + 2,\ \dfrac{dy}{dt} = 1$

$$S = \pi\int_0^4 (t+2)\sqrt{\frac{t^2}{4} + 1}\,dt$$

$$= \pi\int_0^4 (t+2)\sqrt{t^2 + 4}\,dt$$

$$\approx 159.6264$$

65. $x = \cos^2\theta,\ \dfrac{dx}{d\theta} = -2\cos\theta\sin\theta$

$y = \cos\theta,\ \dfrac{dy}{d\theta} = -\sin\theta$

$$S = 2\pi\int_0^{\pi/2} \cos\theta\sqrt{4\cos^2\theta\sin^2\theta + \sin^2\theta}\,d\theta$$

$$\approx 5.3304$$

66. $x = \theta + \sin\theta,\ \dfrac{dx}{d\theta} = 1 + \cos\theta$

$y = \theta + \cos\theta,\ \dfrac{dy}{d\theta} = 1 - \sin\theta$

$$S = 2\pi\int_0^{\pi/2} (\theta + \cos\theta)\sqrt{(1+\cos\theta)^2 + (1-\sin\theta)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2} (\theta + \cos\theta)\sqrt{3 + 2\cos\theta - 2\sin\theta}\,d\theta$$

$$\approx 23.2433$$

67. $x = t,\ y = 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 2$

(a) $$S = 2\pi\int_0^4 2t\sqrt{1+4}\,dt = 4\sqrt{5}\pi\int_0^4 t\,dt$$

$$= \left[2\sqrt{5}\pi t^2\right]_0^4 = 32\pi\sqrt{5}$$

(b) $$S = 2\pi\int_0^4 t\sqrt{1+4}\,dt = 2\sqrt{5}\pi\int_0^4 t\,dt$$

$$= \left[\sqrt{5}\pi t^2\right]_0^4 = 16\pi\sqrt{5}$$

68. $x = t,\ y = 4 - 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = -2$

(a) $S = 2\pi \int_0^2 (4 - 2t)\sqrt{1 + 4}\, dt$

$= \left[2\sqrt{5}\pi(4t - t^2)\right]_0^2 = 8\pi\sqrt{5}$

(b) $S = 2\pi \int_0^2 t\sqrt{1 + 4}\, dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

69. $x = 4\cos\theta,\ y = 4\sin\theta,\ \dfrac{dx}{d\theta} = -4\sin\theta,\ \dfrac{dy}{d\theta} = 4\cos\theta$

$S = 2\pi \int_0^{\pi/2} 4\cos\theta\sqrt{(-4\sin\theta)^2 + (4\cos\theta)^2}\, d\theta$

$= 32\pi \int_0^{\pi/2} \cos\theta\, d\theta = \left[32\pi\sin\theta\right]_0^{\pi/2} = 32\pi$

70. $x = \frac{1}{3}t^3,\ y = t + 1,\ 1 \le t \le 2$, y-axis

$\dfrac{dx}{dt} = t^2,\ \dfrac{dy}{dt} = 1$

$S = 2\pi \int_1^2 \frac{1}{3}t^3\sqrt{t^4 + 1}\, dt = \frac{\pi}{9}\left[(x^4 + 1)^{3/2}\right]_1^2$

$= \frac{\pi}{9}(17^{3/2} - 2^{3/2}) \approx 23.48$

71. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$S = 4\pi \int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta = 12a^2\pi \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta = \frac{12\pi a^2}{5}\left[\sin^5\theta\right]_0^{\pi/2} = \frac{12}{5}\pi a^2$

72. $x = a\cos\theta,\ y = b\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

(a) $S = 4\pi \int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$= 4\pi \int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta}\, d\theta = \frac{-4ab\pi}{e}\int_0^{\pi/2} (-e\sin\theta)\sqrt{1 - e^2\cos^2\theta}\, d\theta$

$= \frac{-2ab\pi}{e}\left[e\cos\theta\sqrt{1 - e^2\cos^2\theta} + \arcsin(e\cos\theta)\right]_0^{\pi/2} = \frac{2ab\pi}{e}\left[e\sqrt{1 - e^2} + \arcsin(e)\right]$

$= 2\pi b^2 + \left(\frac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right)\arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$

$\left(e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}\text{: eccentricity}\right)$

(b) $S = 4\pi \int_0^{\pi/2} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$= 4\pi \int_0^{\pi/2} a\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\, d\theta = \frac{4a\pi}{c}\int_0^{\pi/2} c\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\, d\theta$

$= \frac{2a\pi}{c}\left[c\sin\theta\sqrt{b^2 + c^2\sin^2\theta} + b^2\ln\left|c\sin\theta + \sqrt{b^2 + c^2\sin^2\theta}\right|\right]_0^{\pi/2}$

$= \frac{2a\pi}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left|c + \sqrt{b^2 + c^2}\right| - b^2\ln b\right]$

$= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\left|\frac{a + \sqrt{a^2 - b^2}}{b}\right| = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left|\frac{1 + e}{1 - e}\right|$

73. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

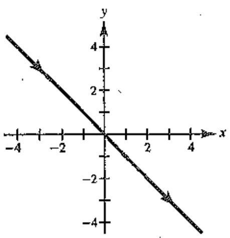
See Theorem 10.7.

74. (a) 0

(b) 4

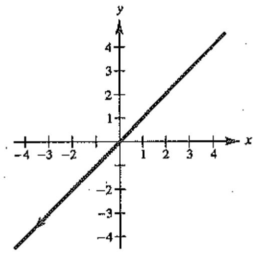
75. One possible answer is the graph given by

$x = t, y = -t.$



76. One possible answer is the graph given by

$x = -t, y = -t.$



77. $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

See Theorem 10.8.

78. (a) $S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

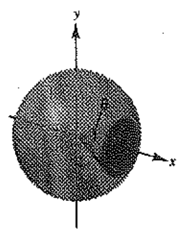
(b) $S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

79. Let y be a continuous function of x on $a \le x \le b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t)\, dt$ and

$$\int_a^b y\, dx = \int_{t_1}^{t_2} g(t)f'(t)\, dt.$$

80. $x = r\cos\phi,\ y = r\sin\phi$

$$S = 2\pi\int_0^\theta r\sin\phi\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}\, d\phi$$
$$= 2\pi r^2\int_0^\theta \sin\phi\, d\phi$$
$$= \Big[-2\pi r^2\cos\phi\Big]_0^\theta$$
$$= 2\pi r^2(1 - \cos\theta)$$

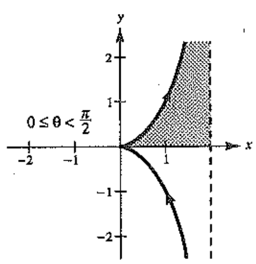


81. $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

$$\frac{dx}{d\theta} = 4\sin\theta\cos\theta$$

$$A = \int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)\, d\theta = 8\int_0^{\pi/2}\sin^4\theta\, d\theta$$
$$= 8\left[\frac{-\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$



82. $x = 2\cot\theta,\ y = 2\sin^2\theta,\ \frac{dx}{d\theta} = -2\csc^2\theta$

$$A = 2\int_{\pi/2}^0 (2\sin^2\theta)(-2\csc^2\theta)\, d\theta = -8\int_{\pi/2}^0 d\theta = \Big[-8\theta\Big]_{\pi/2}^0 = 4\pi$$

83. πab is area of ellipse (d).

84. $\frac{3}{8}\pi a^2$ is area of asteroid (b).

85. $6\pi a^2$ is area of cardioid (f).

86. $2\pi a^2$ is area of deltoid (c).

87. $\frac{8}{3}ab$ is area of hourglass (a).

88. $2\pi ab$ is area of teardrop (e).

89. $x = \sqrt{t},\ y = 4 - t,\ 0 < t < 4$

$$A = \int_0^2 y\,dx = \int_0^4 (4-t)\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^4 (4t^{-1/2} - t^{1/2})\,dt = \left[\frac{1}{2}\left(8\sqrt{t} - \frac{2}{3}t\sqrt{t}\right)\right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{1}{A}\int_0^2 yx\,dx = \frac{3}{16}\int_0^4 (4-t)\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)dt = \frac{3}{32}\int_0^4 (4-t)\,dt = \left[\frac{3}{32}\left(4t - \frac{t^2}{2}\right)\right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{1}{A}\int_0^2 \frac{y^2}{2}\,dx = \frac{3}{32}\int_0^4 (4-t)^2\frac{1}{2\sqrt{t}}\,dt = \frac{3}{64}\int_0^4 [16t^{-1/2} - 8t^{1/2} + t^{3/2}]\,dt = \frac{3}{64}\left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

90. $x = \sqrt{4-t},\ y = \sqrt{t},\ \dfrac{dx}{dt} = -\dfrac{1}{2\sqrt{4-t}},\ 0 \le t \le 4$

$$A = \int_4^0 \sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right)dt = \int_0^2 \sqrt{4-u^2}\,du = \frac{1}{2}\left[u\sqrt{4-u^2} + 4\arcsin\frac{u}{2}\right]_0^2 = \pi$$

Let $u = \sqrt{4-t}$, then $du = -1/(2\sqrt{4-t})\,dt$ and $\sqrt{t} = \sqrt{4-u^2}$.

$$\bar{x} = \frac{1}{\pi}\int_4^0 \sqrt{4-t}\sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right)dt = -\frac{1}{2\pi}\int_4^0 \sqrt{t}\,dt = \left[-\frac{1}{2\pi}\frac{2}{3}t^{3/2}\right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi}\int_4^0 (\sqrt{t})^2\left(-\frac{1}{2\sqrt{4-t}}\right)dt = -\frac{1}{4\pi}\int_4^0 \frac{t}{\sqrt{4-t}}\,dt = -\frac{1}{4\pi}\left[\frac{-2(8+t)}{3}\sqrt{4-t}\right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

91. $x = 3\cos\theta,\ y = 3\sin\theta,\ \dfrac{dx}{d\theta} = -3\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-3\sin\theta)\,d\theta \\ &= -54\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -54\pi\int_{\pi/2}^0 (1-\cos^2\theta)\sin\theta\,d\theta \\ &= -54\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 36\pi \text{ (Sphere)} \end{aligned}$$

92. $x = \cos\theta,\ y = 3\sin\theta,\ \dfrac{dx}{d\theta} = -\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-\sin\theta)\,d\theta \\ &= -18\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -18\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 12\pi \end{aligned}$$

93. $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

(a) $\dfrac{dy}{d\theta} = a\sin\theta,\ \dfrac{dx}{d\theta} = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}\right]\Big/[a(1-\cos\theta)]$$

$$= \frac{\cos\theta - 1}{a(1-\cos\theta)^3} = \frac{-1}{a(\cos\theta - 1)^2}$$

(b) At $\theta = \dfrac{\pi}{6},\ x = a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right),\ y = a\left(1 - \dfrac{\sqrt{3}}{2}\right)\dfrac{dy}{dx} = \dfrac{1/2}{1 - \sqrt{3}/2} = 2 + \sqrt{3}.$

Tangent line: $y - a\left(1 - \dfrac{\sqrt{3}}{2}\right) = (2 + \sqrt{3})\left(x - a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right)\right)$

—CONTINUED—

93. —CONTINUED—

(c) $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1-\cos\theta} = 0 \Rightarrow \sin\theta = 0, 1 - \cos\theta \neq 0$

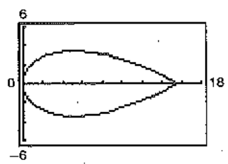
Points of horizontal tangency: $(x, y) = (a(2n+1)\pi, 2a)$

(d) Concave downward on all open θ-intervals:
$\ldots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \ldots$

(e) $$s = \int_0^{2\pi} \sqrt{a^2\sin^2\theta + a^2(1-\cos\theta)^2}\,d\theta$$
$$= a\int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta$$
$$= a\int_0^{2\pi}\sqrt{4\sin^2\theta/2}\,d\theta$$
$$= 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$
$$= -4a\cos\left(\frac{\theta}{2}\right)\Big]_0^{2\pi} = 8a$$

94. $x = t^2\sqrt{3}, y = 3t - \frac{1}{3}t^3$

(a)



(b) $\dfrac{dx}{dt} = 2\sqrt{3}t, \dfrac{dy}{dt} = 3 - t^2, \dfrac{dy}{dx} = \dfrac{3-t^2}{2\sqrt{3}t}$

$$\frac{d^2y}{dx^2} = \left[\frac{2\sqrt{3}(t)(-2t) - (3-t^2)2\sqrt{3}}{12t^2}\right]\Big/\left[2\sqrt{3}t\right]$$
$$= \frac{-2\sqrt{3}t^2 - 6\sqrt{3}}{(12t^2)(2\sqrt{3}t)} = -\frac{t^2+3}{12t^3}$$

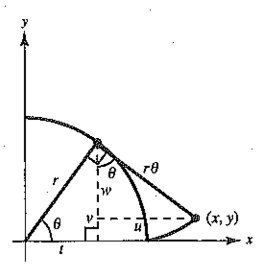
(c) $(x, y) = \left(\sqrt{3}, \frac{8}{3}\right)$ at $t = 1$. $\dfrac{dy}{dx} = \dfrac{2}{2\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$$y - \frac{8}{3} = \frac{\sqrt{3}}{3}\left(x - \sqrt{3}\right)$$
$$y = \frac{\sqrt{3}}{3}x + \frac{5}{3}$$

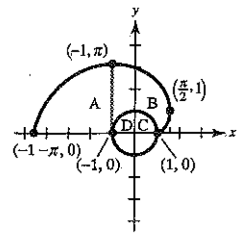
(d) $$s = \int_{-3}^{3}\sqrt{12t^2 + (3-t)^2}\,dt$$
$$= \int_{-3}^{3}\sqrt{t^4 - 6t^2 + 9 + 12t^2}\,dt$$
$$= \int_{-3}^{3}\sqrt{(t^2+3)^2}\,dt$$
$$= \int_{-3}^{3}(t^2+3)\,dt = 36$$

(e) $$S = 2\pi\int_0^3\left(3t - \tfrac{1}{3}t^3\right)(t^2+3)\,dt = 81\pi$$

95. $x = t + u = r\cos\theta + r\theta\sin\theta$
$= r(\cos\theta + \theta\sin\theta)$
$y = v - w = r\sin\theta - r\theta\cos\theta$
$= r(\sin\theta - \theta\cos\theta)$



96. Let's focus on the region above the x-axis. From Exercise 95, the equation of the involute from $(1, 0)$ to $(-1, \pi)$ is



$$x = \cos\theta + \theta\sin\theta$$

$$y = \sin\theta - \theta\cos\theta$$

$$0 \le \theta \le \pi.$$

At $(-1, \pi)$, the string is fully extended and has length π. So, the area of region A is $\frac{1}{4}\pi(\pi^2) = \frac{1}{4}\pi^3$.

We now need to find the area of region B.

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta = 0 \implies \theta = \frac{\pi}{2}. \ (\theta = 0 \text{ is cusp.})$$

Hence, the far right point on the involute is $(\pi/2, 1)$.

The area of the region B + C + D is given by

$$\int_{\theta=\pi}^{\theta=\pi/2} y\,dx - \int_{\theta=0}^{\theta=\pi/2} y\,dx = \int_{\theta=\pi}^{\theta=0} y\,dx$$

where $y = \sin\theta - \theta\cos\theta$ and $dx = \theta\cos\theta\,d\theta$.

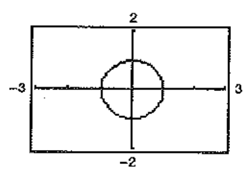
Thus, we can calculate

$$\int_{\pi}^{0} [\sin\theta - \theta\cos\theta]\theta\cos\theta\,d\theta = \frac{\pi}{6}(\pi^2 + 3).$$

Since the area of C + D is $\pi/2$, we have

$$\text{Total area covered} = 2\left[\frac{1}{4}\pi^3 + \frac{\pi}{6}(\pi^2 + 3) - \frac{\pi}{2}\right] = \frac{5}{6}\pi^3.$$

97. (a)



(b) $x = \dfrac{1 - t^2}{1 + t^2}, \ y = \dfrac{2t}{1 + t^2}, \ -20 \le t \le 20$

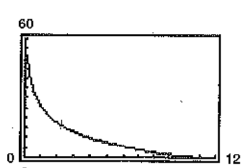
The graph (for $-\infty < t < \infty$) is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

Verify: $x^2 + y^2 = \left(\dfrac{1 - t^2}{1 + t^2}\right)^2 + \left(\dfrac{2t}{1 + t^2}\right)^2 = \dfrac{1 - 2t^2 + t^4 + 4t^2}{(1 + t^2)^2} = \dfrac{(1 + t^2)^2}{(1 + t^2)^2} = 1$

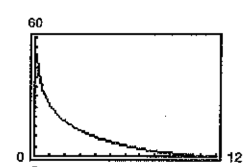
(c) As t increases from -20 to 0, the speed increases, and as t increases from 0 to 20, the speed decreases.

98. (a) $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$0 < x \le 12$



(b) $x = 12\,\text{sech}\,\dfrac{t}{12}, \ y = t - 12\tanh\dfrac{t}{12}, \ 0 \le t$



Same as the graph in (a), but has the advantage of showing the position of the object and any given time t.

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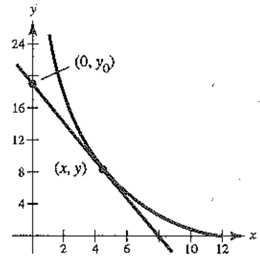
98. —CONTINUED—

(c) $\dfrac{dy}{dx} = \dfrac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12)\tanh(t/12)} = -\sinh\dfrac{t}{12}$



Tangent line: $y - \left(t_0 - 12\tanh\dfrac{t_0}{12}\right) = -\sinh\dfrac{t_0}{12}\left(x - 12\operatorname{sech}\dfrac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh\frac{t_0}{12}\right)x$$

y-intercept: $(0, t_0)$

Distance between $(0, t_0)$ and (x, y): $d = \sqrt{\left(12\operatorname{sech}\dfrac{t_0}{12}\right)^2 + \left(-12\tanh\dfrac{t_0}{12}\right)^2} = 12$

$$d = 12 \text{ for any } t \geq 0.$$

99. False. $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$

100. False. Both dx/dt and dy/dt are zero when $t = 0$. By eliminating the parameter, we have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

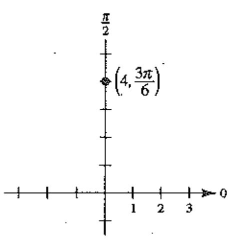
Section 10.4 Polar Coordinates and Polar Graphs

1. $\left(4, \dfrac{\pi}{2}\right)$

$x = 4\cos\left(\dfrac{\pi}{2}\right) = 0$

$y = 4\sin\left(\dfrac{\pi}{2}\right) = 4$

$(x, y) = (0, 4)$

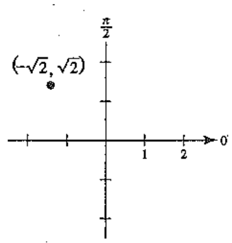


2. $\left(-2, \dfrac{7\pi}{4}\right)$

$x = -2\cos\left(\dfrac{7\pi}{4}\right) = -\sqrt{2}$

$y = -2\sin\left(\dfrac{7\pi}{4}\right) = \sqrt{2}$

$(x, y) = \left(-\sqrt{2}, \sqrt{2}\right)$

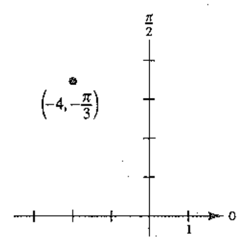


3. $\left(-4, -\dfrac{\pi}{3}\right)$

$x = -4\cos\left(-\dfrac{\pi}{3}\right) = -2$

$y = -4\sin\left(-\dfrac{\pi}{3}\right) = 2\sqrt{3}$

$(x, y) = \left(-2, 2\sqrt{3}\right)$

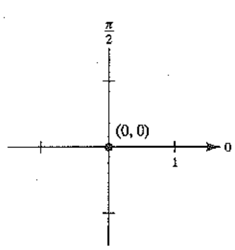


4. $\left(0, -\dfrac{7\pi}{6}\right)$

$x = 0\cos\left(-\dfrac{7\pi}{6}\right) = 0$

$y = 0\sin\left(-\dfrac{7\pi}{6}\right) = 0$

$(x, y) = (0, 0)$

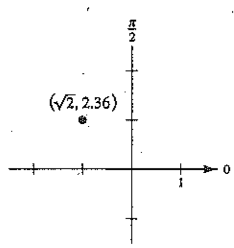


5. $\left(\sqrt{2}, 2.36\right)$

$x = \sqrt{2}\cos(2.36) \approx -1.004$

$y = \sqrt{2}\sin(2.36) \approx 0.996$

$(x, y) = (-1.004, 0.996)$

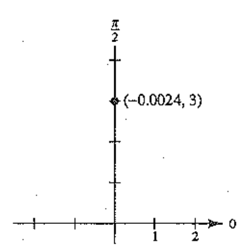


6. $(-3, -1.57)$

$x = -3\cos(-1.57) \approx -0.0024$

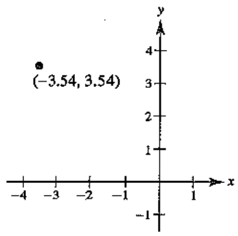
$y = -3\sin(-1.57) \approx 3$

$(x, y) = (-0.0024, 3)$



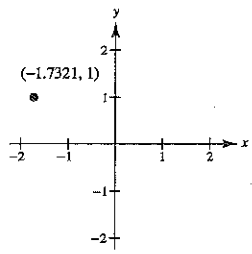
7. $(r, \theta) = \left(5, \dfrac{3\pi}{4}\right)$

$(x, y) = (-3.5355, 3.5355)$



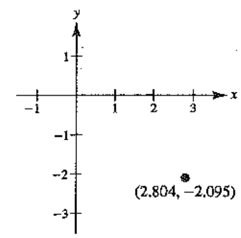
8. $(r, \theta) = \left(-2, \dfrac{11\pi}{6}\right)$

$(x, y) = (-1.7321, 1)$



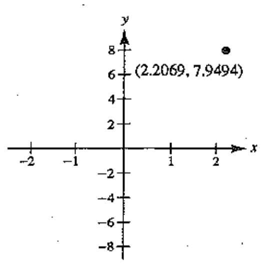
9. $(r, \theta) = (-3.5, 2.5)$

$(x, y) = (2.804, -2.095)$



10. $(r, \theta) = (8.25, 1.3)$

$(x, y) = (2.2069, 7.9494)$

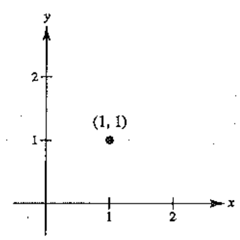


11. $(x, y) = (1, 1)$

$r = \pm\sqrt{2}$

$\tan\theta = 1$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \left(\sqrt{2}, \dfrac{\pi}{4}\right), \left(-\sqrt{2}, \dfrac{5\pi}{4}\right)$

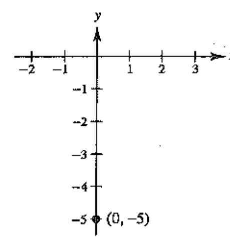


12. $(x, y) = (0, -5)$

$r = \pm 5$

$\tan\theta$ undefined

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \left(5, \dfrac{3\pi}{2}\right), \left(-5, \dfrac{\pi}{2}\right)$

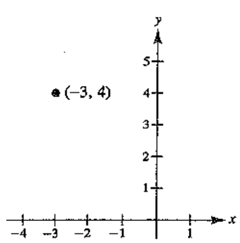


13. $(x, y) = (-3, 4)$

$r = \pm\sqrt{9 + 16} = \pm 5$

$\tan\theta = -\frac{4}{3}$

$\theta \approx 2.214, 5.356, (5, 2.214), (-5, 5.356)$



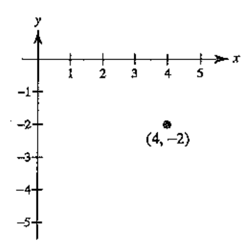
14. $(x, y) = (4, -2)$

$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$

$\tan\theta = -\dfrac{2}{4} = -\dfrac{1}{2}$

$\theta \approx -0.464$

$\left(2\sqrt{5}, -0.464\right), \left(-2\sqrt{5}, 2.678\right)$



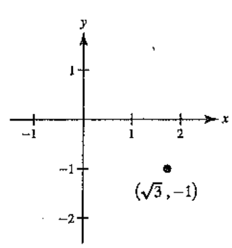
15. $(x, y) = \left(\sqrt{3}, -1\right)$

$r = \sqrt{3 + 1} = 2$

$\tan\theta = -\sqrt{3}/3$

$(r, \theta) = (2, 11\pi/6)$

$= (-2, 5\pi/6)$



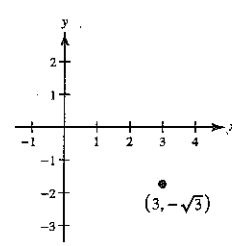
16 $(x, y) = \left(3, \sqrt{3}\right)$

$r = \sqrt{9 + 3} = 2\sqrt{3}$

$\tan\theta = -\sqrt{3}/3$

$(r, \theta) = \left(2\sqrt{3}, 11\pi/6\right)$

$= \left(-2\sqrt{3}, 5\pi/6\right)$



17. $(x, y) = (3, -2)$

$(r, \theta) = (3.606, -0.588)$

18. $(x, y) = (3\sqrt{2}, 3\sqrt{2})$

$(r, \theta) = (6, 0.785)$

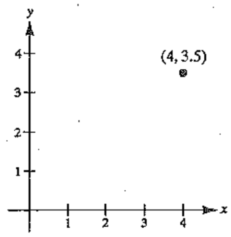
19. $(x, y) = \left(\frac{5}{2}, \frac{4}{3}\right)$

$(r, \theta) = (2.833, 0.490)$

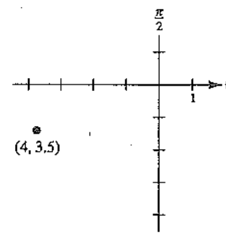
20. $(x, y) = (0, -5)$

$(r, \theta) = (5, -1.571)$

21. (a) $(x, y) = (4, 3.5)$



(b) $(r, \theta) = (4, 3.5)$



22. (a) Moving horizontally, the x-coordinate changes. Moving vertically, the y-coordinate changes.

