Calculus BC

Test #4 Review

Eliminate the parameter and find a corresponding rectangular equation.

1. $x = 3t^2 \quad y = 2t + 1$
2. $x = 3 + \cos\theta \quad y = 2 - \sin\theta$

Find $\frac{dy}{dx}$ for the following parametric curves.

3. $x = t^2 \quad y = \sqrt{t-1}$
4. $x = \sqrt{t} \quad y = (t-1)^3$
5. $x = 2\cos\theta \quad y = 2 + \sin\theta$

Find the equation of the tangent line for the following curves at the given values.

6. $x = 2t \quad y = t^2 + 5 \quad$ at $t = 1$
7. $x = \sqrt{t} \quad y = \frac{1}{2}t^2 \quad$ at $t = 4$

Find $\frac{d^2y}{dx^2}$ for the following parametric curves.

8. $x = 2\cos\theta \quad y = \sin\theta$
9. $x = t^3 + 2 \quad y = t^2 + t$

Find the arc length of the curves on the given interval.

10. $x = t^2 \quad y = 2t^2 - 1, \quad 1 < t < 4$
11. $x = \frac{8}{3}t^{3/2} \quad y = 2t - t^2, \quad 1 < t < 3$

Convert the following equation to ~~rectangular~~ form. PARAMETRIC

12. $r = 3\sec\theta$

13. $r = 2\sin\theta$

For the following, find $\frac{dy}{dx}$ for the given value of θ.

14. $r = 2 + 3\sin\theta,\ \theta = \dfrac{3\pi}{2}$

15. $r = 2\sin(3\theta),\ \theta = \dfrac{\pi}{4}$

16. Find the point of horizontal and vertical tangency for $r = 1 + \sin\theta$. Give answers in polar and rectangular form.

17. Find the equation of the tangent line to the curve $r = 2 - 3\sin\theta$ at the point $(2, \pi)$.

For each of the following, find the area of the described region.

18. one petal of $r = 2\cos(3\theta)$

19. interior of $r = 2 + 2\cos\theta$

20. inner loop of $r = 1 + 2\cos\theta$

21. between the loops of $r = 1 + 2\cos\theta$

22. inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$

23. A particle's position at time t on the coordinate plane xy is given by the vector $\langle t^2 + t, 1 - t^3 \rangle$.
    a. Find the velocity vector of the particle at time t = 4.
    b. Find the acceleration vector of the particle at time t = 4.
    c. Find the particle's speed at time t = 4.
    d. Find the equation of the line tangent to the motion of the particle at t = 2.
    e. Find the total distance traveled by the particle in the time interval $0 < t < 4$.

24. The velocity of a moving particle at time t is given by the vector$\langle t^3 - 4t, t \rangle$. At time t = 1, the position vector of the particle was $\langle 0, \frac{1}{2} \rangle$.
    a. Find the position vector of the particle at any time t.
    b. Find the velocity vector at t = 2. Interpret the values found for both the horizontal and vertical velocity of the particle at time t = 2.
    c. Find the particle's speed at t = 1.
    d. Find the acceleration vector of the particle at any time t.
    e. Find the total distance traveled by the particle in the time interval $1 < t < 3$.\

25. A moving particle has position $\langle x(t), y(t) \rangle$ at time t. The position of the particle at time t =1 is (2, 6), and the velocity vector at any time t > 0 is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.
    a. Find the acceleration vector at time t = 3.
    b. Find the position of the particle at any time t. Find the position of the particle at time t = 3.
    c. For what time t > 0 does the line tangent to the path of the particle have a slope of 8?
    d. The particle approaches a line as $t \to \infty$. Find the slope of this line. Show your work that leads to your conclusion.

