Applications of Derivatives Review :

Related Rates :

1. $A = \pi r^2$ $\boxed{B}$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \Rightarrow \left.\frac{dA}{dt}\right|_{r=5} = 2\pi(5)\cdot\frac{1}{\pi} = 10$$

2. $V = x^3$ $\qquad SA = 6x^2$ $\boxed{D}$

$$\frac{dV}{dt} = 3x^2\cdot\frac{dx}{dt} \qquad \frac{d(SA)}{dt} = 12x\cdot\frac{dx}{dt}$$

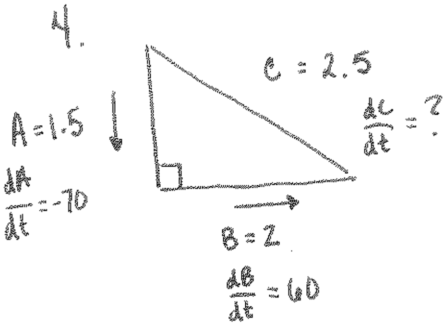
$$12 = 3(20)^2\cdot\frac{dx}{dt} \qquad \left.\frac{d(SA)}{dt}\right|_{x=20} = 12(20)\cdot\frac{1}{100} =$$

$$\frac{1}{100} = \frac{dx}{dt}$$

3. $\tan\theta = \frac{1}{300}y$ $\boxed{C}$

$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{300}\cdot\frac{dy}{dt} \Rightarrow \sec^2(\pi/6)\cdot\frac{d\theta}{dt} = \frac{1}{300}\cdot 80$$

$$\frac{d\theta}{dt} = \frac{1}{5}$$

4. 

$\boxed{C}$

$$A^2 + B^2 = C^2$$

$$A\cdot\frac{dA}{dt} + B\cdot\frac{dB}{dt} = C\cdot\frac{dC}{dt}$$

$$1.5(-70) + 2(60) = 2.5\cdot\frac{dc}{dt}$$

$$\frac{dc}{dt} = 6$$

5. $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$ $\boxed{A}$

$$\frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt} \Rightarrow 1 = 8\pi r$$
$$r = \frac{1}{8\pi}$$

6. $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2 \cdot \frac{dh}{dt} + h \cdot 2r \cdot \frac{dr}{dt}\right]$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2 \cdot \frac{dh}{dt} + 2rh \cdot \frac{dr}{dt}\right]$$ $\boxed{C}$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[4 \cdot r^2 - 4rh\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi r\left[r - h\right]$$ Positive when $r > h$

7. (a) $y = \sqrt{x}$

$$\frac{dy}{dt} = \frac{1}{2}x^{-1/2} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{y=2} = \frac{1}{8} \cdot 4 = \frac{1}{2}$$

(b) $D = \sqrt{(x-0)^2 + (y-0)^2}$

$$D = \sqrt{x^2 + y^2} = \left(x^2 + (\sqrt{x})^2\right)^{1/2} = \left(x^2 + x\right)^{1/2}$$

$$\frac{dD}{dt} = \frac{1}{2}\left(x^2 + x\right)^{-1/2}\left(2x \cdot \frac{dx}{dt} + \frac{dx}{dt}\right)$$

$$\left.\frac{dD}{dt}\right|_{y=2} = \frac{1}{2\sqrt{20}}(36) = \frac{9}{\sqrt{5}}$$

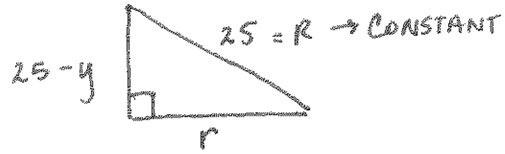
(c) $\tan\theta = \frac{y}{x} = \frac{\sqrt{x}}{x} = x^{-1/2}$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{1}{2}x^{-3/2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{-1}{2x^{3/2}} \cdot \frac{dx}{dt}$$

$$\left.\frac{d\theta}{dt}\right|_{y=2} = \left(\frac{2}{\sqrt{5}}\right)^2 \cdot \frac{-1}{2(4)^{3/2}} \cdot 4 = -\frac{1}{5}$$

8. (a) $V = \frac{\pi}{3} y^2 (3R - y)$



$V = \frac{\pi}{3} y^2 (75 - y)$

$V = \frac{\pi}{3}(75y^2 - y^3)$

$\frac{dV}{dt} = \frac{\pi}{3}\left(150y \cdot \frac{dy}{dt} - 3y^2 \cdot \frac{dy}{dt}\right)$

$-12 = \frac{\pi}{3}\left(2700 \frac{dy}{dt} - 972 \cdot \frac{dy}{dt}\right) \Rightarrow \frac{dy}{dt} = \frac{1}{48\pi}$ ft/min

