Calculus BC

Test #1

Use the appropriate method to find each of the following integrals.

1. $\int x \cos 2x \, dx$

$$\int x\cos(2x)dx$$

$u = x \qquad dv = \cos(2x)\,dx$

$du = dx \qquad v = \frac{1}{2}\sin(2x)$

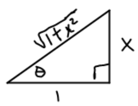
$$= \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)$$

$$= \boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

2. $\int \frac{\sqrt{1+x^2}}{x}\,dx$

$$\int \frac{\sqrt{1+x^2}}{x}dx$$

$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$



$$= \int \frac{\sec\theta \cdot \sec^2\theta}{\tan\theta}d\theta = \int \csc\theta \sec^2\theta\, d\theta$$

$u = \csc\theta \qquad dv = \sec^2\theta\, d\theta$

$du = -\csc\theta\cot\theta\, d\theta \qquad v = \tan\theta$

$$= \csc\theta\tan\theta + \int \csc\theta\cot\theta\tan\theta\, d\theta$$

$$= \sec\theta + \int \csc\theta\, d\theta$$

$$= \sec\theta - \ln|\csc\theta + \cot\theta| + C$$

$$\boxed{= \sqrt{1+x^2} - \ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C}$$

3. $\int \frac{x^4 + 2x + 7}{x^2 + 3x + 2}dx$

$$\int \frac{x^4+2x+7}{x^2+3x+2}dx$$

$$= \int\left(x^2 - 3x + 7 - \frac{13x+7}{x^2+3x+2}\right)dx$$

$$= \int(x^2-3x+7)dx - \int\frac{13x+7}{(x+2)(x+1)}dx$$

$$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 7x - 19\int\frac{1}{x+2}dx + 6\int\frac{1}{x+1}dx$$

$$\boxed{\begin{array}{c} = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 7x - 19\ln|x+2| + 6\ln|x+1| + C \\ \text{OR} \\ \frac{1}{3}x^3 - \frac{3}{2}x^2 + 7x + \ln\left|\frac{(x+1)^6}{(x+2)^{19}}\right| + C \end{array}}$$

$$\begin{array}{r} x^2 - 3x + 7 \\ x^2+3x+2 \overline{)\, x^4 + 2x + 7} \\ -(x^4 + 3x^3 + 2x^2) \\ \hline -3x^3 - 2x^2 + 2x + 7 \\ -(-3x^3 - 9x^2 - 6x) \\ \hline 7x^2 + 8x + 7 \\ -(7x^2 + 21x + 14) \\ \hline -13x - 7 \end{array}$$

$$\frac{13x+7}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$13x + 7 = Ax + A + Bx + 2B$

$= (A+B)x + (A+2B)$

$A + B = 13 \qquad A + 2B = 7$

$\boxed{A = 19} \qquad \boxed{B = -6}$

4. $\displaystyle\int \frac{2x+3}{x^2-9}\,dx$

$$\int \frac{2x+3}{x^2-9}\,dx = \int \frac{2x+3}{(x-3)(x+3)}\,dx$$

$$\frac{2x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$2x+3 = Ax+3A+Bx-3B$$

$$= (A+B)x + (3A-3B)$$

$$A+B=2 \qquad 3A-3B=3$$

$$\boxed{A=\tfrac{3}{2}} \qquad \boxed{B=\tfrac{1}{2}}$$

$$= \frac{3}{2}\int \frac{1}{x-3}\,dx + \frac{1}{2}\int \frac{1}{x+3}\,dx$$

$$\boxed{= \frac{3}{2}\ln|x-3| + \frac{1}{2}\ln|x+3| + C \quad \text{OR} \quad \ln\left|\sqrt{(x-3)^3(x+3)}\right| + C}$$

5. $\displaystyle\int x^2 e^{-x}\,dx$

$$\int x^2 e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx$$

$u = x^2 \quad dv = e^{-x}\,dx$

$du = 2x\,dx \quad v = -e^{-x}$

$$= -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right]$$

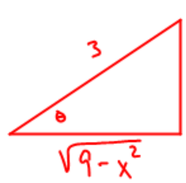
$u = x \quad dv = e^{-x}$

$du = dx \quad v = -e^{-x}$

$$= \boxed{-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C \quad \text{OR} \quad -\frac{1}{e^x}\left(x^2+2x+2\right) + C}$$