Test #4 Review Answers:

1. $y = 2t + 1 \Rightarrow \frac{y-1}{2} = t$

$x = 3t^2 \Rightarrow x = 3\left(\frac{y-1}{2}\right)^2 \Rightarrow x = \frac{3}{4}(y-1)^2$

2. $x = 3 + \cos\theta$ $\quad y = 2 - \sin\theta$ $\quad \sin^2\theta + \cos^2\theta = 1$

$x - 3 = \cos\theta$ $\quad -(y-2) = \sin\theta$ $\quad [-(y-2)]^2 + (x-3)^2 = 1$

$(y-2)^2 + (x-3)^2 = 1$

3. $\frac{dy}{dt} = \frac{1}{2}(t-1)^{-\frac{1}{2}}$ $\quad \frac{dy}{dx} = \frac{\frac{1}{2}(t-1)^{-\frac{1}{2}}}{2t} = \frac{1}{4t\sqrt{t-1}}$

$\frac{dx}{dt} = 2t$

4. $\frac{dy}{dt} = 3(t-1)^2$ $\quad \frac{dy}{dx} = \frac{3(t-1)^2}{\frac{1}{2}(t)^{-\frac{1}{2}}} = 6\sqrt{t}(t-1)^2$

$\frac{dx}{dt} = \frac{1}{2}(t)^{-\frac{1}{2}}$

5. $\frac{dy}{dt} = \cos\theta$ $\quad \frac{dy}{dx} = \frac{\cos\theta}{-2\sin\theta} = -\frac{1}{2}\cot\theta$

$\frac{dx}{dt} = -2\sin\theta$

6. @ $t = 1$, $x = 2$ & $y = 6$

$\frac{dy}{dt} = 2t$ $\quad \frac{dy}{dx} = \frac{2t}{2} = t$

$\frac{dx}{dt} = 2$

$\left.\frac{dy}{dx}\right|_{t=1} = 1$

Tangent Line.

$y - 6 = x - 2$

$y = x + 4$

7. @ $t=4$, $x=2$ & $y=8$

$$\frac{dy}{dt} = t \qquad \frac{dy}{dx} = \frac{t}{\frac{1}{2}(t)^{-\frac{1}{2}}} = 2t^{3/2}$$

$$\frac{dx}{dt} = \frac{1}{2}(t)^{-\frac{1}{2}} \qquad \left.\frac{dy}{dx}\right|_{t=4} = 16$$

TANGENT LINE

$$y - 8 = 16(x-2)$$
$$y = 16x - 24$$

8. $$\frac{dy}{dt} = \cos\theta \qquad \frac{dy}{dx} = \frac{\cos\theta}{-2\sin\theta} = -\frac{1}{2}\cot\theta$$

$$\frac{dx}{dt} = -2\sin\theta \qquad \frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{1}{2}\csc^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{2}\csc^2\theta}{-2\sin\theta} = -\frac{1}{4}\csc^3\theta$$

9. $$\frac{dy}{dt} = 2t+1 \qquad \frac{dy}{dx} = \frac{2t+1}{3t^2} = \frac{2}{3}t^{-1} + \frac{1}{3}t^{-2}$$

$$\frac{dx}{dt} = 3t^2 \qquad \frac{d}{dt}\left[\frac{dy}{dx}\right] = -\frac{2}{3}t^{-2} - \frac{2}{3}t^{-3}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{2}{3}t^{-2} - \frac{2}{3}t^{-3}}{3t^2} = -\frac{2}{9}\left(\frac{1}{t^4} + \frac{1}{t^5}\right)$$

10. $$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 4t$$

$$s = \int_1^4 \sqrt{(2t)^2 + (4t)^2}\, dt = \int_1^4 \sqrt{20t^2}\, dt = 15\sqrt{5} \approx 33.541$$

11. $$\frac{dx}{dt} = 4t^{1/2} \qquad \frac{dy}{dt} = 2-2t$$

$$s = \int_1^3 \sqrt{(4t^{1/2})^2 + (2-2t)^2}\, dt = \int_1^3 \sqrt{4(t+1)^2}\, dt = 12$$

12. $x = r\cos\theta \qquad y = r\sin\theta$

$x = (3\sec\theta)\cos\theta \qquad y = (3\sec\theta)\sin\theta$

$x = 3 \qquad y = 3\tan\theta$

13. $x = (2\sin\theta)\cos\theta \qquad y = (2\sin\theta)\sin\theta$

$x = 2\sin\theta\cos\theta \qquad y = 2\sin^2\theta$

$x = \sin(2\theta)$

14. $x = (2 + 3\sin\theta)\cos\theta \qquad y = (2 + 3\sin\theta)\sin\theta$

$x = 2\cos\theta + 3\sin\theta\cos\theta \qquad y = 2\sin\theta + 3\sin^2\theta$

$$\frac{dx}{dt} = -2\sin\theta + 3[\sin\theta \cdot -\sin\theta + \cos\theta \cdot \cos\theta] \qquad \frac{dy}{dt} = 2\cos\theta + 6\sin\theta\cos\theta$$

$$= -2\sin\theta - 3\sin^2\theta + 3\cos^2\theta$$

$$\frac{dy}{dx} = \frac{2\cos\theta + 6\sin\cos\theta}{3\cos^2\theta - 3\sin^2\theta - 2\sin\theta} \qquad \left.\frac{dy}{dx}\right|_{\theta = \frac{3\pi}{2}} = 0$$