(b) $(25-y)^2 + r^2 = 25^2$

$r^2 = 625 - (25-y)^2$

$r = \sqrt{625 - (25-y)^2} = \sqrt{50y - y^2}$

(c) $\frac{dr}{dt} = \frac{1}{2}(50y - y^2)^{-1/2}\left(50 \cdot \frac{dy}{dt} - 2y \cdot \frac{dy}{dt}\right)$

$\left.\frac{dr}{dt}\right|_{y=18} = \frac{1}{2\sqrt{576}}\left(\frac{25}{24\pi} - \frac{3}{4\pi}\right) = \frac{1}{48}\left(\frac{7}{24\pi}\right) = \frac{7}{1152\pi}$ ft/min

9. (a) $x^2 + y^2 = 400$ $\qquad x^2 + 12^2 = 400$

$x = 16$

$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$

$x \frac{dx}{dt} = -y \cdot \frac{dy}{dt} \Rightarrow 16\left(\frac{dx}{dt}\right) = -12(-2)$

$\left.\frac{dx}{dt}\right|_{y=12} = \frac{3}{2}$ ft/sec

(b) $A = \frac{1}{2} x \cdot y$

$\frac{dA}{dt} = \frac{1}{2}\left[x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right] \Rightarrow \left.\frac{dA}{dt}\right|_{y=12} = \frac{1}{2}\left[16(-2) + 12\left(\frac{3}{2}\right)\right] = -7$ ft²/sec

(c) $\sin\theta = \frac{1}{20} y$

$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{20} \cdot \frac{dy}{dt} \Rightarrow \left.\frac{d\theta}{dt}\right|_{y=12} = \sec\theta \cdot \frac{1}{20} \cdot (-2)$

$= \frac{20}{16} \cdot \frac{1}{20} \cdot (-2) = -1/8$ Rad/sec

10. (a) $4y \cdot \frac{dy}{dx} + 3\left[x \cdot \frac{dy}{dx} + y\right] = 0$

$$4y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$\frac{dy}{dx} = \frac{-3y}{4y+3x}$$

(b) $\frac{dy}{dx} = -\frac{3}{4} \Longrightarrow \frac{-3y}{4y+3x} = \frac{-3}{4}$

$$-12y = -3(4y+3x)$$

$$4y = 4y+3x$$

$$0 = x$$

$$2y^2+3(0)y = 1$$

$$y^2 = 1/2$$

$$y = \pm\sqrt{1/2}$$

$(0, \sqrt{1/2})$

$(0, -\sqrt{1/2})$

(c) $2y^2 + 3xy = 1$

$$4y \cdot \frac{dy}{dt} + 3\left[x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right] = 0$$

$$4(2)(-2) + 3\left[\frac{-7}{6}(-2) + 2\frac{dx}{dt}\right] = 0$$

$$-16 + 7 + 6\frac{dx}{dt} = 0 \Longrightarrow \left.\frac{dx}{dt}\right|_{t=3} = \frac{3}{2}$$

$$2(2)^2 + 3x(2) = 1$$

$$8 + 6x = 1$$

$$6x = -7$$

$$x = -7/6$$

Particle Motion:

1. $v(t) = \frac{1}{2}\sin t - 3$ $\boxed{C}$

$a(t) = \frac{1}{2}\cos t \Rightarrow a(\pi/3) = \frac{1}{2}\cos(\pi/3) = \frac{1}{4}$

2. $v(t) = x^{1/2} \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2}x^{-1/2}$

$= x^{-1/2} + \frac{1}{2}x^{-1/2}\ln x$

$= \frac{1}{2}x^{-1/2}\left[2 + \ln x\right]$

Note: For $x(t)$, $t > 0$

$\boxed{B}$

$v(t) = 0 \Rightarrow 2 + \ln x = 0 \Rightarrow x = e^{-2}$

3. $v(t) = 3\cos t + 2t$ $\boxed{D}$

$a(t) = -3\sin t + 2$ $v(.730) = 3.696$

$a(t) = 0 \Rightarrow \sin t = 2/3$

$t = .730$

4. $\frac{d}{dt}\left[x_1(t)\right] = 2\sin t\cos t$ SAME VELOCITY $\Downarrow$ $\boxed{C}$

$\frac{d}{dt}\left[x_2(t)\right] = -e^{-t}$ $2\sin t\cos t = -e^{-t}$ → LOOK AT GRAPHS OF BOTH SIDES