(b) Both r and θ values change.

(c) In polar mode, horizontal (or vertical) changes result in changes in both r and θ.

23. $r = 2 \sin \theta$ circle

Matches (c)

24. $r = 4 \cos 2\theta$

Rose curve

Matches (b)

25. $r = 3(1 + \cos \theta)$

Cardioid

Matches (a)

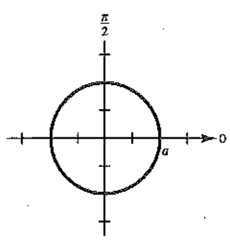
26. $r = 2 \sec \theta$

Line

Matches (d)

27. $x^2 + y^2 = a^2$

$r = a$

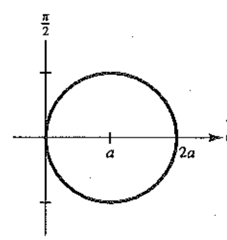


28. $x^2 + y^2 - 2ax = 0$

$r^2 - 2ar \cos \theta = 0$

$r(r - 2a \cos \theta) = 0$

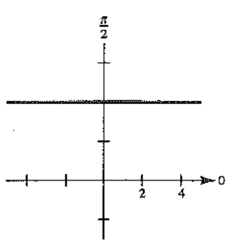
$r = 2a \cos \theta$



29. $y = 4$

$r \sin \theta = 4$

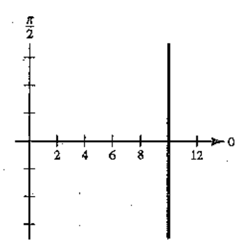
$r = 4 \csc \theta$



30. $x = 10$

$r \cos \theta = 10$

$r = 10 \sec \theta$

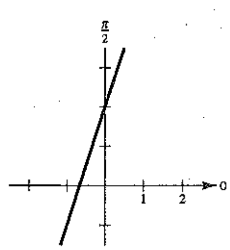


31. $3x - y + 2 = 0$

$3r \cos \theta - r \sin \theta + 2 = 0$

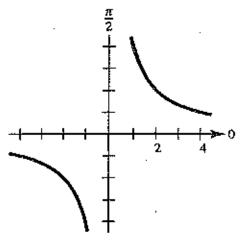
$r(3 \cos \theta - \sin \theta) = -2$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$



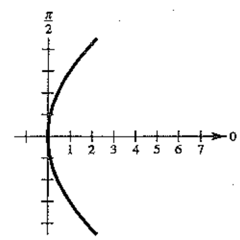
32.

$$xy = 4$$
$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2 = 4\sec\theta\csc\theta$$
$$= 8\csc 2\theta$$



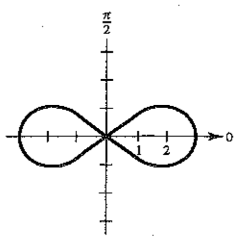
33.

$$y^2 = 9x$$
$$r^2\sin^2\theta = 9r\cos\theta$$
$$r = \frac{9\cos\theta}{\sin^2\theta}$$
$$r = 9\csc^2\theta\cos\theta$$



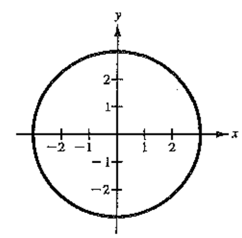
34.

$$(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$$
$$(r^2)^2 - 9(r^2\cos^2\theta - r^2\sin^2\theta) = 0$$
$$r^2[r^2 - 9(\cos 2\theta)] = 0$$
$$r^2 = 9\cos 2\theta$$



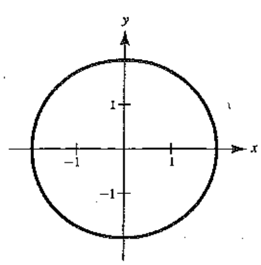
35.

$$r = 3$$
$$r^2 = 9$$
$$x^2 + y^2 = 9$$



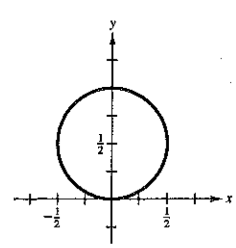
36.

$$r = -2$$
$$r^2 = 4$$
$$x^2 + y^2 = 4$$



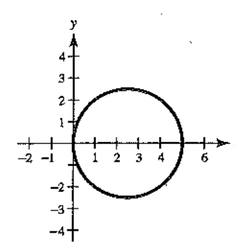
37.

$$r = \sin\theta$$
$$r^2 = r\sin\theta$$
$$x^2 + y^2 = y$$
$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$
$$x^2 + y^2 - y = 0$$



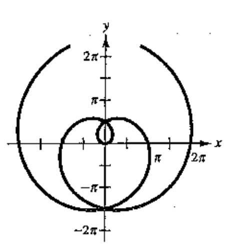
38.

$$r = 5\cos\theta$$
$$r^2 = 5r\cos\theta$$
$$x^2 + y^2 = 5x$$
$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$
$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$



39.

$$r = \theta$$
$$\tan r = \tan\theta$$
$$\tan\sqrt{x^2 + y^2} = \frac{y}{x}$$
$$\sqrt{x^2 + y^2} = \arctan\frac{y}{x}$$

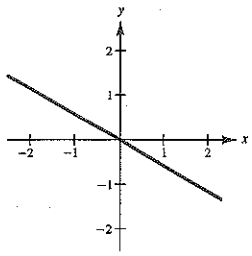


40. $\theta = \dfrac{5\pi}{6}$

$\tan\theta = \tan\dfrac{5\pi}{6}$

$\dfrac{y}{x} = -\dfrac{\sqrt{3}}{3}$

$y = -\dfrac{\sqrt{3}}{3}x$

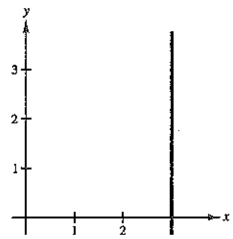


41. $r = 3\sec\theta$

$r\cos\theta = 3$

$x = 3$

$x - 3 = 0$

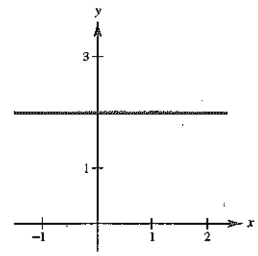


42. $r = 2\csc\theta$

$r\sin\theta = 2$

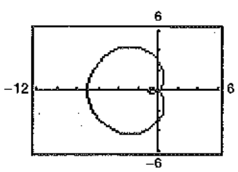
$y = 2$

$y - 2 = 0$



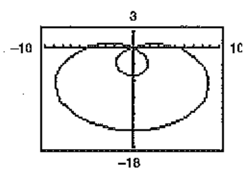
43. $r = 3 - 4\cos\theta$

$0 \le \theta < 2\pi$



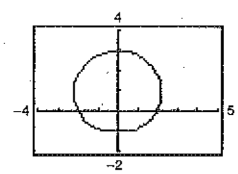
44. $r = 5(1 - 2\sin\theta)$

$0 \le \theta < 2\pi$



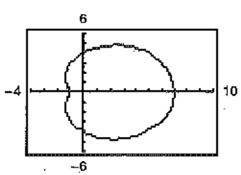
45. $r = 2 + \sin\theta$

$0 \le \theta < 2\pi$



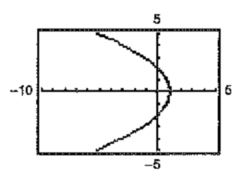
46. $r = 4 + 3\cos\theta$

$0 \le \theta < 2\pi$



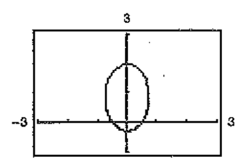
47. $r = \dfrac{2}{1 + \cos\theta}$

Traced out once on $-\pi < \theta < \pi$



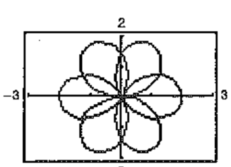
48. $r = \dfrac{2}{4 - 3\sin\theta}$

Traced out once on $0 \le \theta \le 2\pi$



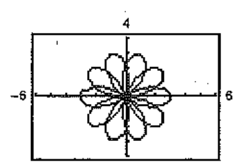
49. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



50. $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$

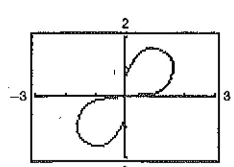


51. $r^2 = 4\sin 2\theta$

$r_1 = 2\sqrt{\sin 2\theta}$

$r_2 = -2\sqrt{\sin 2\theta}$

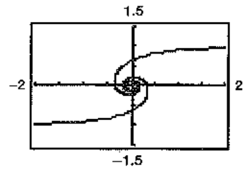
$0 \le \theta < \dfrac{\pi}{2}$



52. $r^2 = \dfrac{1}{\theta}$

Graph as $r_1 = \dfrac{1}{\sqrt{\theta}},\ r_2 = -\dfrac{1}{\sqrt{\theta}}$.

It is traced out once on $[0, \infty)$.



53.

$$\begin{aligned} r &= 2(h\cos\theta + k\sin\theta) \\ r^2 &= 2r(h\cos\theta + k\sin\theta) \\ r^2 &= 2[h(r\cos\theta) + k(r\sin\theta)] \\ x^2 + y^2 &= 2(hx + ky) \\ x^2 + y^2 - 2hx - 2ky &= 0 \\ (x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) &= 0 + h^2 + k^2 \\ (x - h)^2 + (y - k)^2 &= h^2 + k^2 \end{aligned}$$

Radius: $\sqrt{h^2 + k^2}$

Center: (h, k)

54. (a) The rectangular coordinates of (r_1, θ_1) are $(r_1\cos\theta_1, r_1\sin\theta_1)$. The rectangular coordinates of (r_2, θ_2) are $(r_2\cos\theta_2, r_2\sin\theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\ &= r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2^2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1 \\ &= r_2^2(\cos^2\theta_2 + \sin^2\theta_2) + r_1^2(\cos^2\theta_1 + \sin^2\theta_1) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|. \end{aligned}$$

(c) If $\theta_1 - \theta_2 = 90°$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = (2, \pi/2)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2)\cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

Using $(r_1, \theta_1) = (-1, \pi)$ and $(r_2, \theta_2) = [2, (5\pi/2)]$, $d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2)\cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}$.

You always obtain the same distance.

55. $\left(4, \dfrac{2\pi}{3}\right), \left(2, \dfrac{\pi}{6}\right)$

$$d = \sqrt{4^2 + 2^2 - 2(4)(2)\cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)}$$

$$= \sqrt{20 - 16\cos\frac{\pi}{2}} = 2\sqrt{5} \approx 4.5$$

56. $\left(10, \dfrac{7\pi}{6}\right), (3, \pi)$

$$d = \sqrt{10^2 + 3^2 - 2(10)(3)\cos\left(\frac{7\pi}{6} - \pi\right)}$$

$$= \sqrt{109 - 60\cos\frac{\pi}{6}} = \sqrt{109 - 30\sqrt{3}} \approx 7.6$$

57. $(2, 0.5),\ (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28\cos(-0.7)} \approx 5.6$$

58. $(4, 2.5),\ (12, 1)$

$$d = \sqrt{4^2 + 12^2 - 2(4)(12)\cos(2.5 - 1)}$$

$$= \sqrt{160 - 96\cos 1.5} \approx 12.3$$

59. $r = 2 + 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + \cos\theta(2 + 3\sin\theta)}{3\cos\theta\cos\theta - \sin\theta(2 + 3\sin\theta)}$$

$$= \frac{2\cos\theta(3\sin\theta + 1)}{3\cos 2\theta - 2\sin\theta} = \frac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$$

At $\left(5, \dfrac{\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

At $(2, \pi)$, $\dfrac{dy}{dx} = -\dfrac{2}{3}$.

At $\left(-1, \dfrac{3\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

60. $r = 2(1 - \sin\theta)$

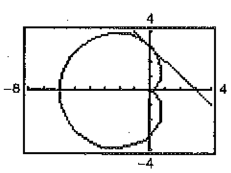
$$\frac{dy}{dx} = \frac{-2\cos\theta\sin\theta + 2\cos\theta(1 - \sin\theta)}{-2\cos\theta\cos\theta - 2\sin\theta(1 - \sin\theta)}$$

At $(2, 0)$, $\dfrac{dy}{dx} = -1$.

At $\left(3, \dfrac{7\pi}{6}\right)$, $\dfrac{dy}{dx}$ is undefined.

At $\left(4, \dfrac{3\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

61. (a), (b) $r = 3(1 - \cos\theta)$



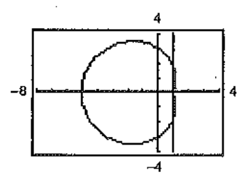
$(r, \theta) = \left(3, \dfrac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$

Tangent line: $y - 3 = -1(x - 0)$

$y = -x + 3$

(c) At $\theta = \dfrac{\pi}{2}, \dfrac{dy}{dx} = -1.0$.

62. (a), (b) $r = 3 - 2\cos\theta$

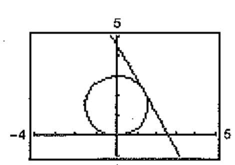


$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$

Tangent line: $x = 1$

(c) At $\theta = 0$, $\dfrac{dy}{dx}$ does not exist (vertical tangent).

63. (a), (b) $r = 3\sin\theta$



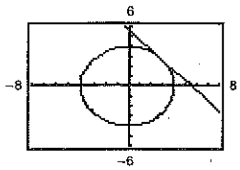
$(r, \theta) = \left(\dfrac{3\sqrt{3}}{2}, \dfrac{\pi}{3}\right) \Rightarrow (x, y) = \left(\dfrac{3\sqrt{3}}{4}, \dfrac{9}{4}\right)$

Tangent line: $y - \dfrac{9}{4} = -\sqrt{3}\left(x - \dfrac{3\sqrt{3}}{4}\right)$

$y = -\sqrt{3}x + \dfrac{9}{2}$

(c) At $\theta = \dfrac{\pi}{3}, \dfrac{dy}{dx} = -\sqrt{3} \approx -1.732$.

64. (a), (b) $r = 4$



$(r, \theta) = \left(4, \frac{\pi}{4}\right) \Rightarrow (x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

Tangent line: $y - 2\sqrt{2} = -1\left(x - 2\sqrt{2}\right)$

$y = -x + 4\sqrt{2}$

(c) At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -1.$

65. $r = 1 - \sin\theta$

$$\frac{dy}{d\theta} = (1 - \sin\theta)\cos\theta - \cos\theta\sin\theta$$

$$= \cos\theta(1 - 2\sin\theta) = 0$$

$$\cos\theta = 0 \text{ or } \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Horizontal tangents: $\left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin\theta)\sin\theta - \cos\theta\cos\theta$$

$$= -\sin\theta + \sin^2\theta + \sin^2\theta - 1$$

$$= 2\sin^2\theta - \sin\theta - 1$$

$$= (2\sin\theta + 1)(\sin\theta - 1) = 0$$

$$\sin\theta = 1 \text{ or } \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Vertical tangents: $\left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$

66. $r = a\sin\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos\theta + a\cos\theta\sin\theta$$

$$= 2a\sin\theta\cos\theta = 0$$

$$\theta = 0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a\sin^2\theta + a\cos^2\theta = a(1 - 2\sin^2\theta) = 0$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}, \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Horizontal: $(0, 0), \left(a, \frac{\pi}{2}\right)$

Vertical: $\left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

67. $r = 2\csc\theta + 3$

$$\frac{dy}{d\theta} = (2\csc\theta + 3)\cos\theta + (-2\csc\theta\cot\theta)\sin\theta$$

$$= 3\cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Horizontal: $\left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$

68. $r = a\sin\theta\cos^2\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos^3\theta + [-2a\sin^2\theta\cos\theta + a\cos^3\theta]\sin\theta$$

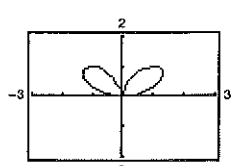
$$= 2a[\sin\theta\cos^3\theta - \sin^3\theta\cos\theta]$$

$$= 2a\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = 0$$

$$\theta = 0, \ \tan^2\theta = 1, \ \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Horizontal: $\left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$

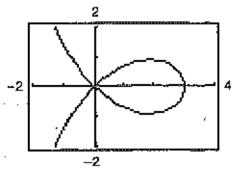
69. $r = 4\sin\theta\cos^2\theta$



Horizontal tangents:

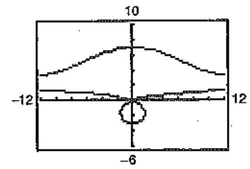
$(r, \theta) = (0, 0), (1.4142, 0.7854), (1.4142, 2.3562)$

70. $r = 3\cos 2\theta \sec\theta$



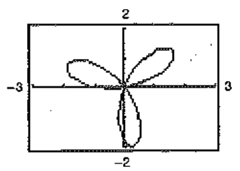
Horizontal tangents: $(r, \theta) = (2.061, \pm 0.452)$

71. $r = 2\csc\theta + 5$



Horizontal tangents: $(r, \theta) = \left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$

72. $r = 2\cos(3\theta - 2)$

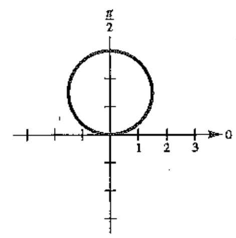


Horizontal tangents:

$(r, \theta) = (1.894, 0.776), (1.755, 2.594), (1.998, -1.442), (-0.423, 0.072)$

73.

$$r = 3\sin\theta$$
$$r^2 = 3r\sin\theta$$
$$x^2 + y^2 = 3y$$
$$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$



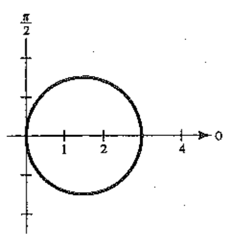
Circle $r = \frac{3}{2}$

Center: $\left(0, \frac{3}{2}\right)$

Tangent at the pole: $\theta = 0$

74.

$$r = 3\cos\theta$$
$$r^2 = 3r\cos\theta$$
$$x^2 + y^2 = 3x$$
$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$



Circle: $r = \frac{3}{2}$

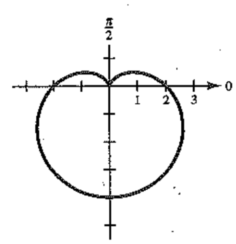
Center: $\left(\frac{3}{2}, 0\right)$

Tangent at pole: $\theta = \frac{\pi}{2}$

75. $r = 2(1 - \sin\theta)$

Cardioid

Symmetric to y-axis, $\theta = \frac{\pi}{2}$

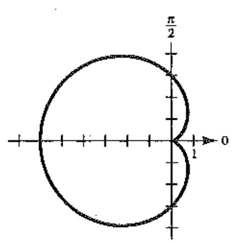


76. $r = 3(1 - \cos\theta)$

Cardioid

Symmetric to polar axis since r is a function of $\cos\theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6



77. $r = 2\cos(3\theta)$

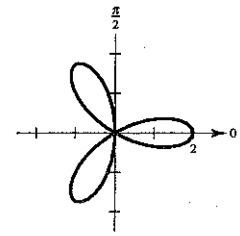
Rose curve with three petals

Symmetric to the polar axis

Relative extrema: $(2, 0), \left(-2, \frac{\pi}{3}\right), \left(2, \frac{2\pi}{3}\right)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	0	$-\sqrt{2}$	-2	0	2	0	-2

Tangents at the pole: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



78. $r = -\sin(5\theta)$

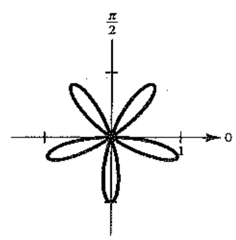
Rose curve with five petals

Symmetric to $\theta = \frac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5\cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

Tangents at the pole: $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$



79. $r = 3\sin 2\theta$

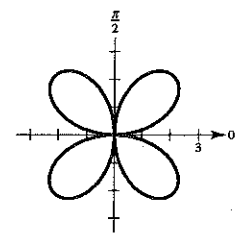
Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 3, \frac{\pi}{4}\right), \left(\pm 3, \frac{5\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$

($\theta = \pi,\ 3\pi/2$ give the same tangents.)