15. $x = (2\sin(3\theta))\cos\theta \qquad y = (2\sin(3\theta))\sin\theta$

$x = 2\sin(3\theta)\cos\theta \qquad y = 2\sin(3\theta)\sin\theta$

$$\frac{dx}{d\theta} = 2\sin(3\theta) \cdot -\sin\theta + \cos\theta \cdot 6\cos(3\theta) \qquad \frac{dy}{d\theta} = 2\sin(3\theta)\cos\theta + \sin\theta \cdot 6\cos(3\theta)$$

$$= 6\cos\theta\cos(3\theta) - 2\sin\theta\sin(3\theta) \qquad = 2\sin(3\theta)\cos\theta + 6\cos(3\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{2\sin(3\theta)\cos\theta + 6\cos(3\theta)\sin\theta}{6\cos\theta\cos(3\theta) - 2\sin\theta\sin(3\theta)} \qquad \left.\frac{dy}{dx}\right|_{\theta = \pi/4} = \frac{1}{2}$$

16 $x = (1+\sin\theta)\cos\theta \quad y = (1+\sin\theta)\sin\theta$

$x = \cos\theta + \sin\theta\cos\theta \quad y = \sin\theta + \sin^2\theta$

$\frac{dx}{d\theta} = -\sin\theta + \sin\theta\cdot -\sin\theta + \cos\theta\cos\theta \quad \frac{dy}{d\theta} = \cos\theta + 2\sin\theta\cos\theta$

$= \cos^2\theta - \sin^2\theta - \sin\theta$

$\frac{dx}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \quad \frac{dy}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

HORIZ: $(2, \frac{\pi}{2})\ (\frac{1}{2}, \frac{7\pi}{6})\ (\frac{1}{2}, \frac{11\pi}{6})$
$(0,2)\ (-\frac{\sqrt{3}}{4}, -\frac{1}{4})\ (\frac{\sqrt{3}}{4}, -\frac{1}{4})$

VERT: $(\frac{3}{2}, \frac{\pi}{6})\ (\frac{3}{2}, \frac{5\pi}{6})$
$(\frac{3\sqrt{3}}{4}, \frac{3}{4})\ (-\frac{3\sqrt{3}}{4}, \frac{3}{4})$

* $\frac{3\pi}{2}$ NOT POSSIBLE SINCE $\frac{dx}{d\theta}$ & $\frac{dy}{d\theta}$ BOTH $= 0$

17. $x = (2-3\sin\theta)\cos\theta \quad y = (2-3\sin\theta)\sin\theta \quad x(\pi) = -2 \quad y(\pi) = 0$

$x = 2\cos\theta - 3\sin\theta\cos\theta \quad y = 2\sin\theta - 3\sin^2\theta$

$\frac{dx}{d\theta} = -2\sin\theta - 3[\sin\theta\cdot -\sin\theta + \cos\theta\cdot\cos\theta] \quad \frac{dy}{d\theta} = 2\cos\theta - 6\sin\theta\cos\theta$

$= -2\sin\theta + 3\sin^2\theta - 3\cos^2\theta$

TANGENT LINE: $y = \frac{2}{3}(x+2)$

$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{2}{3}$

18. $2\cos(3\theta) = 0$

$3\theta = \frac{\pi}{2} \quad 3\theta = \frac{3\pi}{2}$

$\theta = \frac{\pi}{6} \quad \theta = \frac{\pi}{2}$

$\frac{1}{2}\int_{\pi/6}^{\pi/2} (2\cos(3\theta))^2\, d\theta = \frac{\pi}{3} \approx 1.047$

19. $\frac{1}{2}\int_0^{2\pi} (2+2\cos\theta)^2\, d\theta = 6\pi \approx 18.850$

20. $1+2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1+2\cos\theta)^2\, d\theta = \pi - \frac{3\sqrt{3}}{2} \approx 0.544$