6. $\displaystyle\int x^3\sqrt{9-x^2}\,dx$

$$\int x^3\sqrt{9-x^2}\,dx = \int 27\sin^3\theta \cdot \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\,d\theta$$



$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$= 243\int \sin^3\theta \cdot \cos^2\theta\,d\theta$$

$$= 243\int \sin\theta \cdot (1-\cos^2\theta)\cdot\cos^2\theta\,d\theta$$

$$= 243\int \sin\theta\,(\cos^2\theta - \cos^4\theta)\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= -243\int (u^2-u^4)\,du$$

$$= -243\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$= -81\cos^3\theta + \frac{243}{5}\cos^5\theta + C$$

$$= -81\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + \frac{243}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 + C = \boxed{-3\sqrt{(9-x^2)}^3 + \frac{\sqrt{(9-x^2)}^5}{5} + C}$$

7. $\int e^{4x}\sin(2x)\,dx$

$$\int e^{4x}\sin(2x)\,dx = -\frac{1}{2}e^{4x}\cos(2x) + 2\int e^{4x}\cos(2x)$$

$u = e^{4x}$ $dv = \sin(2x)\,dx$

$du = 4e^{4x}dx$ $v = \frac{-1}{2}\cos(2x)$

$$= -\frac{1}{2}e^{4x}\cos(2x) + 2\left[\frac{1}{2}e^{4x}\sin(2x) - 2\int e^{4x}\sin(2x)\,dx\right.$$

$u = e^{4x}$ $dv = \cos(2x)\,dx$

$du = 4e^{4x}dx$ $v = \frac{1}{2}\sin(2x)\,dx$

$$\int e^{4x}\sin(2x)\,dx = -\frac{1}{2}e^{4x}\cos(2x) + e^{4x}\sin(2x) - 4\int e^{4x}\sin(2x)\,dx$$

$$5\int e^{4x}\sin(2x)\,dx = -\frac{1}{2}e^{4x}\cos(2x) + e^{4x}\sin(2x)$$

$$\int e^{4x}\sin(2x)\,dx = \boxed{-\frac{1}{10}e^{4x}\cos(2x) + \frac{1}{5}e^{4x}\sin(2x) + C}$$

8. $\int \frac{2x}{x^2+3x+2}\,dx$

$$\int \frac{2x}{x^2+3x+2}\,dx = \int \frac{2x}{(x+2)(x+1)}\,dx$$

$$= 4\int \frac{1}{x+2}\,dx - 2\int \frac{1}{x+1}\,dx$$

$$= \boxed{4\ln|x+2| - 2\ln|x+1| + C \quad \text{OR} \quad \ln\left|\frac{(x+2)^4}{(x+1)^2}\right| + C}$$

$$\frac{2x}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{(x+1)}$$

$$2x = Ax + A + Bx + 2B$$

$$= (A+B)x + (A+2B)$$

$A+B=2$ $A+2B=0$

$\boxed{A=4}$ $\boxed{B=-2}$

9. $\int \frac{(\ln 5x)^2}{x}\,dx$

$$\int \frac{(\ln 5x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}(\ln|5x|)^3 + C}$$

$u = \ln 5x$

$du = \frac{1}{5x}\cdot 5\,dx$

$= \frac{dx}{x}$

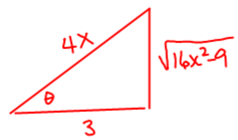
10. $$\int \frac{dx}{x^2\sqrt{16x^2-9}}$$

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \frac{3}{4}\int \frac{\cancel{\sec\theta\tan\theta}\, d\theta}{\frac{9}{16}\sec^{\cancel{2}}\theta \cdot 3\,\cancel{\tan\theta}} = \frac{4}{9}\int \cos\theta \, d\theta$$

$4x = 3\sec\theta$

$x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta$



$$= \frac{4}{9}\sin\theta + C$$

$$= \frac{4}{9}\left(\frac{\sqrt{16x^2-9}}{4x}\right) + C$$

$$= \boxed{\frac{\sqrt{16x^2-9}}{9x} + C}$$

11. $$\int_0^1 \ln(x+1)\, dx$$

$$\int_0^1 \ln(x+1)\, dx = \left[x\ln(x+1) - \int \frac{x}{x+1}\, dx\right]_0^1$$