80. $r = 3\cos 2\theta$

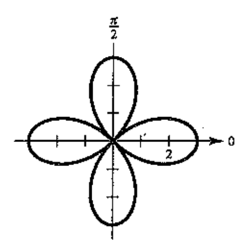
Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $(3, 0), \left(-3, \frac{\pi}{2}\right), (3, \pi), \left(-3, \frac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

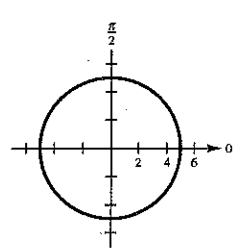
$\theta = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ given the same tangents.



81. $r = 5$

Circle radius: 5

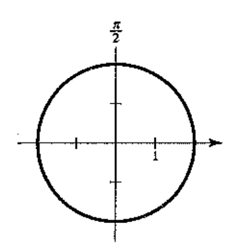
$x^2 + y^2 = 25$



82. $r = 2$

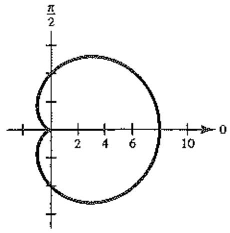
Circle radius: 2

$x^2 + y^2 = 4$



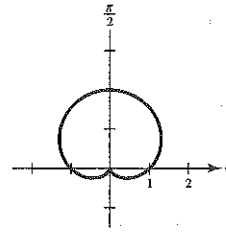
83. $r = 4(1 + \cos\theta)$

Cardioid



84. $r = 1 + \sin\theta$

Cardioid

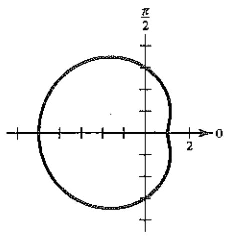


85. $r = 3 - 2\cos\theta$

Limaçon

Symmetric to polar axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	2	3	4	5

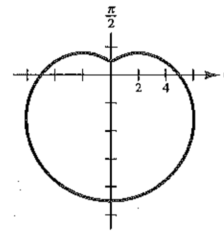


86. $r = 5 - 4\sin\theta$

Limaçon

Symmetric to $\theta = \dfrac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	9	7	5	3	1

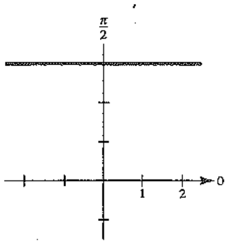


87. $r = 3\csc\theta$

$r\sin\theta = 3$

$y = 3$

Horizontal line

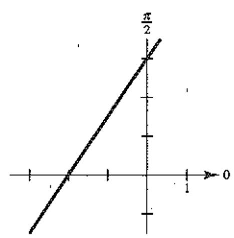


88. $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

$2r\sin\theta - 3r\cos\theta = 6$

$2y - 3x = 6$

Line



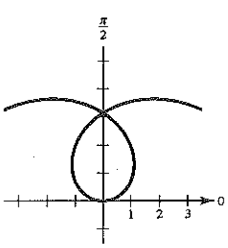
89. $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \dfrac{\pi}{2}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

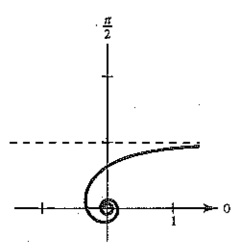
Tangent at the pole: $\theta = 0$



90. $r = \dfrac{1}{\theta}$

Hyperbolic spiral

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	$\frac{4}{\pi}$	$\frac{2}{\pi}$	$\frac{4}{3\pi}$	$\frac{1}{\pi}$	$\frac{4}{5\pi}$	$\frac{2}{3\pi}$



91. $r^2 = 4\cos(2\theta)$

$r = 2\sqrt{\cos 2\theta}, \quad 0 \le \theta \le 2\pi$

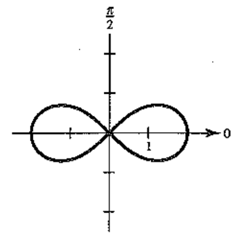
Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(\pm 2, 0)$

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$



92. $r^2 = 4\sin\theta$

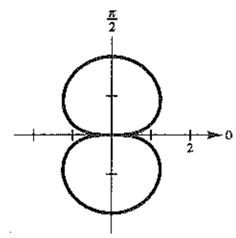
Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{2}\right)$

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{2}$	$\dfrac{5\pi}{6}$	π
r	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0

Tangent at the pole: $\theta = 0$

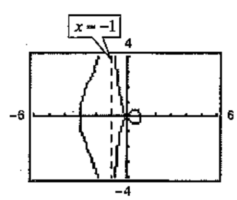


93. Since

$$r = 2 - \sec\theta = 2 - \frac{1}{\cos\theta},$$

the graph has polar axis symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, \frac{-\pi}{3}.$$



Furthermore,

$$r \Longrightarrow -\infty \text{ as } \theta \Longrightarrow \frac{\pi}{2^-}$$

$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow -\frac{\pi}{2^+}.$$

Also, $r = 2 - \dfrac{1}{\cos\theta} = 2 - \dfrac{r}{r\cos\theta} = 2 - \dfrac{r}{x}$

$$rx = 2x - r$$

$$r = \frac{2x}{1 + x}.$$

Thus, $r \Longrightarrow \pm\infty$ as $x \Longrightarrow -1$.

94. Since

$$r = 2 + \csc\theta = 2 + \frac{1}{\sin\theta},$$

the graphs has symmetry with respect to $\theta = \pi/2$. Furthermore,

$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow 0^+$$

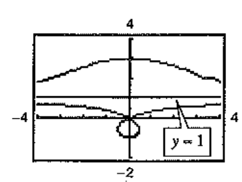
$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow \pi^-.$$

Also, $r = 2 + \dfrac{1}{\sin\theta} = 2 + \dfrac{r}{\sin\theta} = 2 + \dfrac{r}{y}$

$$ry = 2y + r$$

$$r = \frac{2y}{y - 1}.$$

Thus, $r \Longrightarrow \pm\infty$ as $y \Longrightarrow 1$.



95. $r = \dfrac{2}{\theta}$

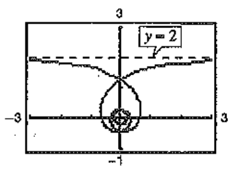
Hyperbolic spiral

$r \Longrightarrow \infty$ as $\theta \Longrightarrow 0$

$$r = \frac{2}{\theta} \Longrightarrow \theta = \frac{2}{r} = \frac{2\sin\theta}{r\sin\theta} = \frac{2\sin\theta}{y}$$

$$y = \frac{2\sin\theta}{\theta}$$

$$\lim_{\theta\to 0}\frac{2\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{2\cos\theta}{1} = 2$$



96. $r = 2\cos 2\theta \sec\theta$

Strophoid

$r \Longrightarrow -\infty$ as $\theta \Longrightarrow \dfrac{\pi^-}{2}$

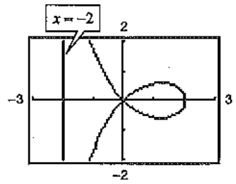
$r \Longrightarrow \infty$ as $\theta \Longrightarrow \dfrac{-\pi^+}{2}$

$$r = 2\cos 2\theta \sec\theta = 2(2\cos^2\theta - 1)\sec\theta$$

$$r\cos\theta = 4\cos^2\theta - 2$$

$$x = 4\cos^2\theta - 2$$

$$\lim_{\theta\to\pm\pi/2}(4\cos^2\theta - 2) = -2$$



97. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y-axis to the point, and y is the directed distance from the x-axis to the point. Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x-axis, measured clockwise.

Points do not have a unique polar representation.

98. $x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2,\ \tan\theta = \dfrac{y}{x}$

99. $r = a$, circle

$\theta = b$, line

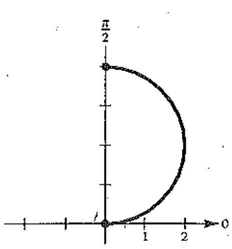
100. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

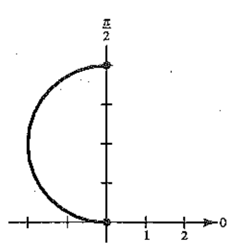
If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

101. $r = 4\sin\theta$

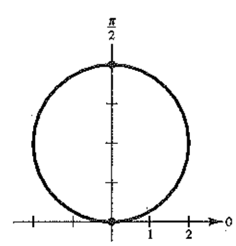
(a) $0 \le \theta \le \dfrac{\pi}{2}$



(b) $\dfrac{\pi}{2} \le \theta \le \pi$

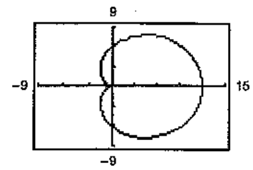


(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

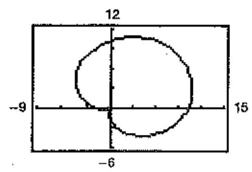


102. $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0$, $r = 6[1 + \cos\theta]$



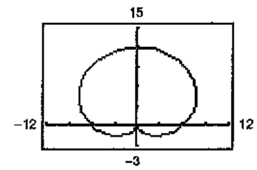
(b) $\theta = \frac{\pi}{4}$, $r = 6\left[1 + \cos\left(\theta - \frac{\pi}{4}\right)\right]$



The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \frac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$
$$= 6\left[1 + \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}\right]$$
$$= 6[1 + \sin\theta]$$



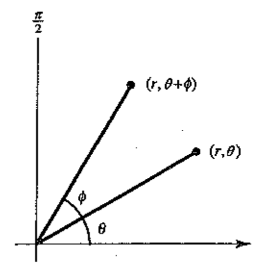
The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/2$.

103. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$



104. (a) $\sin\left(\theta - \frac{\pi}{2}\right) = \sin\theta\cos\left(\frac{\pi}{2}\right) - \cos\theta\sin\left(\frac{\pi}{2}\right)$

$$= -\cos\theta$$
$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$
$$= f(-\cos\theta)$$

(b) $\sin(\theta - \pi) = \sin\theta\cos\pi - \cos\theta\sin\pi$

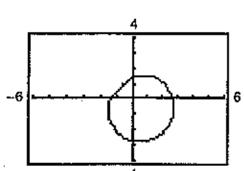
$$= -\sin\theta$$
$$r = f[\sin(\theta - \pi)]$$
$$= f(-\sin\theta)$$

(c) $\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta\cos\left(\frac{3\pi}{2}\right) - \cos\theta\sin\left(\frac{3\pi}{2}\right)$

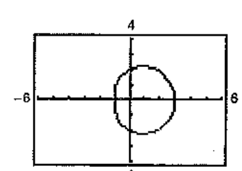
$$= \cos\theta$$
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos\theta)$$

105. $r = 2 - \sin\theta$

(a) $r = 2 - \sin\left(\theta - \frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$



(b) $r = 2 - \sin\left(\theta - \frac{\pi}{2}\right) = 2 - (-\cos\theta) = 2 + \cos\theta$



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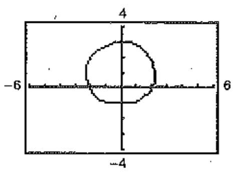
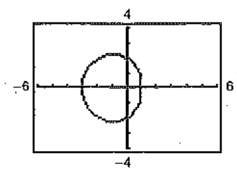
105. —CONTINUED—

(c) $r = 2 - \sin(\theta - \pi) = 2 - (-\sin\theta) = 2 + \sin\theta$

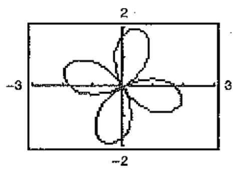


(d) $r = 2 - \sin\left(\theta - \frac{3\pi}{2}\right) = 2 - \cos\theta$

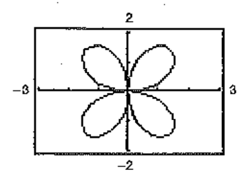


106. $r = 2\sin 2\theta = 4\sin\theta\cos\theta$

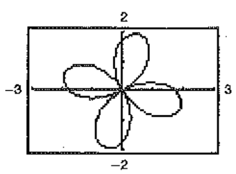
(a) $r = 4\sin\left(\theta - \frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right)$



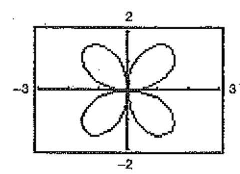
(b) $r = 4\sin\left(\theta - \frac{\pi}{2}\right)\cos\left(\theta - \frac{\pi}{2}\right) = -4\sin\theta\cos\theta$



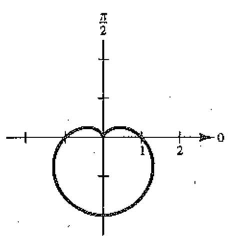
(c) $r = 4\sin\left(\theta - \frac{2\pi}{3}\right)\cos\left(\theta - \frac{2\pi}{3}\right)$



(d) $r = 4\sin(\theta - \pi)\cos(\theta - \pi) = 4\sin\theta\cos\theta$



107. (a) $r = 1 - \sin\theta$

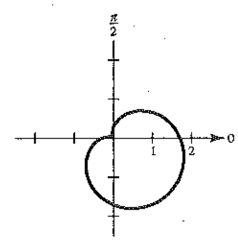


(b) $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

Rotate the graph of

$r = 1 - \sin\theta$

through the angle $\pi/4$.

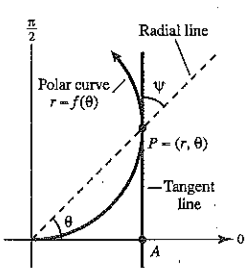


108. By Theorem 9.11, the slope of the tangent line through A and P is

$$\frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta}.$$

This is equal to

$$\tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi} = \frac{\sin\theta + \cos\theta\tan\psi}{\cos\theta - \sin\theta\tan\psi}.$$



Equating the expressions and cross-multiplying, you obtain

$$(f\cos\theta + f'\sin\theta)(\cos\theta - \sin\theta\tan\psi) = (\sin\theta + \cos\theta\tan\psi)(-f\sin\theta + f'\cos\theta)$$

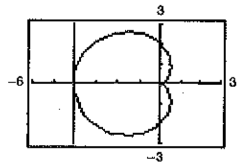
$$f\cos^2\theta - f\cos\theta\sin\theta\tan\psi + f'\sin\theta\cos\theta - f'\sin^2\theta\tan\psi = -f\sin^2\theta - f\sin\theta\cos\theta\tan\psi + f'\sin\theta\cos\theta + f'\cos^2\theta\tan\psi$$

$$f(\cos^2\theta + \sin^2\theta) = f'\tan\psi(\cos^2\theta + \sin^2\theta)$$

$$\tan\psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$

109. $\tan \psi = \dfrac{r}{dr/d\theta} = \dfrac{2(1 - \cos \theta)}{2 \sin \theta}$

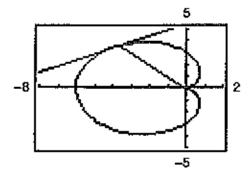
At $\theta = \pi$, $\tan \psi$ is undefined $\implies \psi = \dfrac{\pi}{2}$.



110. $\tan \psi = \dfrac{r}{dr/d\theta} = \dfrac{3(1 - \cos \theta)}{3 \sin \theta}$

At $\theta = \dfrac{3\pi}{4}$, $\tan \psi = \dfrac{1 + \left(\sqrt{2}/2\right)}{\sqrt{2}/2} = \dfrac{2 + \sqrt{2}}{\sqrt{2}}$.

$\psi = \arctan\left(\dfrac{2 + \sqrt{2}}{\sqrt{2}}\right) \approx 1.178 (\approx 67.5°)$

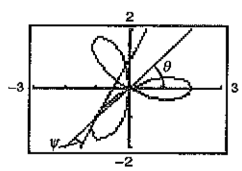


111. $r = 2 \cos 3\theta$

$\tan \psi = \dfrac{r}{dr/d\theta} = \dfrac{2 \cos 3\theta}{-6 \sin 3\theta} = -\dfrac{1}{3} \cot 3\theta$

At $\theta = \dfrac{\pi}{4}$, $\tan \psi = -\dfrac{1}{3} \cot\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{3}$.

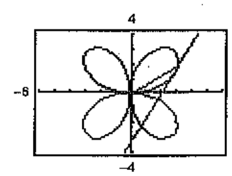
$\psi = \arctan\left(\dfrac{1}{3}\right) \approx 18.4°$



112. $\tan \psi = \dfrac{r}{dr/d\theta} = \dfrac{4 \sin 2\theta}{8 \cos 2\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan \psi = \dfrac{\sin(\pi/3)}{2 \cos(\pi/3)} = \dfrac{\sqrt{3}}{2}$.

$\psi = \arctan\left(\dfrac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89°)$

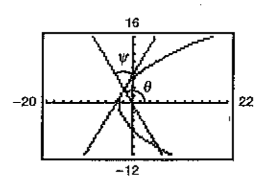


113. $r = \dfrac{6}{1 - \cos \theta} = 6(1 - \cos \theta)^{-1} \implies \dfrac{dr}{d\theta} = \dfrac{6 \sin \theta}{(1 - \cos \theta)^2}$

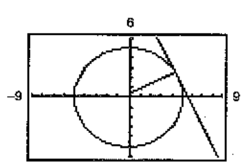
$\tan \psi = \dfrac{r}{\frac{dr}{d\theta}} = \dfrac{\frac{6}{(1 - \cos \theta)}}{\frac{-6 \sin \theta}{(1 - \cos \theta)^2}} = \dfrac{1 - \cos \theta}{-\sin \theta}$

At $\theta = \dfrac{2\pi}{3}$, $\tan \psi = \dfrac{1 - \left(-\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2}} = -\sqrt{3}$.

$\psi = \dfrac{\pi}{3}, (60°)$



114. $\tan \psi = \dfrac{r}{dr/d\theta} = \dfrac{5}{0}$ undefined $\implies \psi = \dfrac{\pi}{2}$



115. True

116. True

117. True

118. True

Section 10.5 Area and Arc Length in Polar Coordinates

1. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} (2\sin\theta)^2\,d\theta$

$= 2\int_{\pi/2}^{\pi} \sin^2\theta\,d\theta$

2. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{3\pi/4}^{5\pi/4} (\cos 2\theta)^2\,d\theta$

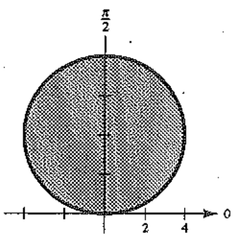
3. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} (1 - \sin\theta)^2\,d\theta$

4. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{0}^{\pi} (1 - \cos 2\theta)^2\,d\theta$

5. (a) $r = 8\sin\theta$



$A = \pi(4)^2 = 16\pi$

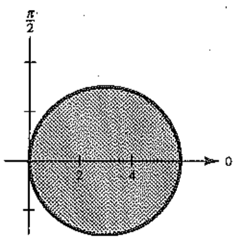
(b) $A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2} [8\sin\theta]^2\,d\theta$

$= 64\int_0^{\pi/2} \sin^2\theta\,d\theta$

$= 32\int_0^{\pi/2} (1 - \cos 2\theta)\,d\theta$

$= 32\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 16\pi$

6. (a) $r = 3\cos\theta$



$A = \pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$

(b) $A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2} [3\cos\theta]^2\,d\theta$

$= 9\int_0^{\pi/2} \cos^2\theta\,d\theta$

$= \frac{9}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{9\pi}{4}$

7. $A = 2\left[\frac{1}{2}\int_0^{\pi/6} (2\cos 3\theta)^2\,d\theta\right] = 2\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{3}$

8. $A = 2\left[\frac{1}{2}\int_0^{\pi/4} (6\sin 2\theta)^2\,d\theta\right] = 36\int_0^{\pi/4} \sin^2 2\theta\,d\theta$

$= 36\int_0^{\pi/4} \frac{1 - \cos 4\theta}{2}\,d\theta$

$= 18\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$

$= 18\left[\frac{\pi}{4}\right] = \frac{9\pi}{2}$

9. $A = 2\left[\frac{1}{2}\int_0^{\pi/4} (\cos 2\theta)^2\, d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}$

10. $A = 2\left[\frac{1}{2}\int_0^{\pi/10} (\cos 5\theta)^2\, d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{10}\sin(10\theta)\right]_0^{\pi/10} = \frac{\pi}{20}$

11. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 - \sin\theta)^2\, d\theta\right]$

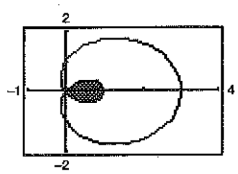
$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$

12. $A = 2\left[\frac{1}{2}\int_0^{\pi/2} (1 - \sin\theta)^2\, d\theta\right]$

$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi - 8}{4}$

13. $A = 2\frac{1}{2}\left[\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta\right]$

$= \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$

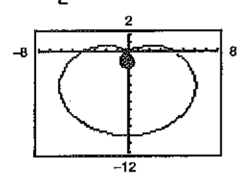


14. $A = 2\left[\frac{1}{2}\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\, d\theta\right]$

$= \int_{\arcsin(2/3)}^{\pi/2} [16 - 48\sin\theta + 36\sin^2\theta]\, d\theta$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$

$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$

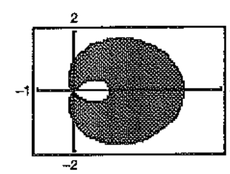


15. The area inside the outer loop is

$$2\left[\frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$$

From the result of Exercise 13, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$



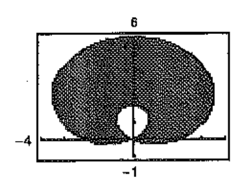
16. Four times the area in Exercise 15, $A = 4(\pi + 3\sqrt{3})$. More specifically, we see that the area inside the outer loop is

$$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} (2(1 + 2\sin\theta))^2\, d\theta\right] = \int_{-\pi/6}^{\pi/2} (4 + 16\sin\theta + 16\sin^2\theta)\, d\theta = 8\pi + 6\sqrt{3}.$$

The area inside the inner loop is

$$2\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} (2(1 + 2\sin\theta))^2\, d\theta\right] = 4\pi - 6\sqrt{3}.$$

Thus, the area between the loops is $(8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.