21. $\frac{1}{2}\int_{4\pi/3}^{8\pi/3}(1+2\cos\theta)^2\,d\theta - \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1+2\cos\theta)^2\,d\theta = \pi + 3\sqrt{3} \approx 8.338$

22. $3\cos\theta = 2-\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{-\pi}{3}, \frac{\pi}{3}$

$\frac{1}{2}\int_{-\pi/3}^{\pi/3}(3\cos\theta)^2\,d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2-\cos\theta)^2\,d\theta = 3\sqrt{3} \approx 5.196$

23. a. $v(t) = \langle 2t+1, -3t^2\rangle$

$v(4) = \langle 9, -48\rangle$

b. $a(t) = \langle 2, -6t\rangle$

$a(4) = \langle 2, -24\rangle$

c. $\text{SPEED} = \sqrt{(2t+1)^2+(-3t^2)^2}$

$\text{SPEED}\Big|_{t=4} = \sqrt{9^2+(-48)^2} = \sqrt{2385} \approx 48.837$

d. $x'(t) = 2t+1$ $\quad x'(2) = 5$ $\quad \frac{dy}{dx}\Big|_{t=2} = \frac{-12}{5}$

$y'(t) = -3t^2$ $\quad y'(2) = -12$

$x(2) = 6$

$y(2) = -7$

TANGENT LINE: $y+7 = -\frac{12}{5}(x+6)$

e. $\int_0^4 \sqrt{(2t+1)^2+(-3t^2)^2}\,dt \approx 68.209$

24.

a. $x(t) = \int (t^3 - 4t)\,dt = \frac{1}{4}t^4 - 2t^2 + C$ $\qquad x(t) = \frac{1}{4}t^4 - 2t^2 + \frac{7}{4}$

$0 = \frac{1}{4} - 2 + C$

$\frac{7}{4} = C$

$y(t) = \int t\,dt = \frac{1}{2}t^2 + C$

$\frac{1}{2} = \frac{1}{2} + C$ $\qquad y(t) = \frac{1}{2}t^2$

$0 = C$

<u>Position</u>: $\langle \frac{1}{4}t^4 - 2t^2 + \frac{7}{4}, \frac{1}{2}t^2 \rangle$

b. $v(2) = \langle 0, 2 \rangle$

Based upon $v(2)$, the partical is not moving horizontally when $t=2$. The particle is traveling upwards along a vertical path at a rate of 2.

c. $\text{Speed} = \sqrt{(t^3 - 4t)^2 + (t)^2}$

$\text{Speed}\big|_{t=1} = \sqrt{9 + 1} = \sqrt{10} \approx 3.162$

d. $a(t) = \langle 3t^2 - 4, 1 \rangle$

e. $\int_1^3 \sqrt{(t^3 - 4t)^2 + (t)^2}\,dt \approx 9.778$

25. a. $a(t) = \left\langle \frac{2}{t^3}, -\frac{2}{t^3} \right\rangle$

$a(3) = \left\langle \frac{2}{27}, -\frac{2}{27} \right\rangle$

b. $x(t) = \int \left(1 - \frac{1}{t^2}\right) dt = t + \frac{1}{t} + C$ $\quad x(t) = t + \frac{1}{t}$

$2 = 1 + 1 + C$
$0 = C$

$y(t) = \int \left(2 + \frac{1}{t^2}\right) dt = 2t - \frac{1}{t} + C$ $\quad y(t) = 2t - \frac{1}{t} + 5$

$6 = 2 - 1 + C$
$5 = C$

Position: $\left\langle t + \frac{1}{t}, 2t - \frac{1}{t} + 5 \right\rangle$

@ $t=3$, $\left\langle \frac{10}{3}, \frac{32}{3} \right\rangle$

c. $\frac{dy}{dx} = \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{\frac{2t^2+1}{t^2}}{\frac{t^2-1}{t^2}} = \frac{2t^2+1}{t^2-1}$

$\frac{2t^2+1}{t^2-1} = 8 \rightarrow 2t^2+1 = 8t^2-8 \rightarrow 6t^2 = 9 \rightarrow t^2 = \frac{3}{2} \rightarrow t = \sqrt{\frac{3}{2}}$

d. $\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{2t^2+1}{t^2-1} = 2$.

The slope of the line is 2.