$u = \ln(x+1)$ $\quad dv = dx$

$du = \frac{1}{x+1}dx$ $\quad v = x$

$u = x+1 \Rightarrow x = u-1$

$du = dx$

$\int \frac{u-1}{u}\, du$

$\int (1 - u^{-1})\, du$

$u - \ln|u|$

$(x+1) - \ln|x+1|$

$$= \left[x\ln(x+1) - \left(x+1-\ln|x+1|\right)\right]_0^1$$

$$= \left[-x-1+x\ln(x+1)+\ln|x+1|\right]_0^1$$

$$= \left[(-2+2\ln(2)) - (-1)\right]$$

$$= \boxed{2\ln(2) - 1}$$

12. $$\int_1^2 x^3\ln(x)\, dx$$

$$\int_1^2 x^3\ln(x)\, dx = \left[\frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\, dx\right]_1^2$$

$u = \ln x$ $\quad dv = x^3$

$du = \frac{1}{x}dx$ $\quad v = \frac{1}{4}x^4$

$$= \left[\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right]_1^2$$

$$= \left[(4\ln 2 - 1) - \left(-\frac{1}{16}\right)\right]$$

$$= \boxed{4\ln 2 - \frac{15}{16}}$$

13. Given the differential equation $\frac{dy}{dx} = x + 2$ and $y(0) = 3$. Find an approximation for $y(1)$ by using Euler's method with two equal steps. Sketch your solution.

$(x_0, y_0) \Rightarrow (0, 3) \quad h = 0.5$

$x_1 = 0.5 \qquad y_1 = 3 + (0.5)(0+2) = 4$

$x_2 = 1 \qquad y_2 = 4 + (0.5)(0.5+2) = 5.250$

$\boxed{y(1) \approx 5.250}$

14. The curve passing through (2, 0) satisfies the differential equation $\frac{dy}{dx} = 4x + y$. Find an approximation to $y(3)$ using Euler's Method with two equal steps.

$(x_0, y_0) \Rightarrow (2, 0) \quad h = 0.5$

$x_1 = 2.5 \qquad y_1 = 0 + (0.5)(4(2) + 0) = 4$

$x_2 = 3.0 \qquad y_2 = 4 + (0.5)(4(2.5) + 4) = 11$

$\boxed{y(3) \approx 11}$

15. Suppose you are in charge of stocking a fish pond with fish for which the rate of population growth is modeled by the differential equation $\frac{dP}{dt} = 8P - 0.02P^2$.

Given $P(0) = 50$.

(i) Find $\lim_{t \to \infty} P(t)$.

(ii) What is the range of the solution curve?

(iii) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(iv) Find $\frac{d^2P}{dt^2}$ and use it to find the values of P for which the solution curve is concave up and concave down. Justify your answer.

(v) Does the solution curve have an inflection point? Justify your answer.

(vi) Use the information you found to sketch the graph of $P(t)$.

$$\frac{dp}{dt} = 8p\left(1 - \frac{P}{400}\right)$$

$p(0) = 50$

(i) $\lim_{t \to \infty} P(t) = 400$

(ii) $[50, 400)$

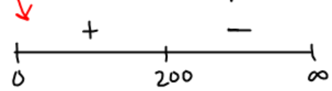
(iii) $[0, \infty)$

(iv) $\frac{d^2p}{dt^2} = 8 \cdot \frac{dp}{dt} - 0.04p \frac{dp}{dt}$

$= \frac{dp}{dt}(8 - 0.04p) = 0$

$\Downarrow$

$p = 200$

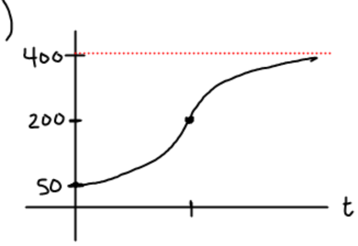


• CONCAVE UP ON $(0, 200)$ bc $\frac{d^2p}{dt^2} > 0$ &

CONCAVE DOWN ON $(200, \infty)$ bc $\frac{d^2p}{dt^2} < 0$.

(V) POINT OF INFLECTION WHEN $P = 200$ BC $P(t)$ CHANGES FROM CONCAVE UP TO CONCAVE DOWN.