17. $r = 1 + \cos\theta$

$r = 1 - \cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \cos\theta$, $\cos\theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right), (0, 0)$

18. $r = 3(1 + \sin\theta)$

$r = 3(1 - \sin\theta)$

Solving simultaneously,

$$3(1 + \sin\theta) = 3(1 - \sin\theta)$$
$$2\sin\theta = 0$$
$$\theta = 0, \pi.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin\theta) = 3(1 - \sin\theta)$, $\sin\theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0), (3, \pi), (0, 0)$

19. $r = 1 + \cos\theta$

$r = 1 - \sin\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \sin\theta$$
$$\cos\theta = -\sin\theta$$
$$\tan\theta = -1$$
$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \sin\theta$, $\sin\theta + \cos\theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$

20. $r = 2 - 3\cos\theta$

$r = \cos\theta$

Solving simultaneously,

$$2 - 3\cos\theta = \cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (0, 0)$

21. $r = 4 - 5\sin\theta$

$r = 3\sin\theta$

Solving simultaneously,

$$4 - 5\sin\theta = 3\sin\theta$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$

22. $r = 1 + \cos\theta$

$r = 3\cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 3\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

23. $r = \dfrac{\theta}{2}$

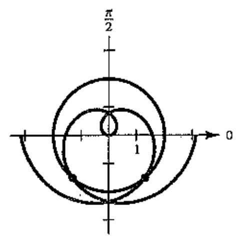
$r = 2$

Solving simultaneously, we have

$\theta/2 = 2, \theta = 4.$

Points of intersection:

$(2, 4), (-2, -4)$



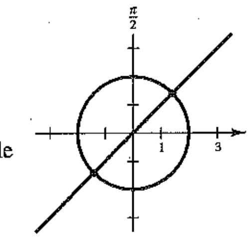
24. $\theta = \dfrac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection:

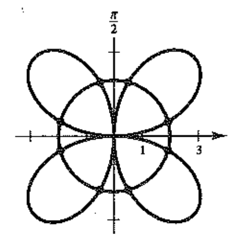
$\left(2, \dfrac{\pi}{4}\right), \left(-2, \dfrac{\pi}{4}\right)$



25. $r = 4 \sin 2\theta$

$r = 2$

$r = 4 \sin 2\theta$ is the equation of a rose curve with four petals and is symmetric to the polar axis, $\theta = \pi/2$, and the pole. Also, $r = 2$ is the equation of a circle of radius 2 centered at the pole. Solving simultaneously,



$4 \sin 2\theta = 2$

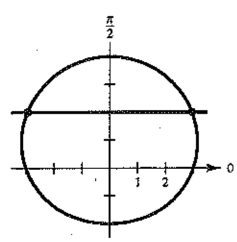
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$

Therefore, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. By symmetry, the other points of intersection are $(2, 7\pi/12)$, $(2, 11\pi/12)$, $(2, 13\pi/12)$, $(2, 17\pi/12)$, $(2, 19\pi/12)$, and $(2, 23\pi/12)$.

26. $r = 3 + \sin\theta$

$r = 2 \csc\theta$



The graph of $r = 3 + \sin\theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2\csc\theta$ is the horizontal line $y = 2$. Therefore, there are two points of intersection. Solving simultaneously,

$$3 + \sin\theta = 2\csc\theta$$

$$\sin^2\theta + 3\sin\theta - 2 = 0$$

$$\sin\theta = \frac{-3 \pm \sqrt{17}}{2}$$

$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

Points of intersection:

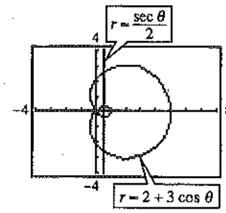
$$\left(\frac{\sqrt{17} + 3}{2}, \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$$\left(\frac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$(3.56, 0.596), (3.56, 2.545)$

27. $r = 2 + 3\cos\theta$

$r = \dfrac{\sec\theta}{2}$



The graph of $r = 2 + 3\cos\theta$ is a limaçon with an inner loop $(b > a)$ and is symmetric to the polar axis. The graph of $r = (\sec\theta)/2$ is the vertical line $x = 1/2$. Therefore, there are four points of intersection. Solving simultaneously,

$$2 + 3\cos\theta = \frac{\sec\theta}{2}$$

$$6\cos^2\theta + 4\cos\theta - 1 = 0$$

$$\cos\theta = \frac{-2 \pm \sqrt{10}}{6}$$

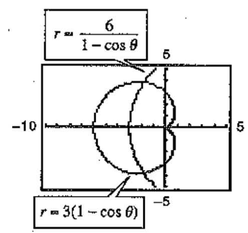
$$\theta = \arccos\left(\frac{-2+\sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2-\sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection: $(-0.581, \pm 2.607), (2.581, \pm 1.376)$

28. $r = 3(1 - \cos\theta)$

$r = \dfrac{6}{1 - \cos\theta}$



The graph of $r = 3(1 - \cos\theta)$ is a cardioid with polar axis symmetry. The graph of

$r = 6/(1 - \cos\theta)$

is a parabola with focus at the pole, vertex$(3, \pi)$, and polar axis symmetry. Therefore, there are two points of intersection. Solving simultaneously,

$$3(1 - \cos\theta) = \frac{6}{1 - \cos\theta}$$

$$(1 - \cos\theta)^2 = 2$$

$$\cos\theta = 1 \pm \sqrt{2}$$

$$\theta = \arccos\left(1 - \sqrt{2}\right).$$

Points of intersection: $\left(3\sqrt{2}, \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 1.998), \left(3\sqrt{2}, 2\pi - \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 4.285)$

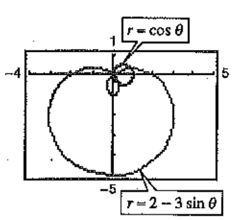
29. $r = \cos\theta$

$r = 2 - 3\sin\theta$

Points of intersection:

$(0, 0), (0.935, 0.363), (0.535, -1.006)$

The graphs reach the pole at different times (θ values).

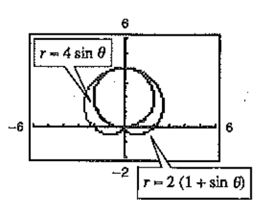


30. $r = 4\sin\theta$

$r = 2(1 + \sin\theta)$

Points of intersection: $(0, 0), \left(4, \dfrac{\pi}{2}\right)$

The graphs reach the pole at different times (θ values).

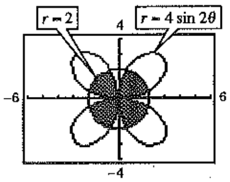


31. From Exercise 25, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12} (4\sin 2\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2\,d\theta$$

$$= 16\int_0^{\pi/12} \sin^2(2\theta)\,d\theta + 2\int_{\pi/12}^{5\pi/12} d\theta \text{ (by symmetry of the petal)}$$

$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$

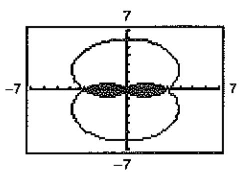


$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}(4\pi - 3\sqrt{3})$$

32. $A = 4\left[\frac{1}{2}\int_0^{\pi/2} 9(1 - \sin\theta)^2\,d\theta\right]$

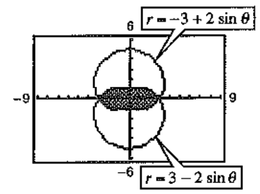
$$= 18\int_0^{\pi/2} (1 - \sin\theta)^2\,d\theta = \frac{9}{2}(3\pi - 8)$$

(from Exercise 14)



33. $A = 4\left[\frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2\,d\theta\right]$

$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$



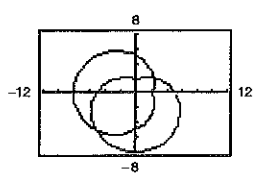
34. $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$$A = 2\left[\frac{1}{2}\int_{\pi/4}^{5\pi/4} (5 - 3\sin\theta)^2\,d\theta\right]$$

$$= \left[\frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{59}{2}\left(\frac{5\pi}{4}\right) - 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right) - \left(\frac{59}{2}\left(\frac{\pi}{4}\right) + 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right)$$

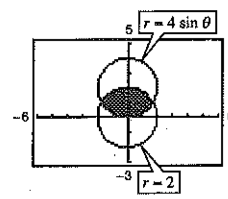
$$= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251$$



35. $A = 2\left[\frac{1}{2}\int_0^{\pi/6} (4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (2)^2\,d\theta\right]$

$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + \left[4\theta\right]_{\pi/6}^{\pi/2}$$

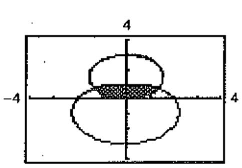
$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}(4\pi - 3\sqrt{3})$$



36. $A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (2 - \sin\theta)^2\,d\theta\right]$

$$= \int_{\pi/6}^{\pi/2} (-4\cos 2\theta + 4\sin\theta)\,d\theta$$

$$= \left[-2\sin(2\theta) - 4\cos\theta\right]_{\pi/6}^{\pi/2} = 3\sqrt{3}$$



37. $A = 2\left[\frac{1}{2}\int_0^{\pi} [a(1 + \cos\theta)]^2\,d\theta\right] - \frac{a^2\pi}{4}$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - \frac{a^2\pi}{4}$$

$$= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$

38. Area = Area of $r = 2a\cos\theta$ − Area of sector − twice area between $r = 2a\cos\theta$ and the lines

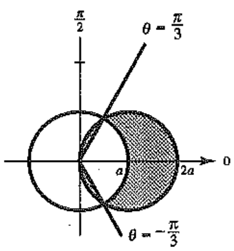
$$\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}.$$

$$A = \pi a^2 - \left(\frac{\pi}{3}\right)a^2 - 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2} (2a\cos\theta)^2\,d\theta\right]$$

$$= \frac{2\pi a^2}{3} - 2a^2\int_{\pi/3}^{\pi/2} (1 + \cos 2\theta)\,d\theta$$

$$= \frac{2\pi a^2}{3} - 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{2\pi a^2}{3} - 2a^2\left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6}$$

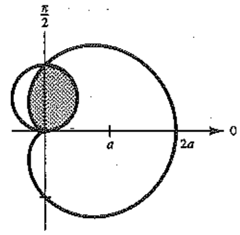


39. $$A = \frac{\pi a^2}{8} + \frac{1}{2}\int_{\pi/2}^{\pi} [a(1 + \cos\theta)]^2\,d\theta$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right] = \frac{a^2}{2}[\pi - 2]$$



40. $r = a\cos\theta, r = a\sin\theta$

$\tan\theta = 1, \theta = \pi/4$

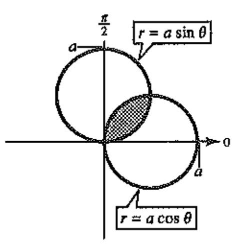
$$A = 2\left[\frac{1}{2}\int_0^{\pi/4} (a\sin\theta)^2\,d\theta\right]$$

$$= a^2\int_0^{\pi/4} \frac{1 - \cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}a^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}a^2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{1}{8}a^2\pi - \frac{1}{4}a^2$$

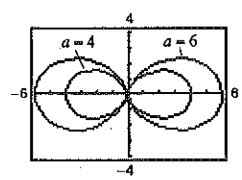


41. (a) $r = a\cos^2\theta$

$r^3 = ar^2\cos^2\theta$

$(x^2 + y^2)^{3/2} = ax^2$

(b)



(c) $$A = 4\left(\frac{1}{2}\right)\int_0^{\pi/2} [(6\cos^2\theta)^2 - (4\cos^2\theta)^2]\,d\theta$$

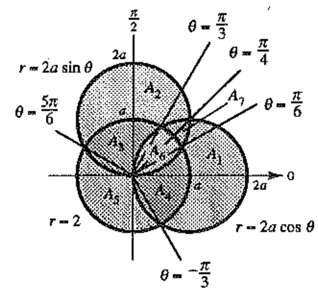
$$= 40\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 10\int_0^{\pi/2} (1 + \cos 2\theta)^2\,d\theta$$

$$= 10\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1 - \cos 4\theta}{2}\right)d\theta$$

$$= 10\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi}{2}$$

42. By symmetry, $A_1 = A_2$ and $A_3 = A_4$.



$$A_1 = A_2 = \frac{1}{2}\int_{-\pi/3}^{\pi/6} [(2a\cos\theta)^2 - (a)^2]\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4} [(2a\cos\theta)^2 - (2a\sin\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/3}^{\pi/6} (4\cos^2\theta - 1)\,d\theta + 2a^2\int_{\pi/6}^{\pi/4} \cos 2\theta\,d\theta$$

$$= \frac{a^2}{2}\Big[\theta + \sin 2\theta\Big]_{-\pi/3}^{\pi/6} + a^2\Big[\sin 2\theta\Big]_{\pi/6}^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{2} + \sqrt{3}\right) + a^2\left(1 - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{4} + 1\right)$$

$$A_3 = A_4 = \frac{1}{2}\left(\frac{\pi}{2}\right)a^2 = \frac{\pi a^2}{4}$$

$$A_5 = \frac{1}{2}\left(\frac{5\pi}{6}\right)a^2 - 2\left(\frac{1}{2}\right)\int_{5\pi/6}^{\pi} (2a\sin\theta)^2\,d\theta$$

$$= \frac{5\pi a^2}{12} - 2a^2\int_{5\pi/6}^{\pi} (1 - \cos 2\theta)\,d\theta$$

$$= \frac{5\pi a^2}{12} - a^2\Big[2\theta - \sin 2\theta\Big]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$

$$A_6 = 2\left(\frac{1}{2}\right)\int_0^{\pi/6} (2a\sin\theta)^2\,d\theta + 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4} a^2\,d\theta$$

$$= 2a^2\int_0^{\pi/6} (1 - \cos 2\theta)\,d\theta + \Big[a^2\theta\Big]_{\pi/6}^{\pi/4}$$

$$= a^2\Big[2\theta - \sin 2\theta\Big]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi a^2}{12} = a^2\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$$

$$A_7 = 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4} [(2a\sin\theta)^2 - (a)^2]\,d\theta$$

$$= a^2\int_{\pi/6}^{\pi/4} (4\sin^2\theta - 1)\,d\theta = a^2\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{\pi/4} = a^2\left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)$$

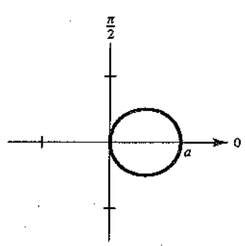
[**Note:** $A_1 + A_6 + A_7 + A_4 = \pi a^2 =$ area of circle of radius a]

43. $r = a\cos(n\theta)$

For $n = 1$:

$r = a\cos\theta$

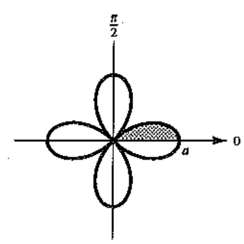
$$A = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$



For $n = 2$:

$r = a\cos 2\theta$

$$A = 8\left(\frac{1}{2}\right)\int_0^{\pi/4} (a\cos 2\theta)^2\,d\theta = \frac{\pi a^2}{2}$$



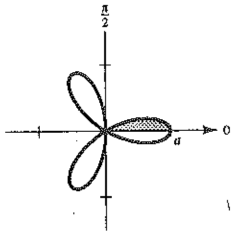
—CONTINUED—

43. —CONTINUED—

For $n = 3$:

$r = a \cos 3\theta$

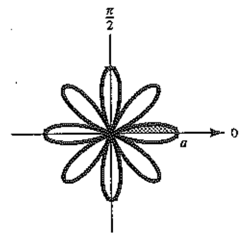
$$A = 6\left(\frac{1}{2}\right)\int_0^{\pi/6} (a \cos 3\theta)^2\, d\theta = \frac{\pi a^2}{4}$$



For $n = 4$:

$r = a \cos 4\theta$

$$A = 16\left(\frac{1}{2}\right)\int_0^{\pi/8} (a \cos 4\theta)^2\, d\theta = \frac{\pi a^2}{2}$$



In general, the area of the region enclosed by $r = a \cos(n\theta)$ for $n = 1, 2, 3, \ldots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

44. $r = \sec \theta - 2 \cos \theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

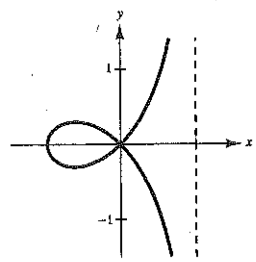
$r \cos \theta = 1 - 2 \cos^2 \theta$

$$x = 1 - 2\frac{r^2 \cos^2 \theta}{r^2} = 1 - 2\left(\frac{x^2}{x^2 + y^2}\right)$$

$(x^2 + y^2)x = x^2 + y^2 - 2x^2$

$y^2(x - 1) = -x^2 - x^3$

$$y^2 = \frac{x^2(1 + x)}{1 - x}$$



$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4} (\sec \theta - 2 \cos \theta)^2\, d\theta$$

$$= \int_0^{\pi/4} (\sec^2 \theta - 4 + 4 \cos^2 \theta)\, d\theta = \int_0^{\pi/4} (\sec^2 \theta - 4 + 2(1 + \cos 2\theta))\, d\theta = \Big[\tan \theta - 2\theta + \sin 2\theta\Big]_0^{\pi/4} = 2 - \frac{\pi}{2}$$

45. $r = a$

$r' = 0$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2}\, d\theta = \Big[a\theta\Big]_0^{2\pi} = 2\pi a$$

(circumference of circle of radius a)

46. $r = 2a \cos \theta$

$r' = -2a \sin \theta$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a \cos \theta)^2 + (-2a \sin \theta)^2}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2a\, d\theta = \Big[2a\theta\Big]_{-\pi/2}^{\pi/2} = 2\pi a$$

47. $r = 1 + \sin \theta$

$r' = \cos \theta$

$$s = 2\int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin \theta)^2 + (\cos \theta)^2}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin \theta}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos \theta}{\sqrt{1 - \sin \theta}}\, d\theta$$

$$= \Big[4\sqrt{2}\sqrt{1 - \sin \theta}\Big]_{\pi/2}^{3\pi/2}$$

$$= 4\sqrt{2}\left(\sqrt{2} - 0\right) = 8$$

48. $r = 8(1 + \cos\theta), 0 \le \theta \le 2\pi$

$r' = -8\sin\theta$

$$s = 2\int_0^{\pi} \sqrt{[8(1+\cos\theta)]^2 + (-8\sin\theta)^2}\,d\theta$$

$$= 16\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\,d\theta$$

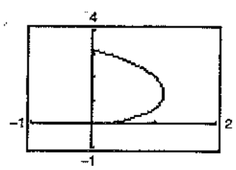
$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\cdot\left(\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\right)d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta$$

$$= \left[32\sqrt{2}\sqrt{1-\cos\theta}\right]_0^{\pi}$$

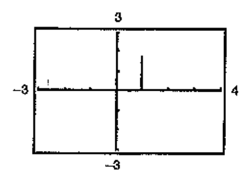
$$= 64$$

49. $r = 2\theta, 0 \le \theta \le \dfrac{\pi}{2}$



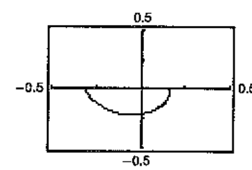
Length ≈ 4.16

50. $r = \sec\theta, 0 \le \theta \le \dfrac{\pi}{3}$



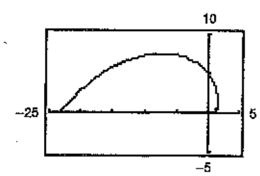
Length ≈ 1.73 (exact $\sqrt{3}$)

51. $r = \dfrac{1}{\theta}, \pi \le \theta \le 2\pi$



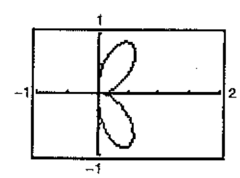
Length ≈ 0.71

52. $r = e^{\theta}, 0 \le \theta \le \pi$



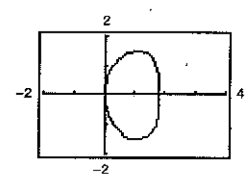
Length ≈ 31.31

53. $r = \sin(3\cos\theta), 0 \le \theta \le \pi$



Length ≈ 4.39

54. $r = 2\sin(2\cos\theta),\ 0 \le \theta \le \pi$



Length ≈ 7.78

55. $r = 6\cos\theta$

$r' = -6\sin\theta$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{36\cos^2\theta + 36\sin^2\theta}\,d\theta$$

$$= 72\pi\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$$

$$= \left[36\pi\sin^2\theta\right]_0^{\pi/2}$$

$$= 36\pi$$

56. $r = a\cos\theta$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi/2} a\cos\theta(\cos\theta)\sqrt{a^2\cos\theta + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi a^2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= \left[\pi a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

57. $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$S = 2\pi \int_0^{\pi/2} e^{a\theta} \cos\theta \sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2}\, d\theta$$

$$= 2\pi\sqrt{1 + a^2} \int_0^{\pi/2} e^{2a\theta} \cos\theta\, d\theta$$

$$= 2\pi\sqrt{1 + a^2}\left[\frac{e^{2a\theta}}{4a^2 + 1}(2a\cos\theta + \sin\theta)\right]_0^{\pi/2}$$

$$= \frac{2\pi\sqrt{1 + a^2}}{4a^2 + 1}(e^{\pi a} - 2a)$$

58. $r = a(1 + \cos\theta)$

$r' = -a\sin\theta$

$$S = 2\pi \int_0^{\pi} a(1 + \cos\theta)\sin\theta\sqrt{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta}\, d\theta = 2\pi a^2 \int_0^{\pi} \sin\theta(1 + \cos\theta)\sqrt{2 + 2\cos\theta}\, d\theta$$

$$= -2\sqrt{2}\pi a^2 \int_0^{\pi} (1 + \cos\theta)^{3/2}(-\sin\theta)\, d\theta = -\frac{4\sqrt{2}\pi a^2}{5}\left[(1 + \cos\theta)^{5/2}\right]_0^{\pi} = \frac{32\pi a^2}{5}$$

59. $r = 4\cos 2\theta$

$r' = -8\sin 2\theta$

$$S = 2\pi \int_0^{\pi/4} 4\cos 2\theta \sin\theta\sqrt{16\cos^2 2\theta + 64\sin^2 2\theta}\, d\theta$$

$$= 32\pi \int_0^{\pi/4} \cos 2\theta \sin\theta\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta \approx 21.87$$

60. $r = \theta$

$r' = 1$

$$S = 2\pi \int_0^{\pi} \theta\sin\theta\sqrt{\theta^2 + 1}\, d\theta \approx 42.32$$

61. $$\text{Area} = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$$\text{Arc length} = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

62. The curves might intersect for different values of θ:

See page 741.

63. (a) is correct: $s \approx 33.124$.

64. (a) $S = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

(b) $S = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

65. Revolve $r = 2$ about the line $r = 5\sec\theta$.

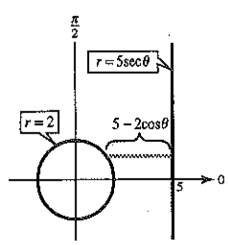
$f(\theta) = 2, f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} (5 - 2\cos\theta)\sqrt{2^2 + 0^2}\, d\theta$$

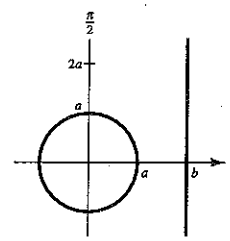
$$= 4\pi \int_0^{2\pi} (5 - 2\cos\theta)\, d\theta$$

$$= 4\pi\left[5\theta - 2\sin\theta\right]_0^{2\pi}$$

$$= 40\pi^2$$



66. Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.



$f(\theta) = a$

$f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} [b - a\cos\theta]\sqrt{a^2 + 0^2}\, d\theta$$

$$= 2\pi a\Big[b\theta - a\sin\theta\Big]_0^{2\pi}$$

$$= 2\pi a(2\pi b) = 4\pi^2 ab$$

67. $r = 8\cos\theta,\ 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} 64\cos^2\theta\, d\theta = 32\int_0^{\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = 16\pi$

(Area circle $= \pi r^2 = \pi 4^2 = 16\pi$)

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c), (d) For $\frac{1}{4}$ of area ($4\pi \approx 12.57$): 0.42

For $\frac{1}{2}$ of area ($8\pi \approx 25.13$): 1.57 ($\pi/2$)

For $\frac{3}{4}$ of area ($12\pi \approx 37.70$): 2.73

(e) No, it does not depend on the radius.

68. $r = 3\sin\theta,\ 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta = \frac{9}{4}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta = \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{9}{4}\pi$

$\left[\textbf{Note: } \text{radius of circle is } \frac{3}{2} \Longrightarrow A = \pi\left(\frac{3}{2}\right)^2\right]$

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	0.0119	0.0930	0.3015	0.6755	1.2270	1.9401	2.7731

(c), (d) For $\frac{1}{8}$ of area $\left(\frac{1}{8}\frac{9}{4}\pi \approx 0.8836\right)$: $\theta \approx 0.88$

For $\frac{1}{4}$ of area $\left(\frac{1}{4}\frac{9}{4}\pi \approx 1.7671\right)$: $\theta \approx 1.15$

For $\frac{1}{2}$ of area $\left(\frac{1}{2}\frac{9}{4}\pi \approx 3.5343\right)$: $\theta = \pi/2 \approx 1.57$

69.

$$r = a\sin\theta + b\cos\theta$$

$$r^2 = ar\sin\theta + br\cos\theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

70. $r = \sin\theta + \cos\theta$, Circle

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$
$$= \frac{1}{2}\int_{0}^{\pi} (\sin\theta + \cos\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_{0}^{\pi} (1 + 2\sin\theta\cos\theta)\,d\theta$$
$$= \frac{1}{2}\Big[\theta + \sin^2\theta\Big]_0^{\pi} = \frac{\pi}{2}$$

Converting to rectangular form:

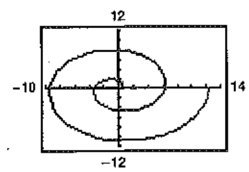
$$r^2 = r\sin\theta + r\cos\theta$$
$$x^2 + y^2 = y + x$$
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{2}$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

Circle of radius $\frac{1}{\sqrt{2}}$ and center $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\text{Area} = \pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$$

71. (a) $r = \theta,\ \theta \geq 0$

As a increases, the spiral opens more rapidly. If $\theta < 0$, the spiral is reflected about the y-axis.



(b) $r = a\theta,\ \theta \geq 0$, crosses the polar axis for $\theta = n\pi$, n and integer. To see this

$$r = a\theta \implies r\sin\theta = y = a\theta\sin\theta = 0$$

for $\theta = n\pi$. The points are $(r, \theta) = (an\pi, n\pi)$, $n = 1, 2, 3, \ldots$.

(c) $f(\theta) = \theta, f'(\theta) = 1$

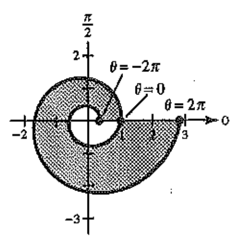
$$s = \int_0^{2\pi} \sqrt{\theta^2 + 1}\,d\theta = \frac{1}{2}\Big[\ln\left(\sqrt{x^2+1} + x\right) + x\sqrt{x^2+1}\Big]_0^{2\pi}$$
$$= \frac{1}{2}\ln\left(\sqrt{4\pi^2+1} + 2\pi\right) + \pi\sqrt{4\pi^2+1} \approx 21.2563$$

(d) $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,dr$

$$= \frac{1}{2}\int_0^{2\pi} \theta^2\,d\theta$$
$$= \frac{\theta^3}{6}\Big]_0^{2\pi} = \frac{4}{3}\pi^3$$

72. $r = e^{\theta/6}$

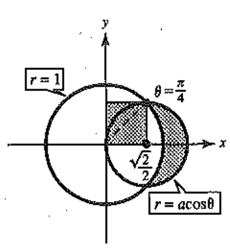
$$A = \frac{1}{2}\int_0^{2\pi} (e^{\theta/6})^2\,d\theta - \frac{1}{2}\int_{-2\pi}^{0} (e^{\theta/6})^2\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} e^{\theta/3}\,d\theta - \frac{1}{2}\int_{-2\pi}^{0} e^{\theta/3}\,d\theta$$
$$= \left[\frac{3}{2}e^{\theta/3}\right]_0^{2\pi} - \left[\frac{3}{2}e^{\theta/3}\right]_{-2\pi}^{0}$$
$$= \frac{3}{2}e^{2\pi/3} - \frac{3}{2} - \frac{3}{2} + \frac{3}{2}e^{-2\pi/3} = \frac{3}{2}\left[e^{2\pi/3} + e^{-2\pi/3} - 2\right]$$
$$\approx 9.3655$$



73. The smaller circle has equation $r = a\cos\theta$.

The area of the shaded lune is:

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4} \left[(a\cos\theta)^2 - 1\right]d\theta$$
$$= \int_0^{\pi/4}\left[\frac{a^2}{2}(1 + \cos 2\theta) - 1\right]d\theta$$
$$= \left[\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \theta\right]_0^{\pi/4}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4}$$



This equals the area of the square, $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

$$\frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4} = \frac{1}{2}$$
$$\pi a^2 + 2a^2 - 2\pi - 4 = 0$$
$$a^2 = \frac{4 + 2\pi}{2 + \pi} = 2$$
$$a = \sqrt{2}$$

Smaller circle: $r = \sqrt{2}\cos\theta$

74. $x = \dfrac{3t}{1+t^3}, y = \dfrac{3t^2}{1+t^3}$

(a) $x^3 + y^3 = \dfrac{27(t^3+t^6)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$

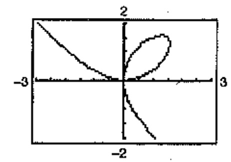
$3xy = \dfrac{27t^3}{(1+t^3)^2}$

Hence, $x^3 + y^3 = 3xy$.

$(r\cos\theta)^3 + (r\sin\theta)^3 = 3(r\cos\theta)(r\sin\theta)$

$r = \dfrac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$

(b)



(c) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} r^2\,d\theta = \dfrac{3}{2}$

75. False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

76. False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

77. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Using θ instead of t, we have $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. Thus,

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \text{ and } \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

Therefore, $s = \displaystyle\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta.$

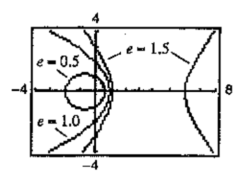
Section 10.6 Polar Equations of Conics and Kepler's Laws

1. $r = \dfrac{2e}{1 + e\cos\theta}$

(a) $e = 1, r = \dfrac{2}{1+\cos\theta}$, parabola

(b) $e = 0.5, r = \dfrac{1}{1+0.5\cos\theta} = \dfrac{2}{2+\cos\theta}$, ellipse

(c) $e = 1.5, r = \dfrac{3}{1+1.5\cos\theta} = \dfrac{6}{2+3\cos\theta}$, hyperbola

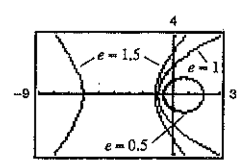


2. $r = \dfrac{2e}{1 - e\cos\theta}$

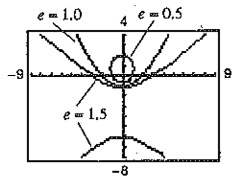
(a) $e = 1, r = \dfrac{2}{1-\cos\theta}$, parabola

(b) $e = 0.5, r = \dfrac{1}{1-0.5\cos\theta} = \dfrac{2}{2-\cos\theta}$, ellipse

(c) $e = 1.5, r = \dfrac{3}{1-1.5\cos\theta} = \dfrac{6}{2-3\cos\theta}$, hyperbola



3. $r = \dfrac{2e}{1 - e\sin\theta}$

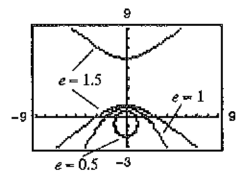


(a) $e = 1, r = \dfrac{2}{1 - \sin\theta}$, parabola

(b) $e = 0.5, r = \dfrac{1}{1 - 0.5\sin\theta} = \dfrac{2}{2 - \sin\theta}$, ellipse

(c) $e = 1.5, r = \dfrac{3}{1 - 1.5\sin\theta} = \dfrac{6}{2 - 3\sin\theta}$, hyperbola

4. $r = \dfrac{2e}{1 + e\sin\theta}$



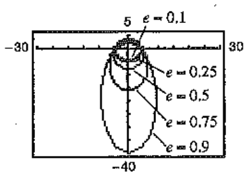
(a) $e = 1, r = \dfrac{2}{1 + \sin\theta}$, parabola

(b) $e = 0.5, r = \dfrac{1}{1 + 0.5\sin\theta} = \dfrac{2}{2 + \sin\theta}$, ellipse

(c) $e = 1.5, r = \dfrac{3}{1 + 1.5\sin\theta} = \dfrac{6}{2 + 3\sin\theta}$, hyperbola

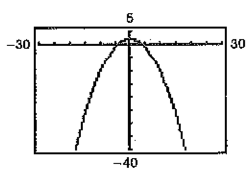
5. $r = \dfrac{4}{1 + e\sin\theta}$

(a)



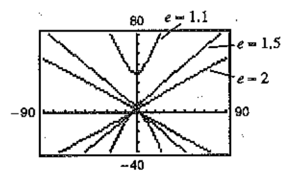
The conic is an ellipse. As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

(b)



The conic is a parabola.

(c)



The conic is a hyperbola. As $e \to 1^+$, the hyperbolas opens more slowly, and as $e \to \infty$, they open more rapidly.

6. $r = \dfrac{4}{1 - 0.4\cos\theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.

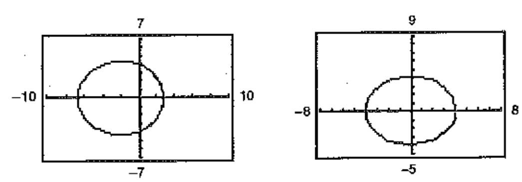
(b) $r = \dfrac{4}{1 + 0.4\cos\theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole

$r = \dfrac{4}{1 - 0.4\sin\theta}$.

The ellipse has a horizontal directrix below the pole.

(c)



7. Parabola; Matches (c)

8. Ellipse; Matches (f)

9. Hyperbola; Matches (a)

10. Parabola; Matches (e)

11. Ellipse; Matches (b)

12. Hyperbola; Matches (d)

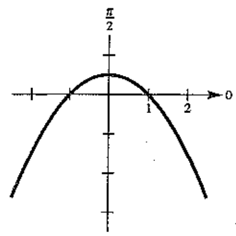
13. $r = \dfrac{-1}{1 - \sin\theta}$

Parabola because $e = 1$; $d = -1$

Distance from pole to directrix: $|d| = 1$

Directrix: $y = 1$

Vertex: $(r, \theta) = \left(-\dfrac{1}{2}, \dfrac{3\pi}{2}\right)$



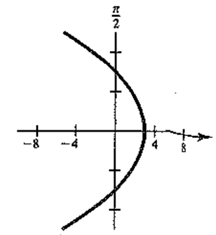
14. $r = \dfrac{6}{1 + \cos\theta}$

Parabola because $e = 1$; $d = 6$

Directrix: $x = 6$

Distance from pole to directrix: $|d| = 6$

Vertex: $(r, \theta) = (3, 0)$



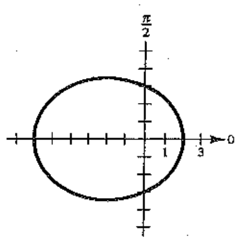
15. $r = \dfrac{6}{2 + \cos\theta} = \dfrac{3}{1 + (1/2)\cos\theta}$

Ellipse because $e = 1/2$; $d = 6$

Directrix: $x = 6$

Distance from pole to directrix: $|d| = 6$

Vertices: $(r, \theta) = (2, 0), (6, \pi)$



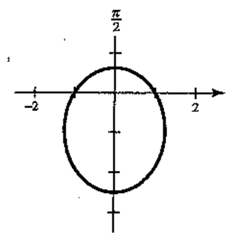
16. $r = \dfrac{5}{5 + 3\sin\theta} = \dfrac{1}{1 + (3/5)\sin\theta}$

Ellipse because $e = 3/5 < 1$; $d = 5/3$

Directrix: $y = 5/3$

Distance from pole to directrix: $|d| = 5/3$

Vertices: $(r, \theta) = (5/8, \pi/2), (5/2, 3\pi/2)$



17. $r(2 + \sin\theta) = 4$

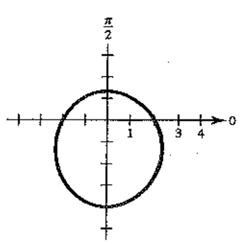
$r = \dfrac{4}{2 + \sin\theta} = \dfrac{2}{1 + (1/2)\sin\theta}$

Ellipse because $e = 1/2$; $d = 4$

Directrix: $y = 4$

Distance from pole to directrix: $|d| = 4$

Vertices: $(r, \theta) = (4/3, \pi/2), (4, 3\pi/2)$



18. $r(3 - 2\cos\theta) = 6$

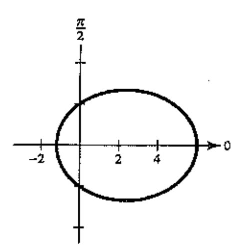
$r = \dfrac{6}{3 - 2\cos\theta} = \dfrac{2}{1 - (2/3)\cos\theta}$

Ellipse because $e = 2/3 < 1$; $d = 3$

Directrix: $x = -3$

Distance from pole to drectrix: $|d| = 3$

Vertices: $(r, \theta) = (6, 0), (6/5, \pi)$



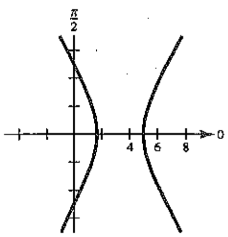
19. $r = \dfrac{5}{-1 + 2\cos\theta} = \dfrac{-5}{1 - 2\cos\theta}$

Hyperbola because $e = 2 > 1$; $d = -5/2$

Directrix: $x = 5/2$

Distance from pole to directrix: $|d| = 5/2$

Vertices: $(r, \theta) = (5, 0), (-5/3, \pi)$



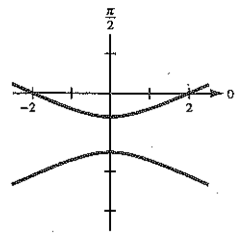
20. $r = \dfrac{-6}{3 + 7\sin\theta} = \dfrac{-2}{1 + (7/3)\sin\theta}$

Hyperbola because $e = 7/3 > 1$; $d = -6/7$

Directrix: $y = -6/7$

Distance from pole to directrix: $|d| = 6/7$

Vertices: $(r, \theta) = (-3/5, \pi/2), (3/2, 3\pi/2)$



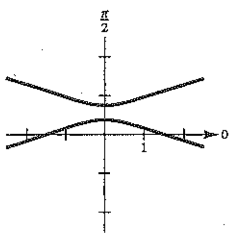
21. $r = \dfrac{3}{2 + 6\sin\theta} = \dfrac{3/2}{1 + 3\sin\theta}$

Hyperbola because $e = 3 > 0$; $d = 1/2$

Directrix: $y = 1/2$

Distance from pole to directrix: $|d| = 1/2$

Vertices: $(r, \theta) = (3/8, \pi/2), (-3/4, 3\pi/2)$



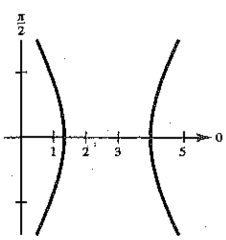
22. $r = \dfrac{4}{1 + 2\cos\theta}$

Hyperbola because $e = 2 > 1$; $d = 2$

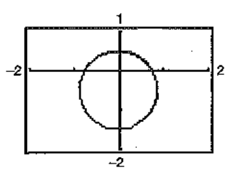
Directrix: $x = 2$

Distance from pole to directrix: $|d| = 2$

Vertices: $(r, \theta) = (4/3, 0), (-4, \pi)$

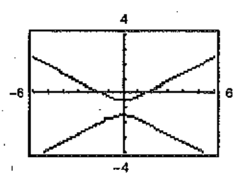


23. $r = 3/(-4 + 2\sin\theta)$



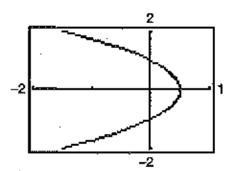
Ellipse

24. $r = -3/(2 + 4\sin\theta)$



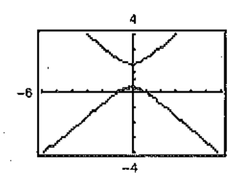
Hyperbola

25. $r = -1/(1 - \cos\theta)$



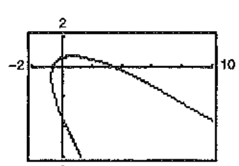
Parabola

26. $r = 2/(2 + 3\sin\theta)$



Hyperbola

27. $r = \dfrac{-1}{1 - \sin\left(\theta - \dfrac{\pi}{4}\right)}$

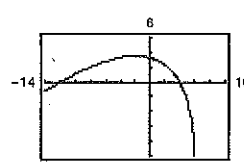


Rotate the graph of

$$r = \frac{-1}{1 - \sin\theta}$$

counterclockwise through the angle $\dfrac{\pi}{4}$.

28. $r = \dfrac{6}{1 + \cos\left(\theta - \dfrac{\pi}{3}\right)}$

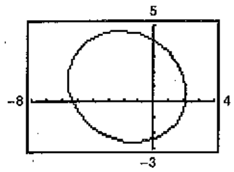


Rotate the graph of

$$r = \frac{6}{1 + \cos\theta}$$

counterclockwise through the angle $\dfrac{\pi}{3}$.

29. $r = \dfrac{6}{2 + \cos\left(\theta + \frac{\pi}{6}\right)}$

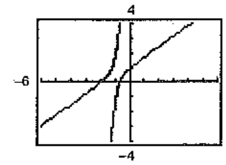


Rotate the graph of

$r = \dfrac{6}{2 + \cos\theta}$

clockwise through the angle $\dfrac{\pi}{6}$.

30. $r = \dfrac{-6}{3 + 7\sin(\theta + (2\pi/3))}$



Rotate graph of

$r = \dfrac{-6}{3 + 7\sin\theta}$

clockwise through angle of $2\pi/3$.

31. Change θ to $\theta + \dfrac{\pi}{4}$: $r = \dfrac{5}{5 + 3\cos\left(\theta + \frac{\pi}{4}\right)}$

32. Change θ to $\theta - \dfrac{\pi}{6}$: $r = \dfrac{2}{1 + \sin\left(\theta - \frac{\pi}{6}\right)}$

33. Parabola

$e = 1, x = -1, d = 1$

$r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{1}{1 - \cos\theta}$

34. Parabola

$e = 1, y = 1, d = 1$

$r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{1}{1 + \sin\theta}$

35. Ellipse

$e = \dfrac{1}{2}, y = 1, d = 1$

$r = \dfrac{ed}{1 + e\sin\theta}$

$= \dfrac{1/2}{1 + (1/2)\sin\theta}$

$= \dfrac{1}{2 + \sin\theta}$

36. Ellipse

$e = \dfrac{3}{4}, y = -2, d = 2$

$r = \dfrac{ed}{1 - e\sin\theta}$

$= \dfrac{2(3/4)}{1 - (3/4)\sin\theta}$

$= \dfrac{6}{4 - 3\sin\theta}$

37. Hyperbola

$e = 2, x = 1, d = 1$

$r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{2}{1 + 2\cos\theta}$

38. Hyperbola

$e = \dfrac{3}{2}, x = -1, d = 1$

$r = \dfrac{ed}{1 - e\cos\theta}$

$= \dfrac{3/2}{1 - (3/2)\cos\theta}$

$= \dfrac{3}{2 - 3\cos\theta}$

39. Parabola

Vertex: $\left(1, -\dfrac{\pi}{2}\right)$

$e = 1, d = 2, r = \dfrac{2}{1 - \sin\theta}$

40. Parabola

Vertex: $(5, \pi)$

$e = 1, d = 10$

$r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{10}{1 - \cos\theta}$

41. Ellipse

Vertices: $(2, 0), (8, \pi)$

$e = \dfrac{3}{5}, d = \dfrac{16}{3}$

$r = \dfrac{ed}{1 + e\cos\theta}$

$= \dfrac{16/5}{1 + (3/5)\cos\theta}$

$= \dfrac{16}{5 + 3\cos\theta}$

42. Ellipse

Vertices: $\left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$

$e = \frac{1}{3}, d = 8$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{8/3}{1 + (1/3)\sin\theta} = \frac{8}{3 + \sin\theta}$$

43. Hyperbola

Vertices: $\left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$

$e = \frac{5}{4}, d = \frac{9}{5}$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{9/4}{1 - (5/4)\sin\theta} = \frac{9}{4 - 5\sin\theta}$$

44. Hyperbola

Vertices: $(2, 0), (10, 0)$

$e = \frac{3}{2}, d = \frac{10}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{5}{1 + (3/2)\cos\theta} = \frac{10}{2 + 3\cos\theta}$$

45. Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

46. $r = \dfrac{4}{1 + \sin\theta}$ is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left of pole.

(b) Parabola with horizontal directrix below pole.

(c) Parabola with vertical directrix to right of pole.

(d) Parabola (b) rotated counterclockwise $\pi/4$.

47. (a) Hyperbola $(e = 2 > 1)$

(b) Ellipse $\left(e = \frac{1}{10} < 1\right)$

(c) Parabola $(e = 1)$

(d) Rotated hyperbola $(e = 3)$

48. If the foci are fixed and $e \to 0$, then $d \to \infty$. To see this, compare the ellipses

$$r = \frac{1/2}{1 + (1/2)\cos\theta}, \; e = 1/2, d = 1$$

$$r = \frac{5/16}{1 + (1/4)\cos\theta}, \; e = 1/4, d = 5/4.$$

49.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2b^2 + y^2a^2 = a^2b^2$$

$$b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$$

$$r^2[b^2\cos^2\theta + a^2(1 - \cos^2\theta)] = a^2b^2$$

$$r^2[a^2 + \cos^2\theta(b^2 - a^2)] = a^2b^2$$

$$r^2 = \frac{a^2b^2}{a^2 + (b^2 - a^2)\cos^2\theta} = \frac{a^2b^2}{a^2 - c^2\cos^2\theta}$$

$$= \frac{b^2}{1 - (c/a)^2\cos^2\theta} = \frac{b^2}{1 - e^2\cos^2\theta}$$

50.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x^2b^2 - y^2a^2 = a^2b^2$$

$$b^2r^2\cos^2\theta - a^2r^2\sin^2\theta = a^2b^2$$

$$r^2[b^2\cos^2\theta - a^2(1 - \cos^2\theta) = a^2b^2$$

$$r^2[-a^2 + \cos^2\theta(a^2 + b^2)] = a^2b^2$$

$$r^2 = \frac{a^2b^2}{-a^2 + c^2\cos^2\theta} = \frac{b^2}{-1 + (c^2/a^2)\cos^2\theta}$$

$$= \frac{-b^2}{1 - e^2\cos^2\theta}$$

51. $a = 5, c = 4, e = \frac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25)\cos^2\theta}$$

52. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16)\cos^2\theta}$$

53. $a = 3, b = 4, c = 5, e = \frac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9)\cos^2\theta}$$

54. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4)\cos^2\theta}$$

55. $A = 2\left[\frac{1}{2}\int_0^{\pi}\left(\frac{3}{2 - \cos\theta}\right)^2 d\theta\right]$

$$= 9\int_0^{\pi}\frac{1}{(2 - \cos\theta)^2}\,d\theta \approx 10.88$$

56. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(\frac{2}{3 - 2\sin\theta}\right)^2 d\theta\right] = 4\int_{-\pi/2}^{\pi/2}\frac{1}{(3 - 2\sin\theta)^2}\,d\theta \approx 3.37$

57. Vertices: $(123{,}000 + 4000, 0) = (127{,}000, 0)$

$(119 + 4000, \pi) = (4119, \pi)$

$$a = \frac{127{,}000 + 4119}{2} = 65{,}559.5$$

$$c = 65{,}559.5 - 4119 = 61{,}440.5$$

$$e = \frac{c}{a} = \frac{122{,}881}{131{,}119} \approx 0.93717$$

$$r = \frac{ed}{1 - e\cos\theta}$$

$$\theta = 0\colon\ r = \frac{ed}{1 - e},\ \theta = \pi\colon\ r = \frac{ed}{1 + e}$$

$$2a = 2(65{,}559.5) = \frac{ed}{1 - e} + \frac{ed}{1 + e}$$

$$131{,}119 = d\left(\frac{e}{1 - e} + \frac{e}{1 + e}\right) = d\left(\frac{2e}{1 - e^2}\right)$$

$$d = \frac{131{,}119(1 - e^2)}{2e} \approx 8514.1397$$

$$r = \frac{7979.21}{1 - 0.93717\cos\theta} = \frac{1{,}046{,}226{,}000}{131{,}119 - 122{,}881\cos\theta}$$

When $\theta = 60° = \frac{\pi}{3}$, $r \approx 15{,}015$.