(vi)



16. Suppose a virus is spreading at a local hospital holding 200 patients. The virus is spreading at a rate that is directly proportional to both the number of patients who have contracted the virus and the number of patients who have not contracted the virus. Let P be the number of patients who have contracted the virus, and let t be the time in minutes since the virus began to spread.

(a) Write a differential equation to model this rate of change.
(b) If $P(0) = 10$ and $P(15) = 50$, solve for P as a function of t.
(c) Use your solution to (b) to find the number of patients who have contracted the virus after 1 hour.
(d) Use your solution to (b) to find the time it takes for 175 patients to contract the virus.

$P(0)=10$
$P(15)=50$
$L = 200$

(a) $\frac{dP}{dt} = kP\left(1-\frac{P}{200}\right)$

(b) GENERAL SOLUTION: $P = \frac{L}{1+be^{-kt}}$

$10 = \frac{200}{1+be^{0}}$

$10+10b=200$

$b=19$

$50 = \frac{200}{1+19e^{-15k}}$

$50+950e^{-15k}=200$

$e^{-15k} = \frac{3}{19}$

$-15k = \ln\left(\frac{3}{19}\right)$

$k = 0.123$

$P(t) = \frac{200}{1+19e^{-0.123t}}$

(c) $P(60) = 198$ PATIENTS

(d) $175 = \frac{200}{1+19e^{-0.123t}} \Rightarrow t = 40$ MINUTES

Determine if each of the following improper integrals converge or diverge. If the integral converges, find the value it converges to.

17. $\int_2^{\infty} \frac{2}{t^2-t}\,dt$

$$\int_2^{\infty}\frac{2}{t^2-t}\,dt = \lim_{b\to\infty}\int_2^b \frac{2}{t(t-1)}\,dt$$

$$= \lim_{b\to\infty}\left[-2\int_2^b \frac{1}{t}\,dt + 2\int_2^b \frac{1}{t-1}\,dt\right]$$

$$= \lim_{b\to\infty}\Big[-2\ln|t| + 2\ln|t-1|\Big]_2^b$$

$$= \lim_{b\to\infty}\left[\ln\left|\left(\frac{t-1}{t}\right)^2\right|\right]_2^b$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{b-1}{b}\right)^2 - \ln\left(\frac{1}{4}\right)\right] = \boxed{0-\ln\left(\tfrac{1}{4}\right) \approx 1.386}$$

CONVERGES

$\frac{2}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$2 = At - A + Bt$
$= (A+B)t + (-A)$

$-A = 2$ $\quad$ $A+B=0$

$\boxed{A=-2}$ $\quad$ $\boxed{B=2}$

18. $$\int_0^\infty x\ln x\,dx = \lim_{a\to0^+}\int_a^1 x\ln x\,dx + \lim_{b\to\infty}\int_1^b x\ln x\,dx$$

$u=\ln x \qquad dv = x\,dx$

$du=\frac{1}{x}dx \qquad v=\frac{1}{2}x^2$

$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$

$= \frac{1}{4}x^2(2\ln x - 1)$

$$= \lim_{a\to0^+}\left[\frac{1}{4}x^2(2\ln x-1)\right]_a^1 + \lim_{b\to\infty}\left[\frac{1}{4}x^2(2\ln x-1)\right]_1^b$$

$$= \lim_{a\to0^+}\left[\left(-\frac{1}{4}\right)-\left(\frac{1}{4}a^2(2\ln a-1)\right)\right] + \lim_{b\to\infty}\left[\frac{1}{4}b^2(2\ln b-1)+\frac{1}{4}\right]$$

$$= \lim_{a\to0^+}\left[-\frac{1}{4}-\frac{1}{2}a^2\ln a+\frac{1}{4}a^2\right] + \lim_{b\to\infty}\left[\frac{1}{4}b^2(2\ln b-1)+\frac{1}{4}\right]$$