Distance between earth and the satellite is $r - 4000 \approx 11{,}015$ miles.

58. (a) $r = \dfrac{ed}{1 - e\cos\theta}$

When $\theta = 0$, $r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \dfrac{(1 - e^2)a}{1 - e\cos\theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi$, $r = \dfrac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0$, $r = \dfrac{(1 - e^2)a}{1 - e} = a(1 + e)$.

59. $a = 1.496 \times 10^8$, $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{149{,}558{,}278.1}{1 - 0.0167\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 147{,}101{,}680$ km

Aphelion distance: $a(1 + e) \approx 152{,}098{,}320$ km

60. $a = 1.427 \times 10^9$, $e = 0.0542$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{1{,}422{,}807{,}988}{1 - 0.0542\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 1{,}349{,}656{,}600$ km

Aphelion distance: $a(1 + e) \approx 1{,}504{,}343{,}400$ km

61. $a = 5.906 \times 10^9$, $e = 0.2488$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{5{,}540{,}410{,}095}{1 - 0.2488\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 4{,}436{,}587{,}200$ km

Aphelion distance: $a(1 + e) \approx 7{,}375{,}412{,}800$ km

62. $a = 5.791 \times 10^7$, $e = 0.2056$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{55{,}462{,}065.54}{1 - 0.2056\cos\theta}$$

Perihelion distance $\approx a(1 - e) \approx 46{,}003{,}704$ km

Aphelion distance $\approx a(1 + e) \approx 69{,}816{,}296$ km

63. $r = \dfrac{5.540 \times 10^9}{1 - 0.2488 \cos \theta}$

(a) $A = \dfrac{1}{2}\int_0^{\pi/9} r^2 \, d\theta \approx 9.368 \times 10^{18} \text{ km}^2$

$$248\left[\frac{\frac{1}{2}\int_0^{\pi/9} r^2\, d\theta}{\frac{1}{2}\int_0^{2\pi} r^2\, d\theta}\right] \approx 21.89 \text{ yrs}$$

(b) $\dfrac{1}{2}\int_{\pi}^{\alpha} r^2 \, d\theta = 9.368 \times 10^8$

By trial and error, $\alpha = \pi + 0.9018$

$0.9018 > \pi/9 \approx 0.3491$ because the rays in part (a) are longer than those in part (b).

(c) For part (a)

$$s = \int_0^{\pi/9} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \approx 2.563 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{2.563 \times 10^9}{21.89} \approx 1.17 \times 10^8 \text{ km/yr}$$

For part (b)

$$s = \int_{\pi}^{\pi + 0.9018} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \approx 4.133 \times 10^9$$

$$\text{Average per year} = \frac{4.133 \times 10^9}{21.89} \approx 1.89 \times 10^8 \text{ km/yr}$$

64. $a = \dfrac{1}{2}(250) = 125$ au, $e = 0.995$

(a) $e = \dfrac{c}{a} \Rightarrow c = 124.375$

$b^2 = a^2 - c^2 \Rightarrow b \approx 12.4844$

length of minor axis: $2b \approx 24.97$ au

(b) $r = \dfrac{(1 - e^2)a}{1 - e\cos\theta} = \dfrac{1.246875}{1 - 0.995\cos\theta}$

(c) Perihelion distance: $a(1 - e) = 0.625$ au

Aphelion distance: $a(1 + e) = 249.375$ au

65. $r_1 = a + c,\ r_0 = a - c,\ r_1 - r_0 = 2c,\ r_1 + r_0 = 2a$

$$e = \frac{c}{a} = \frac{r_1 - r_0}{r_1 + r_0}$$

$$\frac{1 + e}{1 - e} = \frac{1 + \frac{c}{a}}{1 - \frac{c}{a}} = \frac{a + c}{a - c} = \frac{r_1}{r_0}$$

66. For a hyperbola,

$r_0 = c - a$ and $r_1 = c + a$.

Thus $r_1 + r_0 = 2c$ and $r_1 - r_0 = 2a$.

$$e = \frac{c}{a} = \frac{r_1 + r_0}{r_1 - r_0}$$

$$\frac{e + 1}{e - 1} = \frac{c/a + 1}{c/a - 1} = \frac{c + a}{c - a} = \frac{r_1}{r_0}$$

67. $r_1 = \dfrac{ed}{1 + \sin\theta}$ and $r_2 = \dfrac{ed}{1 - \sin\theta}$

Points of intersection: $(ed, 0)$, (ed, π)

$$r_1\!: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin\theta}\right)(\cos\theta) + \left(\frac{-ed\cos\theta}{(1 + \sin\theta)^2}\right)(\sin\theta)}{\left(\frac{-ed}{1 + \sin\theta}\right)(\sin\theta) + \left(\frac{-ed\cos\theta}{(1 + \sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = -1$. At (ed, π), $\dfrac{dy}{dx} = 1$.

$$r_2\!: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin\theta}\right)(\cos\theta) + \left(\frac{ed\cos\theta}{(1 - \sin\theta)^2}\right)(\sin\theta)}{\left(\frac{-ed}{1 - \sin\theta}\right)(\sin\theta) + \left(\frac{ed\cos\theta}{(1 - \sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = 1$. At (ed, π), $\dfrac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

68. $r_1 = \dfrac{c}{1 + \cos\theta}, r_2 = \dfrac{d}{1 - \cos\theta}$ (Parabolas)

To find the intersection points:

$$\frac{c}{1+\cos\theta} = \frac{d}{1-\cos\theta}$$

$$c - c \cdot \cos\theta = d + d \cdot \cos\theta$$

$$\cos\theta = \frac{c-d}{c+d}$$

$$r_1 = \frac{c}{1 + \left(\dfrac{c-d}{c+d}\right)} = \frac{c(c+d)}{2c} = \frac{c+d}{2} = r_2$$

$$\frac{dr_1}{d\theta} = \frac{c \cdot \sin\theta}{(1+\cos\theta)^2}, \frac{dr_2}{d\theta} = \frac{-d \cdot \sin\theta}{(1-\cos\theta)^2}$$

For the first parabola,

$$\frac{dy}{dx} = \frac{r_1\cos\theta + r'_1\sin\theta}{-r_1\sin\theta + r'_1\cos\theta} = \frac{c \cdot \cos\theta(1+\cos\theta) + c \cdot \sin^2\theta}{-c \cdot \sin\theta(1+\cos\theta) + c \cdot \sin\theta\cos\theta} = \frac{1+\cos\theta}{-\sin\theta}.$$

Similarly for the second parabola,

$$\frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta}.$$

Since the product of the slopes is -1, they intersect at right angles.

Review Exercises for Chapter 10

1. $4x^2 + y^2 = 4$

Ellipse

Vertex: $(1, 0)$.

Matches (e)

2. $4x^2 - y^2 = 4$

Hyperbola

Vertex: $(1, 0)$

Matches (c)

3. $y^2 = -4x$

Parabola opening to left.

Matches (b)

4. $y^2 - 4x^2 = 4$

Hyperbola

Vertex: $(0, 2)$

Matches (d)

5. $x^2 + 4y^2 = 4$

Ellipse

Vertex: $(0, 1)$

Matches (a)

6. $x^2 = 4y$

Parabola opening upward.

Matches (f)

7. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

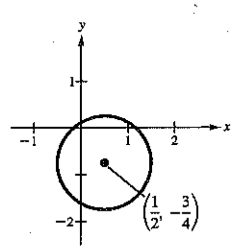
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, \frac{3}{4}\right)$

Radius: 1



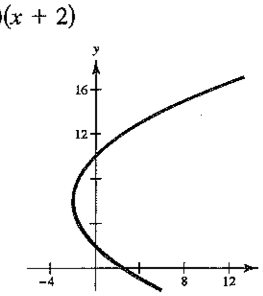
8. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y-6)^2 = 4(2)(x+2)$$

Parabola

Vertex: $(-2, 6)$



9. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$

$$\frac{(x+4)^2}{2} - \frac{(y-3)^2}{3} = 1$$

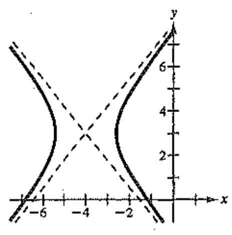
Hyperbola

Center: $(-4, 3)$

Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$

Asymptotes:

$y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$



10. $4x^2 + y^2 - 16x + 15 = 0$

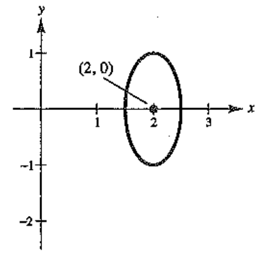
$4(x^2 - 4x + 4) + y^2 = -15 + 16$

$$\frac{(x-2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$



11. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

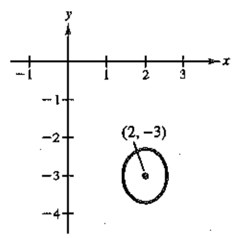
$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$

$$\frac{(x-2)^2}{1/3} + \frac{(y+3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



12. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$

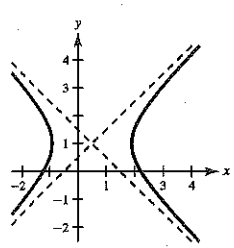
$$\frac{[x - (1/2)]^2}{2} - \frac{(y-1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$



13. Vertex: $(0, 2)$

Directrix: $x = -3$

Parabola opens to the right.

$p = 3$

$(y - 2)^2 = 4(3)(x - 0)$

$y^2 - 4y - 12x + 4 = 0$

14. Vertex: $(4, 2)$

Focus: $(4, 0)$

Parabola opens downward.

$p = -2$

$(x - 4)^2 = 4(-2)(y - 2)$

$x^2 - 8x + 8y = 0$

15. Vertices: $(-3, 0), (7, 0)$

Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$a = 5, c = 2, b = \sqrt{21}$

$$\frac{(x-2)^2}{25} + \frac{y^2}{21} = 1$$

16. Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, \frac{4}{b^2} = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

17. Vertices: $(\pm 4, 0)$

Foci: $(\pm 6, 0)$

Center: $(0, 0)$

Horizontal transverse axis

$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

18. Foci: $(0, \pm 8)$

Asymptotes: $y = \pm 4x$

Center: $(0, 0)$

Vertical transverse axis

$c = 8$

$y = \frac{a}{b}x = 4x$ asymptote $\rightarrow a = 4b$

$b^2 = c^2 - a^2 = 64 - (4b)^2 \implies 17b^2 = 64$

$\implies b^2 = \frac{64}{17} \implies a^2 = \frac{1024}{17}$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

19. $\frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$

By Example 5 of Section 10.1,

$$C = 12\int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right)\sin^2\theta}\, d\theta \approx 15.87.$$

20. $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 10.1,

$$C = 20\int_0^{\pi/2} \sqrt{1 - \frac{21}{25}\sin^2\theta}\, d\theta \approx 23.01.$$

21. $y = x - 2$ has a slope of 1. The perpendicular slope is -1.

$y = x^2 - 2x + 2$

$\frac{dy}{dx} = 2x - 2 = -1$ when $x = \frac{1}{2}$ and $y = \frac{5}{4}$.

Perpendicular line: $y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$

$$4x + 4y - 7 = 0$$

22. $2x + y = 5$ has slope -2. The perpendicular slope is $\frac{1}{2}$.

$y = -3x^2 + x - 6$ Parabola

$y' = -6x + 1 = \frac{1}{2}$

$6x = \frac{1}{2}$

$x = \frac{1}{12}, \quad y = -\frac{95}{16}$

Perpendicular line: $y + \frac{95}{16} = \frac{1}{2}\left(x - \frac{1}{12}\right)$

$$y = \frac{1}{2}x - \frac{287}{48}$$

23. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: $(0, 50)$

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10{,}000}}$

$$S = 2\pi\int_0^{100} x\sqrt{1 + \frac{x^2}{10{,}000}}\, dx \approx 38{,}294.49$$

24. (a) $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

(b) $F = 2(62.4)\int_{-3}^{3}(3-y)\frac{4}{3}\sqrt{9-y^2}\,dy = \frac{8}{3}(62.4)\left[3\int_{-3}^{3}\sqrt{9-y^2}\,dy - \int_{-3}^{3}y\sqrt{9-y^2}\,dy\right]$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(y\sqrt{9-y^2} + 9\arcsin\frac{y}{3}\right) + \frac{1}{3}(9-y^2)^{3/2}\right]_{-3}^{3}$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(\frac{9\pi}{2}\right) - \frac{3}{2}\left(-\frac{9\pi}{2}\right)\right] = \frac{8}{3}(62.4)\left(\frac{27\pi}{2}\right) \approx 7057.274$$

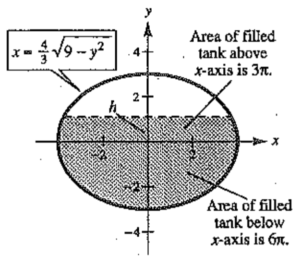
(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$2\int_0^h \frac{4}{3}\sqrt{9-y^2}\,dy = 3\pi$$

$$\int_0^h \sqrt{9-y^2}\,dy = \frac{9\pi}{8}$$

$$\frac{1}{2}\left[y\sqrt{9-y^2} + 9\arcsin\left(\frac{y}{3}\right)\right]_0^h = \frac{9\pi}{8}$$

$$h\sqrt{9-h^2} + 9\arcsin\left(\frac{h}{3}\right) = \frac{9\pi}{4}.$$



By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

(d) Area of ends $= 2(12\pi) = 24\pi$

Area of sides = (Perimeter)(Length)

$$= 16\int_0^{\pi/2}\left(\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\theta}\right)d\theta(16) \text{ [from Example 5 of Section 9.1]}$$

$$\approx 353.656$$

Total area $= 24\pi + 353.656 \approx 429.054$

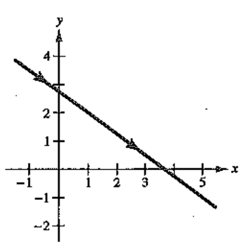
25. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$4y + 3x - 11 = 0$

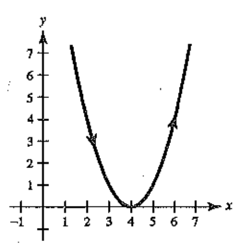
Line



26. $x = t + 4, y = t^2$

$t = x - 4 \Rightarrow y = (x-4)^2$

Parabola

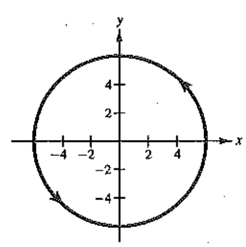


27. $x = 6\cos\theta, y = 6\sin\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$x^2 + y^2 = 36$

Circle

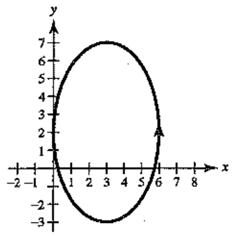


28. $x = 3 + 3\cos\theta, y = 2 + 5\sin\theta$

$$\left(\frac{x-3}{3}\right)^2 + \left(\frac{y-2}{5}\right)^2 = 1$$

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$$

Ellipse

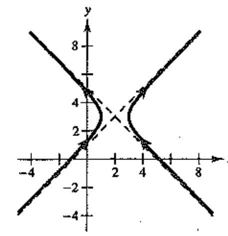


29. $x = 2 + \sec\theta, y = 3 + \tan\theta$

$(x-2)^2 = \sec^2\theta = 1 + \tan^2\theta = 1 + (y-3)^2$

$(x-2)^2 - (y-3)^2 = 1$

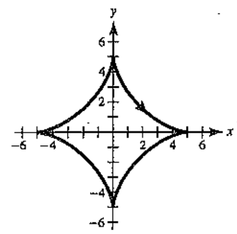
Hyperbola



30. $x = 5\sin^3\theta, y = 5\cos^3\theta$

$$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$$

$$x^{2/3} + y^{2/3} = 5^{2/3}$$



31. $x = 3 + (3 - (-2))t = 3 + 5t$

$y = 2 + (2 - 6)t = 2 - 4t$

(other answers possible)

32. $(x-h)^2 + (y-k)^2 = r^2$

$(x-5)^2 + (y-3)^2 = 2^2 = 4$

$x = 5 + 2\cos t, \quad y = 3 + 2\sin t$

33. $\dfrac{(x+3)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

Let $\dfrac{(x+3)^2}{16} = \cos^2\theta$ and $\dfrac{(y-4)^2}{9} = \sin^2\theta$.

Then $x = -3 + 4\cos\theta$ and $y = 4 + 3\sin\theta$.

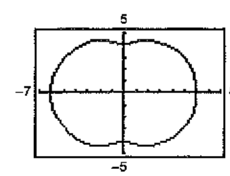
34. $a = 4,\ c = 5,\ b^2 = c^2 - a^2 = 9,\ \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

Let $\dfrac{y^2}{16} = \sec^2\theta$ and $\dfrac{x^2}{9} = \tan^2\theta$.

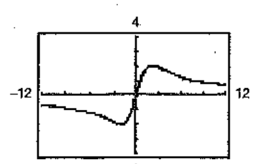
Then $x = 3\tan\theta$ and $y = 4\sec\theta$.

35. $x = \cos 3\theta + 5\cos\theta$

$y = \sin 3\theta + 5\sin\theta$



36. (a) $x = 2\cot\theta, y = 4\sin\theta\cos\theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4\cot^2\theta)4\sin\theta\cos\theta$

$= 16\csc^2\theta \cdot \sin\theta \cdot \cos\theta$

$= 16\dfrac{\cos\theta}{\sin\theta}$

$= 8(2\cot\theta)$

$= 8x$

37. $x = 1 + 4t$

$y = 2 - 3t$

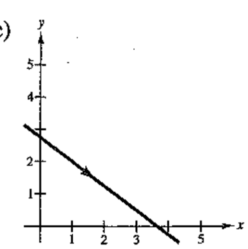
(a) $\dfrac{dy}{dx} = -\dfrac{3}{4}$

No horizontal tangents

(b) $t = \dfrac{x-1}{4}$

$y = 2 - \dfrac{3}{4}(x-1) = \dfrac{-3x+11}{4}$

(c)



38. $x = t + 4$

$y = t^2$

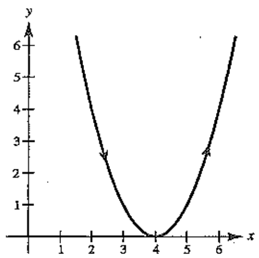
(a) $\frac{dy}{dx} = \frac{2t}{1} = 2t = 0$ when $t = 0$.

Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$y = (x - 4)^2$

(c)



39. $x = \frac{1}{t}$

$y = 2t + 3$

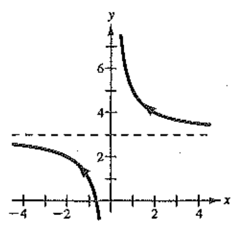
(a) $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

No horizontal tangents, $(t \neq 0)$

(b) $t = \frac{1}{x}$

$y = \frac{2}{x} + 3$

(c)



40. $x = \frac{1}{t}$

$y = t^2$

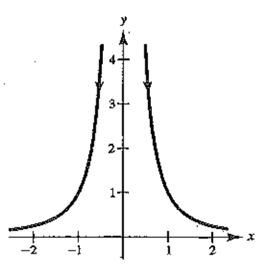
(a) $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents, $(t \neq 0)$

(b) $t = \frac{1}{x}$

$y = \frac{1}{x^2}$

(c)



41. $x = \frac{1}{2t + 1}$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

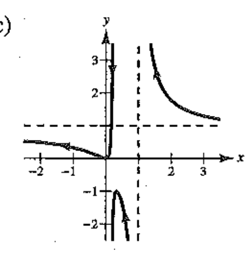
Point of horizontal tangency: $\left(\frac{1}{3}, -1\right)$

(b) $2t + 1 = \frac{1}{x} \implies t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}, \quad (x \neq 0)$$

(c)



42. $x = 2t - 1$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

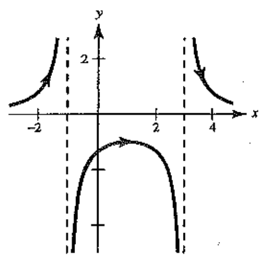
$= \dfrac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \dfrac{x + 1}{2}$

$y = \dfrac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \dfrac{4}{(x - 3)(x + 1)}$

(c)



43. $x = 3 + 2\cos\theta$

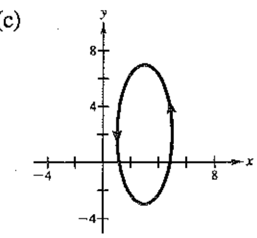
$y = 2 + 5\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{5\cos\theta}{-2\sin\theta} = -2.5\cot\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

(b) $\dfrac{(x - 3)^2}{4} + \dfrac{(y - 2)^2}{25} = 1$

(c)



44. $x = 6\cos\theta$

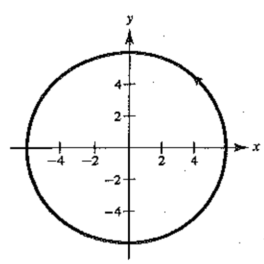
$y = 6\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{6\cos\theta}{-6\sin\theta} = -\cot\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\dfrac{x}{6}\right)^2 + \left(\dfrac{y}{6}\right)^2 = 1$

(c)



45. $x = \cos^3\theta$

$y = 4\sin^3\theta$

(a) $\dfrac{dy}{dx} = \dfrac{12\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)}$

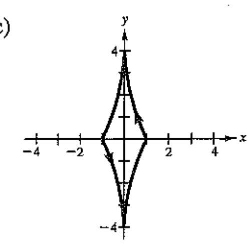
$= \dfrac{-4\sin\theta}{\cos\theta} = -4\tan\theta = 0$

when $\theta = 0, \pi$.

But, $\dfrac{dy}{dt} = \dfrac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\dfrac{y}{4}\right)^{2/3} = 1$

(c)



46. $x = e^t$

$y = e^{-t}$

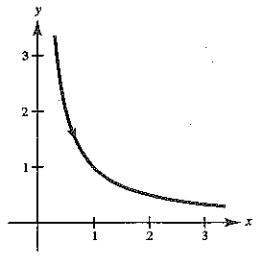
(a) $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$

(c)



47. $x = 4 - t, \quad y = t^2$

$\frac{dx}{dt} = -1, \quad \frac{dy}{dt} = 2t$

$\frac{dy}{dt} = 0$ for $t = 0$.

Horizontal tangent at $t = 0$: $(x, y) = (4, 0)$

No vertical tangents

48. $x = t + 2, y = t^3 - 2t$

$\frac{dx}{dt} = 1, \frac{dy}{dt} = 3t^2 - 2$

$\frac{dy}{dt} = 0$ for $t = \pm\sqrt{\frac{2}{3}} = \frac{\pm\sqrt{6}}{3}$

Horizontal tangents: $t = \frac{\sqrt{2}}{3}$: $(x, y) = \left(\frac{\sqrt{6}}{3} + 2, \frac{2\sqrt{6}}{9} - \frac{2}{3}\sqrt{6}\right)$

$\approx (2.8165, -1.0887)$

$t = -\frac{\sqrt{6}}{3}$: $(x, y) = \left(-\frac{\sqrt{6}}{3} + 2, \frac{2}{3}\sqrt{6} - \frac{2\sqrt{6}}{9}\right)$

$\approx (1.1835, 1.0887)$

No vertical tangents

49. $x = 2 + 2\sin\theta, y = 1 + \cos\theta$

$\frac{dx}{d\theta} = 2\cos\theta, \frac{dy}{d\theta} = -\sin\theta$

$\frac{dy}{d\theta} = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

Horizontal tangents: $(x, y) = (2, 2), (2, 0)$

$\frac{dx}{d\theta} = 0$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Vertical tangents: $(x, y) = (4, 1), (0, 1)$

50. $x = 2 - 2\cos\theta, y = 2\sin 2\theta$

$\frac{dx}{d\theta} = 2\sin\theta, \frac{dy}{d\theta} = 4\cos 2\theta$

$\frac{dy}{d\theta} = 0$ for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

Horizontal tangents: $(x, y) = \left(2 \pm \sqrt{2}, 2\right)$,

$\left(2 \pm \sqrt{2}, -2\right)$

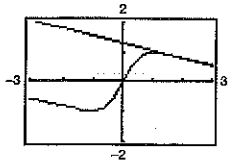
$\frac{dx}{d\theta} = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

Vertical tangents: $(x, y) = (0, 0), (4, 0)$

51. $x = \cot\theta$

$y = \sin 2\theta = 2\sin\theta\cos\theta$

(a), (c)

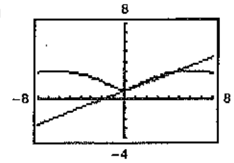


(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} = -4, \dfrac{dy}{d\theta} = 1$, and $\dfrac{dy}{dx} = -\dfrac{1}{4}$.

52. $x = 2\theta - \sin\theta$

$y = 2 - \cos\theta$

(a), (c)



(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} \approx 1.134, \left(2 - \dfrac{\sqrt{3}}{2}\right)$,

$\dfrac{dy}{dt} = 0.5$, and $\dfrac{dy}{dx} \approx 0.441$.

53. $x = r(\cos\theta + \theta\sin\theta)$

$y = r(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = r\theta\cos\theta$$

$$\frac{dy}{d\theta} = r\theta\sin\theta$$

$$s = r\int_0^{\pi} \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\, d\theta$$

$$= r\int_0^{\pi} \theta\, d\theta = \frac{r}{2}\Big[\theta^2\Big]_0^{\pi} = \frac{1}{2}\pi^2 r$$

54. $x = 6\cos\theta$

$y = 6\sin\theta$

$$\frac{dx}{d\theta} = -6\sin\theta$$

$$\frac{dy}{d\theta} = 6\cos\theta$$

$$s = \int_0^{\pi} \sqrt{36\sin^2\theta + 36\cos^2\theta}\, d\theta = \Big[6\theta\Big]_0^{\pi} = 6\pi$$

(one-half circumference of circle)

55. $x = t, y = 3t, 0 \le t \le 2$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -3, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+9} = \sqrt{10}$$

(a) $S = 2\pi\displaystyle\int_0^2 3t\sqrt{10}\, dt = 6\sqrt{10}\,\pi\left[\frac{t^2}{2}\right]_0^2 = 12\sqrt{10}\,\pi$

(b) $S = 2\pi\displaystyle\int_0^2 \sqrt{10}\, dt = 2\pi\sqrt{10}\Big[t\Big]_0^2 = 4\pi\sqrt{10}$

56. $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le \dfrac{\pi}{2}$

$$\frac{dx}{dt} = -2\sin\theta, \frac{dy}{dt} = 2\cos\theta, \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2$$

(a) $S = 2\pi\displaystyle\int_0^{\pi/2} 2\sin\theta(2)d\theta = 8\pi\Big[-\cos\theta\Big]_0^{\pi/2} = 8\pi$

(b) $S = 2\pi\displaystyle\int_0^{\pi/2} 2\cos\theta(2)d\theta = 8\pi\Big[\sin\theta\Big]_0^{\pi/2} = 8\pi$

[**Note:** The surface is a hemisphere: $\frac{1}{2}(4\pi(2^2)) = 8\pi$]

57. $x = 3\sin\theta, y = 2\cos\theta$

$$A = \int_a^b y\, dx = \int_{-\pi/2}^{\pi/2} 2\cos\theta(3\cos\theta)\, d\theta$$

$$= 6\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 3\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 3\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 3\pi$$

58. $A = \displaystyle\int_a^b y\, dx = \int_{\pi}^{0} \sin\theta(-2\sin\theta)\, d\theta$

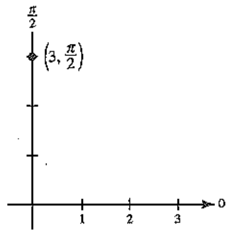
$$= -\int_{\pi}^{0} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi}^{0} = \pi$$

59. $(r, \theta) = \left(3, \frac{\pi}{2}\right)$

$(x, y) = \left(3 \cos \frac{\pi}{2}, 3 \sin \frac{\pi}{2}\right)$

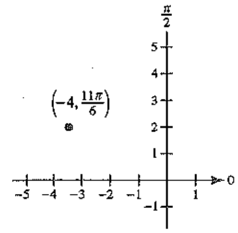
$= (0, 3)$



60. $(r, \theta) = \left(-4, \frac{11\pi}{6}\right)$

$(x, y) = \left(-4 \cos \frac{11\pi}{6}, -4 \sin \frac{11\pi}{6}\right)$

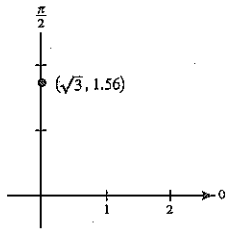
$= \left(-2\sqrt{3}, 2\right)$



61. $(r, \theta) = \left(\sqrt{3}, 1.56\right)$

$(x, y) = \left(\sqrt{3} \cos(1.56), \sqrt{3} \sin(1.56)\right)$

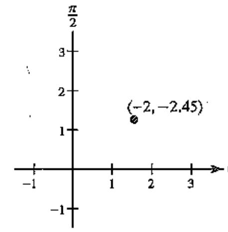
$\approx (0.0187, 1.7319)$



62. $(r, \theta) = (-2, -2.45)$

$(x, y) = (-2 \cos(-2.45), -\sin(-2.45))$

$\approx (1.5405, 1.2755)$

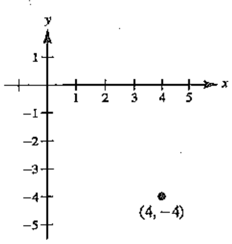


63. $(x, y) = (4, -4)$

$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$

$\theta = \frac{7\pi}{4}$

$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$

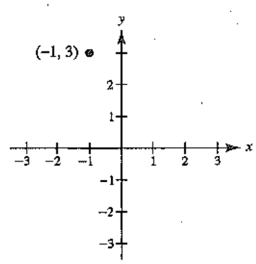


64. $(x, y) = (-1, 3)$

$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

$\theta = \arctan(-3) \approx 1.89(108.43°)$

$(r, \theta) = \left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$



65.

$$r = 3 \cos \theta$$
$$r^2 = 3r \cos \theta$$
$$x^2 + y^2 = 3x$$
$$x^2 + y^2 - 3x = 0$$

66.

$$r = 10$$
$$r^2 = 100$$
$$x^2 + y^2 = 100$$

67.

$$r = -2(1 + \cos \theta)$$
$$r^2 = -2r(1 + \cos \theta)$$
$$x^2 + y^2 = -2\left(\pm\sqrt{x^2 + y^2}\right) - 2x$$
$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

68.

$$r = \frac{1}{2 - \cos \theta}$$
$$2r - r \cos \theta = 1$$
$$2\left(\pm\sqrt{x^2 + y^2}\right) - x = 1$$
$$4(x^2 + y^2) = (x + 1)^2$$
$$3x^2 + 4y^2 - 2x - 1 = 0$$

69.
$$r^2 = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$(x^2 + y^2)^2 = x^2 - y^2$$

70.
$$r = 4\sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$
$$= \frac{4}{(1/2)\cos\theta + \left(\sqrt{3}/2\right)\sin\theta}$$
$$r\left(\cos\theta + \sqrt{3}\sin\theta\right) = 8$$
$$x + \sqrt{3}\,y = 8$$

71.
$$r = 4\cos 2\theta\sec\theta$$
$$= 4(2\cos^2\theta - 1)\left(\frac{1}{\cos\theta}\right)$$
$$r\cos\theta = 8\cos^2\theta - 4$$
$$x = 8\left(\frac{x^2}{x^2 + y^2}\right) - 4$$
$$x^3 + xy^2 = 4x^2 - 4y^2$$
$$y^2 = x^2\left(\frac{4 - x}{4 + x}\right)$$

72.
$$\theta = \frac{3\pi}{4}$$
$$\tan\theta = -1$$
$$\frac{y}{x} = -1$$
$$y = -x$$

73. $(x^2 + y^2)^2 = ax^2y$
$$r^4 = a(r^2\cos^2\theta)(r\sin\theta)$$
$$r = a\cos^2\theta\sin\theta$$

74. $x^2 + y^2 - 4x = 0$
$$r^2 - 4r\cos\theta = 0$$
$$r = 4\cos\theta$$

75. $x^2 + y^2 = a^2\left(\arctan\frac{y}{x}\right)^2$
$$r^2 = a^2\theta^2$$

76. $(x^2 + y^2)\left(\arctan\frac{y}{x}\right)^2 = a^2$
$$r^2\theta^2 = a^2$$

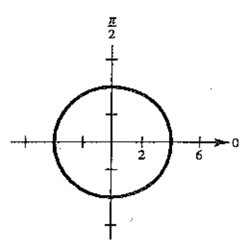
77. $r = 4$

Circle of radius 4

Centered at the pole

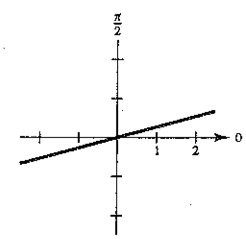
Symmetric to polar axis,

$\theta = \pi/2$, and pole



78. $\theta = \dfrac{\pi}{12}$

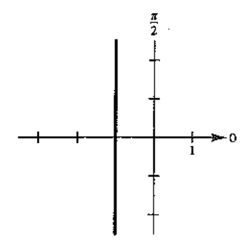
Line



79. $r = -\sec\theta = \dfrac{-1}{\cos\theta}$

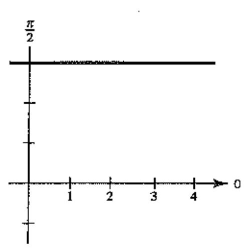
$r\cos\theta = -1, x = -1$

Vertical line



80. $r = 3\csc\theta, r\sin\theta = 3, y = 3$

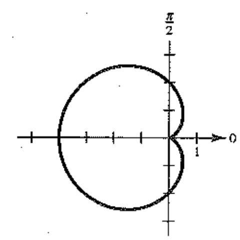
Horizontal line



81. $r = -2(1 + \cos\theta)$

Cardioid

Symmetric to polar axis

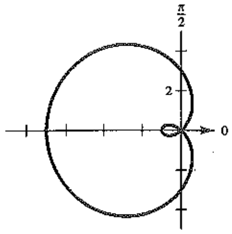


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

82. $r = 3 - 4\cos\theta$

Limaçon

Symmetric to polar axis

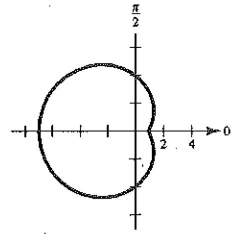


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

83. $r = 4 - 3\cos\theta$

Limaçon

Symmetric to polar axis

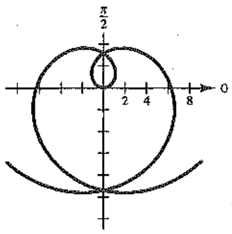


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

84. $r = 2\theta$

Spiral

Symmetric to $\theta = \pi/2$



θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

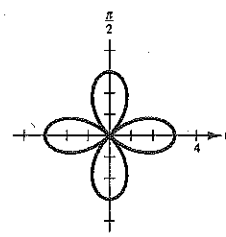
85. $r = -3\cos 2\theta$

Rose curve with four petals

Symmetric to polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $(-3, 0), \left(3, \frac{\pi}{2}\right), (-3, \pi), \left(3, \frac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$



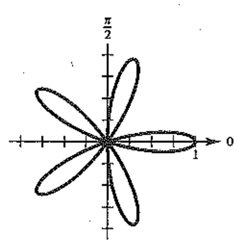
86. $r = \cos 5\theta$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), \left(-1, \frac{\pi}{5}\right), \left(1, \frac{2\pi}{5}\right), \left(-1, \frac{3\pi}{5}\right), \left(1, \frac{4\pi}{5}\right)$

Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$



87. $r^2 = 4\sin^2 2\theta$

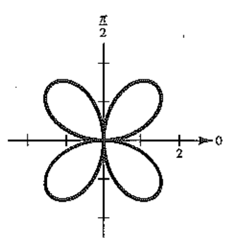
$r = \pm 2\sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \frac{\pi}{4}\right), \left(\pm 2, \frac{3\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$



88. $r^2 = \cos 2\theta$

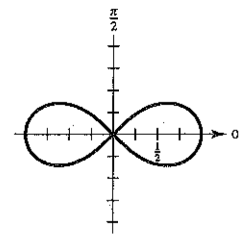
Lemniscate

Symmetric to the polar axis

Relative extrema: $(\pm 1, 0)$

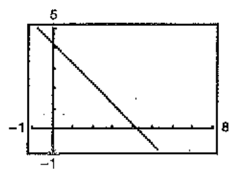
Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm\frac{\sqrt{2}}{2}$	0



89. $r = \dfrac{3}{\cos \theta - (\pi/4)}$

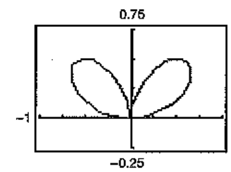
Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$



90. $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to $\theta = \pi/2$



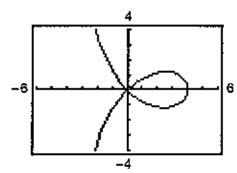
91. $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$r \Longrightarrow \infty$ as $\theta \Longrightarrow \frac{\pi^-}{2}$

$r \Longrightarrow \infty$ as $\theta \Longrightarrow \frac{-\pi^+}{2}$



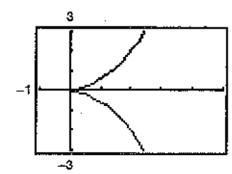
92. $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$r \Longrightarrow \infty$ as $\theta \Longrightarrow \frac{\pi^-}{2}$

$r \Longrightarrow \infty$ as $\theta \Longrightarrow \frac{-\pi^+}{2}$



93. $r = 1 - 2 \cos \theta$

(a) The graph has polar symmetry and the tangents at the pole are $\theta = \frac{\pi}{3}, -\frac{\pi}{3}$.

(b) $\dfrac{dy}{dx} = \dfrac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$

Horizontal tangents: $-4 \cos^2 \theta + \cos \theta + 2 = 0, \cos \theta = \dfrac{-1 \pm \sqrt{1 + 32}}{-8} = \dfrac{1 \pm \sqrt{33}}{8}$

When $\cos \theta = \dfrac{1 \pm \sqrt{33}}{8}, r = 1 - 2\left(\dfrac{1 + \sqrt{33}}{8}\right) = \dfrac{3 \mp \sqrt{33}}{4}$,

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, -0.568)$$

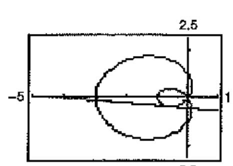
$$\left[\frac{3 + \sqrt{33}}{4}, \arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, -2.206).$$

Vertical tangents: $\sin \theta(4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4}, \theta = 0, \pi, \theta = \pm\arccos\left(\frac{1}{4}\right), (-1, 0), (3, \pi)$

$$\left(\frac{1}{2}, \pm\arccos \frac{1}{4}\right) \approx (0.5, \pm 1.318)$$

(c)



94. $r^2 = 4\sin(2\theta)$

(a) $2r\left(\dfrac{dr}{d\theta}\right) = 8\cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4\cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

(b) $$\frac{dy}{dx} = \frac{r\cos\theta + [(4\cos 2\theta\sin\theta)/r]}{-r\sin\theta + [(4\cos 2\theta\cos\theta)/r]}$$

$$= \frac{\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

Horizontal tangents:

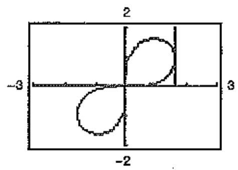
$\dfrac{dy}{dx} = 0$ when $\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta = 0$,

$\tan\theta = -\tan(2\theta)$, $\theta = 0, \dfrac{\pi}{3}$, $(0, 0)$, $\left(\pm\sqrt{2\sqrt{3}}, \dfrac{\pi}{3}\right)$

Vertical tangents when $\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$:

$\tan 2\theta\tan\theta = 1$, $\theta = 0, \dfrac{\pi}{6}$, $(0, 0)$, $\left(\pm\sqrt{2\sqrt{3}}, \dfrac{\pi}{6}\right)$

(c)



95. Circle: $r = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + 3\sin\theta\cos\theta}{3\cos\theta\cos\theta - 3\sin\theta\sin\theta} = \frac{\sin 2\theta}{\cos^2\theta - \sin^2\theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5\sin\theta$

$$\frac{dy}{dx} = \frac{-5\cos\theta\sin\theta + (4 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (4 - 5\sin\theta)\sin\theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan\alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

Therefore, $\alpha = \arctan\left(\dfrac{2\sqrt{3}}{3}\right) \approx 49.1°$.

96. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

97. $r = 1 + \cos\theta$, $r = 1 - \cos\theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos\theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin^2\theta + \cos\theta(1 + \cos\theta)}{-\sin\theta\cos\theta - \sin\theta(1 + \cos\theta)}.$$

At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos\theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2\theta + \cos\theta(1 - \cos\theta)}{\sin\theta\cos\theta - \sin\theta(1 - \cos\theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

98. $r = a\sin\theta$, $r = a\cos\theta$

The points of intersection are $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$. For $r = a\sin\theta$,

$$m_1 = \frac{dy}{dx} = \frac{a\cos\theta\sin\theta + a\sin\theta\cos\theta}{a\cos^2\theta - a\sin^2\theta} = \frac{2\sin\theta\cos\theta}{\cos 2\theta}.$$

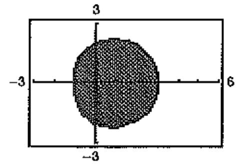
At $(a/\sqrt{2}, \pi/4)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a\cos\theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a\sin^2\theta + a\cos^2\theta}{-a\sin\theta\cos\theta - a\cos\theta\sin\theta} = \frac{\cos 2\theta}{-2\sin\theta\cos\theta}.$$

At $(a/\sqrt{2}, \pi/4)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$.

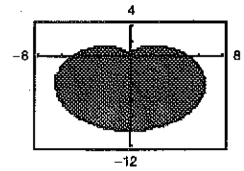
99. $r = 2 + \cos\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi} (2 + \cos\theta)^2\, d\theta\right] \approx 14.14,\ \left(\frac{9\pi}{2}\right)$$



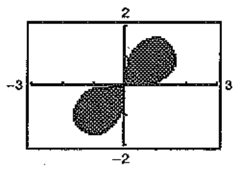
100. $r = 5(1 - \sin\theta)$

$$A = 2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2} [5(1 - \sin\theta)]^2\right] d\theta \approx 117.81,\ \left(\frac{75\pi}{2}\right)$$



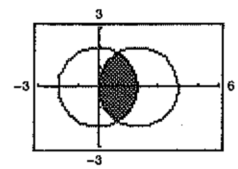
101. $r^2 = 4\sin 2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\, d\theta\right] = 4$$



102. $r = 4\cos\theta,\ r = 2$

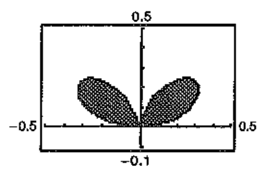
$$A = 2\left[\frac{1}{2}\int_0^{\pi/3} 4\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\, d\theta\right] \approx 4.91$$



103. $r = \sin\theta\cos^2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} (\sin\theta\cos^2\theta)^2\, d\theta\right]$$

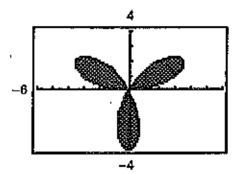
$$\approx 0.10,\ \left(\frac{\pi}{32}\right)$$



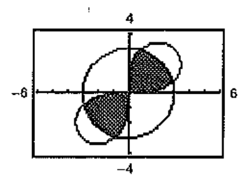
104. $r = 4\sin 3\theta$

$$A = 3\left[\frac{1}{2}\int_0^{\pi/3} (4\sin 3\theta)^2\, d\theta\right]$$

$$\approx 12.57\ (4\pi)$$



105. $r = 3,\ r^2 = 18\sin 2\theta$



$$9 = r^2 = 18\sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

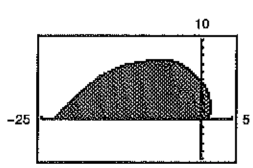
$$\theta = \frac{\pi}{12}$$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta\, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\, d\theta\right]$$

$$\approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$

106. $r = e^{\theta},\ 0 \le \theta \le \pi$

$$A = \frac{1}{2}\int_0^{\pi} (e^{\theta})^2\, d\theta \approx 133.62$$



107. $r = a(1 - \cos\theta),\ 0 \le \theta \le \pi$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$s = \int_0^{\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= \sqrt{2}a\int_0^{\pi}\sqrt{1-\cos\theta}\,d\theta$$

$$= \sqrt{2}a\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$$

$$= -2\sqrt{2}a\Big[(1+\cos\theta)^{1/2}\Big]_0^{\pi} = 4a$$

108. $r = a\cos 2\theta,\ -\pi/4 \le \theta \le \pi/2$

$$\frac{dr}{d\theta} = -2a\sin 2\theta$$

$$s = \int_{-\pi/2}^{\pi/2}\sqrt{a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\,d\theta$$

$$= a\int_{-\pi/2}^{\pi/2}\sqrt{1+3\sin^2 2\theta}\,d\theta$$

Using a graphing utility, $s \approx 4.8442a$.