$$= \left[-\frac{1}{4}-0+0\right]+\left[\infty+\frac{1}{4}\right]$$

$$\boxed{\int_0^\infty x\ln x\,dx \text{ DIVERGES}}$$

CONSIDER

① $\lim_{a\to0^+} a^2\ln a$

$\lim_{a\to0^+}\frac{\ln a}{a^{-2}}$

L'Hops $\lim_{a\to0^+}\frac{\frac{1}{a}}{\frac{-2}{a^3}} = \frac{-a^2}{2} = 0$

② $\lim_{b\to\infty} b^2(2\ln b-1) = \infty$

$\infty(\infty)$

19. $$\int_0^3 \frac{1}{x\sqrt{x}}\,dx$$

$$\int_0^3\frac{1}{x\sqrt{x}}\,dx = \lim_{a\to0^+}\int_a^3 x^{-\frac{3}{2}}\,dx$$

$$= \lim_{a\to0^+}\left[-2x^{-\frac{1}{2}}\right]_a^3 = \lim_{a\to0^+}\left[\frac{-2}{\sqrt{3}}+\frac{2}{\sqrt{a}}\right] = \boxed{\frac{-2}{\sqrt{3}}+\infty \text{ DIVERGES}}$$

20. $$\int_0^1 \frac{1}{x(\ln x)^2}\,dx$$

$x\neq0$

$x\neq1$

$$\int_0^1\frac{1}{x(\ln x)^2}dx = \int_0^{1/2}\frac{1}{x(\ln x)^2}dx + \int_{1/2}^1\frac{1}{x(\ln x)^2}dx$$

$$= \lim_{a\to0^+}\int_a^{1/2}\frac{1}{x(\ln x)^2}dx + \lim_{b\to1^-}\int_{1/2}^b\frac{1}{x(\ln x)^2}dx$$

$$= \lim_{a\to0^+}\left[-\frac{1}{\ln x}\right]_a^{1/2} + \lim_{b\to1^-}\left[-\frac{1}{\ln x}\right]_{1/2}^b$$

$$= \lim_{a\to0^+}\left[-\frac{1}{\ln(\frac{1}{2})}+\frac{1}{\ln(a)}\right] + \lim_{b\to1^-}\left[-\frac{1}{\ln(b)}+\frac{1}{\ln(\frac{1}{2})}\right]$$

$$= \boxed{-\frac{1}{\ln(\frac{1}{2})}+0+\infty+\frac{1}{\ln(\frac{1}{2})} \text{ DIVERGES}}$$

CONSIDER:

$\int\frac{1}{x(\ln x)^2}dx$

$u=\ln x$

$du=\frac{1}{x}dx$

$\int u^{-2}\,du$

$-\frac{1}{u}+C$

21. $\int_{-\infty}^{0} \frac{1}{2x-1}\,dx$

$$\int_{-\infty}^{0} \frac{1}{2x-1}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{2x-1}\,dx$$

$$= \lim_{a\to-\infty} \left[\tfrac{1}{2}\ln|2x-1|\right]_a^0$$

$$= \boxed{\lim_{a\to-\infty} \left[0 - \tfrac{1}{2}\ln|2a-1|\right] = 0-\infty \quad \text{DIVERGES}}$$

22. $\int_{0}^{1} \frac{\ln x}{\sqrt{x}}\,dx$

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to0^+}\int_a^1 x^{-\frac{1}{2}}\ln x\,dx$$

$u = \ln x \qquad dv = x^{-\frac{1}{2}}dx$

$du = \frac{1}{x}dx \qquad v = 2x^{\frac{1}{2}}$

$$= \lim_{a\to0^+}\left[2x^{\frac{1}{2}}\ln x - 2\int x^{-\frac{1}{2}}dx\right]_a^1$$

$$= \lim_{a\to0^+}\left[2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}}\right]_a^1$$

$$= \lim_{a\to0^+}\left[2x^{\frac{1}{2}}(\ln x-2)\right]_a^1 = \lim_{a\to0^+}\left[-4-\left(2a^{\frac{1}{2}}(\ln a-2)\right)\right]$$

$$= \boxed{-4 \quad \text{CONVERGES}}$$

CONSIDER:

$$\lim_{a\to0^+} 2a^{\frac{1}{2}}(\ln a-2)$$

$$= \lim_{a\to0^+}\frac{(\ln a-2)}{2a^{-\frac{1}{2}}}$$

$$= \lim_{a\to0^+}\frac{\frac{1}{a}}{-a^{-\frac{3}{2}}}$$

$$= \lim_{a\to0^+} -a^{\frac{1}{2}} = 0$$