109. $f(\theta) = 1 + 4\cos\theta$

$$f'(\theta) = -4\sin\theta$$

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(1+4\cos\theta)^2 + (-4\sin\theta)^2} = \sqrt{17 + 8\cos\theta}$$

$$S = 2\pi\int_0^{\pi/2}(1+4\cos\theta)\sin\theta\sqrt{17+8\cos\theta}\,d\theta$$

$$= \frac{34\pi\sqrt{17}}{5} \approx 88.08$$

110. $f(\theta) = 2\sin\theta$

$$f'(\theta) = 2\cos\theta$$

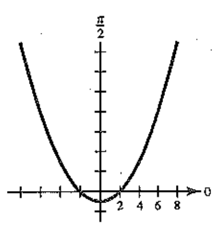
$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2$$

$$S = 2\pi\int_0^{\pi/2} 2\sin\theta\cos\theta(2)\,d\theta$$

$$= 4\pi$$

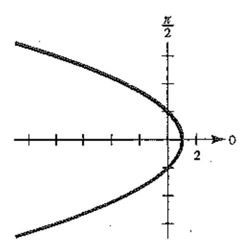
111. $r = \dfrac{2}{1-\sin\theta},\ e = 1$

Parabola



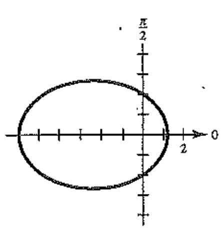
112. $r = \dfrac{2}{1+\cos\theta},\ e = 1$

Parabola



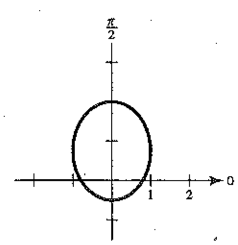
113. $r = \dfrac{6}{3+2\cos\theta} = \dfrac{2}{1+(2/3)\cos\theta},\ e = \dfrac{2}{3}$

Ellipse



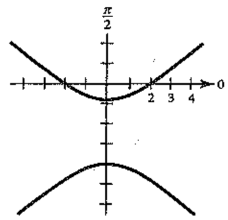
114. $r = \dfrac{4}{5-3\sin\theta} = \dfrac{4/5}{1-(3/5)\sin\theta},\ e = \dfrac{3}{5}$

Ellipse



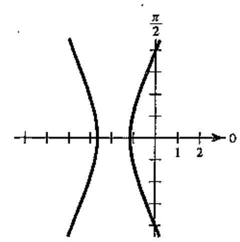
115. $r = \dfrac{4}{2 - 3\sin\theta} = \dfrac{2}{1 - (3/2)\sin\theta}, \; e = \dfrac{3}{2}$

Hyperbola



116. $r = \dfrac{8}{2 - 5\cos\theta} = \dfrac{4}{1 - (5/2)\cos\theta}, \; e = \dfrac{5}{2}$

Hyperbola



117. Circle

Center: $\left(5, \dfrac{\pi}{2}\right) = (0, 5)$ in rectangular coordinates

Solution point: $(0, 0)$

$x^2 + (y - 5)^5 = 25$

$x^2 + y^2 - 10y = 0$

$r^2 - 10r\sin\theta = 0$

$r = 10\sin\theta$

118. Line

Slope: $\sqrt{3}$

Solution point: $(0, 0)$

$y = \sqrt{3}\,x, \; r\sin\theta = \sqrt{3}\,r\cos\theta,$

$\tan\theta = \sqrt{3}, \; \theta = \dfrac{\pi}{3}$

119. Parabola

Vertex: $(2, \pi)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 - \cos\theta}$

120. Parabola

Vertex: $\left(2, \dfrac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 + \sin\theta}$

121. Ellipse

Vertices: $(5, 0), (1, \pi)$

Focus: $(0, 0)$

$a = 3, c = 2, e = \dfrac{2}{3}, d = \dfrac{5}{2}$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right)\cos\theta} = \frac{5}{3 - 2\cos\theta}$$

122. Hyperbola

Vertices: $(1, 0), (7, 0)$

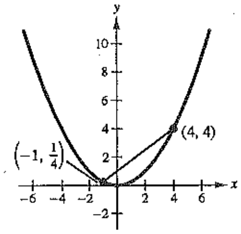
Focus: $(0, 0)$

$a = 3, c = 4, e = \dfrac{4}{3}, d = \dfrac{7}{4}$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right)\cos\theta} = \frac{7}{3 + 4\cos\theta}$$

Problem Solving for Chapter 10

1. (a)



(b) $x^2 = 4y$

$2x = 4y'$

$y' = \frac{1}{2}x$

$y - 4 = 2(x - 4) \implies y = 2x - 4$ Tangent line at $(4, 4)$

$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \implies y = -\frac{1}{2}x - \frac{1}{4}$ Tangent line at $\left(-1, \frac{1}{4}\right)$

Tangent lines have slopes of 2 and $-1/2 \implies$ perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$

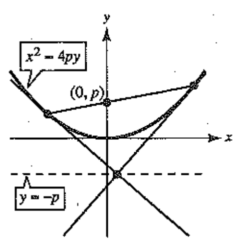
$$8x - 16 = -2x - 1$$

$$10x = 15$$

$$x = \frac{3}{2} \implies \left(\frac{3}{2}, -1\right)$$

Point of intersection, $(3/2, -1)$, is on directrix $y = -1$.

2. Assume $p > 0$.



Let $y = mx + p$ be the equation of the focal chord.

First find x-coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \implies y' = \frac{x}{2p}.$$

(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}\left[2pm + 2p\sqrt{m^2 + 1}\right]\frac{1}{2p}\left[2pm - 2p\sqrt{m^2 + 1}\right] = \frac{1}{4p^2}[4p^2m^2 - 4p^2(m^2 + 1)] = \frac{1}{4p^2}[-4p^2] = -1$$

—CONTINUED—

2. —CONTINUED—

(b) Finally, we show that the tangent lines intersect at a point on the directrix $y = -p$.

Let $b = 2pm + 2p\sqrt{m^2+1}$ and $c = 2pm - 2p\sqrt{m^2+1}$.

$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2+1}$

$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2+1}$

$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2+1}$

$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2+1}$

Tangent line at $x = b$: $y - \frac{b^2}{4p} = \frac{b}{2p}(x - b) \implies y = \frac{bx}{2p} - \frac{b^2}{4p}$

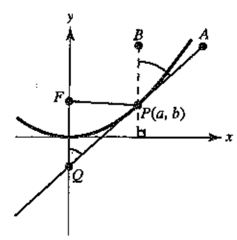
Tangent line at $x = c$: $y - \frac{c^2}{4p} = \frac{c}{2p}(x - c) \implies y = \frac{cx}{2p} - \frac{c^2}{4p}$

Intersection of tangent lines:

$$\frac{bx}{2p} - \frac{b^2}{4p} = \frac{cx}{2p} - \frac{c^2}{4p}$$
$$2bx - b^2 = 2cx - c^2$$
$$2x(b - c) = b^2 - c^2$$
$$2x\left(4p\sqrt{m^2+1}\right) = 16p^2m\sqrt{m^2+1}$$
$$x = 2pm$$

Finally, the corresponding y-value is $y - p$, which shows that the intersection point lies on the directrix.

3. Consider $x^2 = 4py$ with focus $F = (0, p)$.



Let $P(a, b)$ be point on parabola.

$2x = 4py' \implies y' = \frac{x}{2p}$

$y - b = \frac{a}{2p}(x - a)$ Tangent line at P

For $x = 0$, $y = b + \frac{a}{2p}(-a) = b - \frac{a^2}{2p} = b - \frac{4pb}{2p} = -b$.

Thus, $Q = (0, -b)$.

$\triangle FQP$ is isosceles because

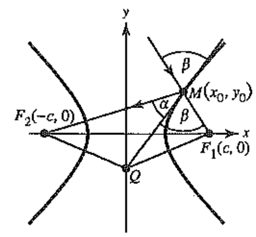
$|FQ| = p + b$

$$|FP| = \sqrt{(a-0)^2 + (b-p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2}$$
$$= \sqrt{4pb + b^2 - 2bp + p^2}$$
$$= \sqrt{(b+p)^2}$$
$$= b + p.$$

Thus, $\measuredangle FQP = \measuredangle BPA = \measuredangle FPQ$.

4. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, a^2 + b^2 = c^2, MF_2 - MF_1 = 2a$

$y' = \dfrac{b^2x}{a^2y}$



Tangent line at $M(x_0, y_0)$: $\quad y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0x - x_0^2}{a^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

At $x = 0, y = -\dfrac{b^2}{y_0} \Rightarrow Q = \left(0, -\dfrac{b^2}{y_0}\right)$.

$QF_2 = QF_1 = \sqrt{c^2 + \dfrac{b^4}{y_0^2}} = d$

$MQ = \sqrt{x_0^2 + \left(y_0 + \dfrac{b^2}{y_0}\right)^2} = f$

By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos\alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos\alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta.$$

$$\cos\alpha = \frac{(MF_2)^2f^2 - d^2}{2f(MF_2)}, \cos\beta = \frac{(MF_1)^2 + f^2 - d^2}{2f(MF_1)}$$

$MF_2 = MF_1 + 2a$. Let $z = MF_1$.

Slopes: MF_1: $\dfrac{y_0}{x_0 - c}$; QF_1: $\dfrac{-b^2}{y_0c}$; QF_2: $\dfrac{b^2}{y_0c}$

To show $\alpha = \beta$, consider

$$[(MF_2)^2 + f^2 - d^2][2f(MF_1)] = [(MF_1)^2 + f^2 - d^2][2f(MF_2)]$$

$$\Leftrightarrow \quad [(z + 2a)^2 + f^2 - d^2][z] = [z^2 + f^2 - d^2][z + 2a]$$

$$\Leftrightarrow \quad z^2 + 2az = f^2 - d^2$$

$$\Leftrightarrow \quad (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right)$$

$$\Leftrightarrow \quad az - x_0c + a^2 = 0$$

$$\Leftrightarrow \quad a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2$$

$$\Leftrightarrow \quad x_0^2b^2 - a^2y_0^2 = a^2b^2$$

$$\Leftrightarrow \quad \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.$$

Thus, $\alpha = \beta$ and the reflective property is verified.

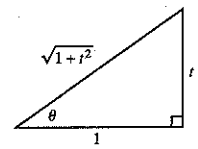
5. (a) In $\triangle OCB$, $\cos\theta = \dfrac{2a}{OB} \implies OB = 2a \cdot \sec\theta.$

In $\triangle OAC$, $\cos\theta = \dfrac{OA}{2a} \implies OA = 2a \cdot \cos\theta.$

$$\begin{aligned} r = OP = AB = OB - OA &= 2a(\sec\theta - \cos\theta) \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) \\ &= 2a \cdot \frac{\sin^2\theta}{\cos\theta} \\ &= 2a \cdot \tan\theta\sin\theta \end{aligned}$$

(b) $x = r\cos\theta = (2a\tan\theta\sin\theta)\cos\theta = 2a\sin^2\theta$

$y = r\sin\theta = (2a\tan\theta\sin\theta)\sin\theta = 2a\tan\theta \cdot \sin^2\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$



Let $t = \tan\theta$, $-\infty < t < \infty$.

Then $\sin^2\theta = \dfrac{t^2}{1+t^2}$ and $x = 2a\dfrac{t^2}{1+t^2}$, $y = 2a\dfrac{t^3}{1+t^2}$.

(c)
$$\begin{aligned} r &= 2a\tan\theta\sin\theta \\ r\cos\theta &= 2a\sin^2\theta \\ r^3\cos\theta &= 2a\,r^2\sin^2\theta \\ (x^2+y^2)x &= 2ay^2 \\ y^2 &= \frac{x^3}{(2a-x)} \end{aligned}$$

6. (a) $A = 4\displaystyle\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\left(\frac{1}{2}\right)\left[x\sqrt{a^2-x^2} + a^2\arcsin\left(\frac{x}{a}\right)\right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi\displaystyle\int_0^b \frac{a^2}{b^2}(b^2-y^2)\,dy = \frac{2\pi a^2}{b^2}\int_0^b (b^2-y^2)\,dy = \frac{2\pi a^2}{b^2}\left[b^2y - \frac{1}{3}y^3\right]_0^b = \frac{4}{3}\pi a^2 b$

$$\begin{aligned} S &= 4\pi\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\left(\frac{\sqrt{b^4+(a^2-b^2)y^2}}{b\sqrt{b^2-y^2}}\right)dy \\ &= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4+c^2y^2}\,dy = \frac{2\pi a}{b^2c}\left[cy\sqrt{b^4+c^2y^2} + b^4\ln\left|cy+\sqrt{b^4+c^2y^2}\right|\right]_0^b \\ &= \frac{2\pi a}{b^2c}\left[b^2c\sqrt{b^2+c^2} + b^4\ln\left|cb+b\sqrt{b^2+c^2}\right| - b^4\ln(b^2)\right] \\ &= 2\pi a^2 + \frac{\pi ab^2}{c}\ln\left(\frac{c+a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left(\frac{1+e}{1-e}\right) \end{aligned}$$

(c) **Disk:** $V = 2\pi\displaystyle\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2-x^2)\,dx = \frac{2\pi b^2}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi ab^2$

$$\begin{aligned} S &= 2(2\pi)\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\left(\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\right)dx \\ &= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4-c^2x^2}\,dx = \frac{2\pi b}{a^2c}\left[cx\sqrt{a^4-c^2x^2} + a^4\arcsin\left(\frac{cx}{a^2}\right)\right]_0^a \\ &= \frac{a\pi b}{a^2c}\left[a^2c\sqrt{a^2-c^2} + a^4\arcsin\left(\frac{c}{a}\right)\right] = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e) \end{aligned}$$

7. (a) $y^2 = \dfrac{t^2(1-t^2)^2}{(1+t^2)^2},\ x^2 = \dfrac{(1-t^2)^2}{(1+t^2)^2}$

$$\frac{1-x}{1+x} = \frac{1-\left(\dfrac{1-t^2}{1+t^2}\right)}{1+\left(\dfrac{1-t^2}{1+t^2}\right)} = \frac{2t^2}{2} = t^2$$

Thus, $y^2 = x^2\left(\dfrac{1-x}{1+x}\right)$.

(b)

$$r^2\sin^2\theta = r^2\cos^2\theta\left(\frac{1-r\cos\theta}{1+r\cos\theta}\right)$$

$$\sin^2\theta(1+r\cos\theta) = \cos^2\theta(1-r\cos\theta)$$

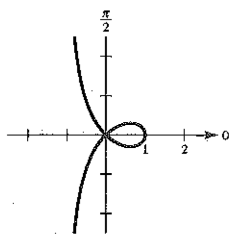
$$r\cos\theta\sin^2\theta + \sin^2\theta = \cos^2\theta - r\cos^3\theta$$

$$r\cos\theta(\sin^2\theta+\cos^2\theta) = \cos^2\theta - \sin^2\theta$$

$$r\cos\theta = \cos 2\theta$$

$$r = \cos 2\theta \cdot \sec\theta$$

(c)



(d) $r(\theta) = 0$ for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

Thus, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

(e) $y'(t) = \dfrac{(1+t^2)(1-3t^2)-(t-t^3)(2t)}{(1+t^2)^2} = \dfrac{1-4t^2-t^4}{(1+t^2)^2} = 0$

$$t^4+4t^2-1=0 \implies t^2 = -2\pm\sqrt{5} \implies x = \frac{1-\left(-2\pm\sqrt{5}\right)}{1+\left(-2\pm\sqrt{5}\right)} = \frac{3\mp\sqrt{5}}{-1\pm\sqrt{5}}$$

$$= \frac{3-\sqrt{5}}{-1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$$

$$\left(\frac{\sqrt{5}-1}{2}, \pm\frac{\sqrt{5}-1}{2}\sqrt{-2+\sqrt{5}}\right)$$

8. $y = a(1-\cos\theta) \implies \cos\theta = \dfrac{a-y}{a}$

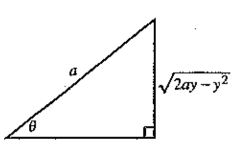
$$\theta = \arccos\left(\frac{a-y}{a}\right)$$

$$x = a(\theta - \sin\theta)$$

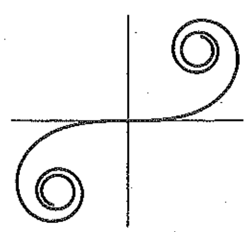
$$= a\left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right)\right)$$

$$= a\left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay-y^2}}{a}\right)$$

$$x = a\cdot\arccos\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2},\ 0 \le y \le 2a$$



9. (a)



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(b) $(-x,-y) = \left(-\displaystyle\int_0^t \cos\frac{\pi u^2}{2}\,du, -\int_0^t \sin\frac{\pi u^2}{2}\,du\right)$ is on the curve whenever (x, y) is on the curve.

(c) $x'(t) = \cos\dfrac{\pi t^2}{2},\ y'(t) = \sin\dfrac{\pi t^2}{2},\ x'(t)^2 + y'(t)^2 = 1$

Thus, $s = \displaystyle\int_0^a dt = a.$

On $[-\pi, \pi]$, $s = 2\pi$.

10. For $t = \frac{\pi}{2}, \frac{3}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \ldots$$

Hence, the curve has length greater that

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \cdots \\ &= \frac{2}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) \\ &> \frac{2}{\pi}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right) \\ &= \infty. \text{ (Harmonic series)} \end{aligned}$$

11. $r = \dfrac{ab}{a \sin\theta + b\cos\theta},\ 0 \le \theta \le \dfrac{\pi}{2}$

$$r(a\sin\theta + b\cos\theta) = ab$$

$$ay + bx = ab$$

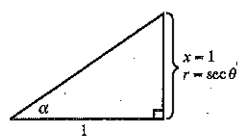
$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

$$\text{Area} = \frac{1}{2}ab$$

12. (a) $\text{Area} = \displaystyle\int_0^\alpha \frac{1}{2}r^2\,d\theta$

$$= \frac{1}{2}\int_0^\alpha \sec^2\theta\,d\theta$$



(b) $\tan\alpha = \dfrac{h}{1} \implies \text{Area} = \dfrac{1}{2}(1)\tan\alpha$

$$\implies \tan\alpha = \int_0^\alpha \sec^2\theta\,d\theta$$

(c) Differentiating, $\dfrac{d}{d\alpha}(\tan\alpha) = \sec^2\alpha.$

13. Let (r, θ) be on the graph.

$$\begin{aligned} \sqrt{r^2 + 1 + 2r\cos\theta}\sqrt{r^2 + 1 - 2r\cos\theta} &= 1 \\ (r^2 + 1)^2 - 4r^2\cos^2\theta &= 1 \\ r^4 + 2r^2 + 1 - 4r^2\cos^2\theta &= 1 \\ r^2(r^2 - 4\cos^2\theta + 2) &= 0 \\ r^2 &= 4\cos^2\theta - 2 \\ r^2 &= 2(2\cos^2\theta - 1) \\ r^2 &= 2\cos 2\theta \end{aligned}$$

14. If a dog is located at (r, θ) in the first quadrant, then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$$(x_1, y_1) = (r\cos\theta, r\sin\theta) \text{ and } (x_2, y_2) = (-r\sin\theta, r\cos\theta).$$

The slope joining these points is

$$\frac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \text{slope of tangent line at } (r, \theta).$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dr}}{\frac{dx}{dr}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$$

$$\begin{aligned} \implies \frac{dr}{d\theta} &= -r \\ \frac{dr}{r} &= -d\theta \\ \ln r &= -\theta + C_1 \\ r &= e^{-\theta + C_1} \\ r &= Ce^{-\theta} \end{aligned}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \implies r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \implies C = \frac{d}{\sqrt{2}}e^{\pi/4}$$

Finally, $r = \dfrac{d}{\sqrt{2}}e^{((\pi/4) - \theta)},\ \theta \ge \dfrac{\pi}{4}.$

15. (a) The first plane makes an angle of 70° with the positive x-axis, and is 150 miles from P:

$$x_1 = \cos 70°(150 - 375t)$$

$$y_1 = \sin 70°(150 - 375t)$$

Similarly for the second plane,

$$x_2 = \cos 135°(190 - 450t)$$

$$= \cos 45°(-190 + 450t)$$

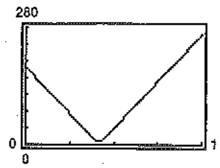
$$y_2 = \sin 135°(190 - 450t)$$

$$= \sin 45°(190 - 450t).$$

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= [[\cos 45(-190 + 450t) - \cos 70(150 - 375t)]^2 + [\sin 45(190 - 450t) - \sin 70(150 - 375t)]^2]^{1/2}$$

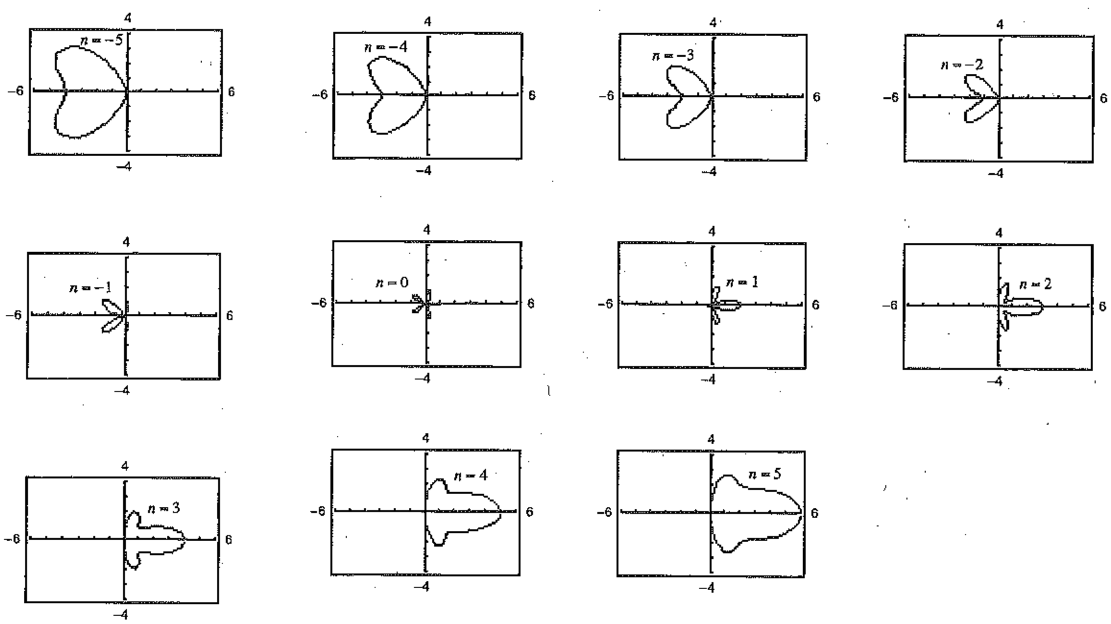
(c)



The minimum distance is 7.59 miles when $t = 0.4145$.

16. The curve is produced over the interval $0 \le \theta \le 10\pi$.

17.



$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".

CHAPTER 11
Vectors and the Geometry of Space