5. $a(t) = -3t^2 + 12t - 15$ $\boxed{A}$

$a'(t) = -6t + 12$ $a''(t) = -6$ ↓ always concave down

CV: $-6(t-2) = 0$

$t = 2$

t	$a(t)$
0	-15
2	-3
5	-30

6. $v'(t) = -t^3 \cdot -e^{-t} + e^{-t} \cdot -3t^2$ $\boxed{B}$

$= t^3e^{-t} - 3t^2e^{-t}$

$= t^2e^{-t}(t-3)$

CV: $t = 0, 3$

t	$v(t)$
0	0
3	-1.344

NOTE: WHEN $t > 3$, $v'(t) > 0$, therefore $t = 3$ has to be a minimum

7. $v(t) = 3t^2 - 24t + 21 = 3(t^2 - 8t + 7) = 3(t-7)(t-1)$

$a(t) = 6t - 24 = 6(t-4)$

$v(t) = 0 \Rightarrow t = 1, 7$

$a(t) = 0 \Rightarrow t = 4$

t	0 to 1	1 to 4	4 to 7	7 to ∞
$v(t)$	+	−	−	+
$a(t)$	−	−	+	+

SAME SIGN, ∴ INCREASING

$\boxed{D}$

8. (a) $v(t) = (t-2)^3 + (t-6)\cdot 3(t-2)^2 = (t-2)^2\left[t-2+3(t-6)\right]$

$= (t-2)^2(4t-20)$

$= 4(t-2)^2(t-5)$

$a(t) = 4\left[(t-2)^2 + (t-5)\cdot 2(t-2)\right]$

$= 4(t-2)\left[t-2+2(t-5)\right] = 4(t-2)(3t-12)$

$= 12(t-2)(t-4)$

(b) $a(t) = 0 \Rightarrow t = 2, 4$

$v(2) = 0 \quad v(4) = -16$

Particle is moving and acceleration is zero when $t=4$ since $v(4) \neq 0$ and $a(4) = 0$.

(c) $v(t) = 0 \Rightarrow t = 2, 5$

t	0 to 2	2 to 5	5 to ∞
$v(t)$	−	−	+

Particle moving right on $5 < t$ since $v(t) > 0$.

(d) $a(t) = 0 \Rightarrow t = 2, 4$

t	0 to 2	2 to 4	4 to ∞
$a(t)$	+	−	+

Velocity is decreasing on $2 < t < 4$ since $a(t) < 0$.

(e) Speed is increasing on $2 < t < 4$ and $5 < t < \infty$ since $v(t)$ and $a(t)$ have the same sign.

Rolle's and Mean Value Theorem :

1. $f'(x) = \cos(\pi x)\cdot\pi$

$f'(x) = 0 \Rightarrow \cos(\pi x) = 0$

$\pi x = \pi/2 \quad \pi x = 3\pi/2$

$x = 1/2 \quad x = 3/2$

$\boxed{D}$

2. $f'(c) = \dfrac{f(3) - f(0)}{3 - 0} = \dfrac{-16 - 2}{3} = -6$

$\boxed{C}$

$f'(x) = -3x^2 + 3$

$-3x^2 + 3 = -6$

$-3x^2 = -9$

$x^2 = 3 \Rightarrow x = \sqrt{3}$

NOTE: ON THE INTERVAL $[0,3]$, $\sqrt{3}$ is ONLY SOLUTION

3. BASED ON THE Graph, THERE ARE 4 POINTS WHERE A TANGENT IS PARALLEL TO THE SECANT DRAWN THROUGH $(a, f(a))$ & $(b, f(b))$

$\boxed{C}$

4. NOTE: MVT APPLIES BC DISCONTINUITY OF $x = -2$ is OUTSIDE $[-1, 2]$

$f'(c) = \dfrac{f(2) - f(-1)}{2 - -1} = \dfrac{\frac{1}{2} + 1}{3} = \dfrac{1}{2}$

$f'(x) = \dfrac{(x+2) - x}{(x+2)^2} = \dfrac{2}{(x+2)^2}$

$\dfrac{2}{(x+2)^2} = \dfrac{1}{2}$

$(x+2)^2 = 4$

$x + 2 = \pm 2$

$x = \cancel{-4}, 0$

Not on $[-1, 2]$

$\boxed{B}$

5. I. $A_{ROC} = \dfrac{f(6) - f(-3)}{6 - -3} = \dfrac{-1 - 0}{9} = \dfrac{-1}{9}$ $\boxed{\text{TRUE}}$

II. MVT DOES NOT APPLY SINCE $f'(2)$ IS UNDEFINED $\boxed{\text{FALSE}}$

III. $h'(x) = f'(\frac{1}{2}x)\cdot\frac{1}{2} \quad h'(6) = f'(3)\cdot\frac{1}{2} = -\frac{1}{2}$ $\boxed{\text{TRUE}}$

$\boxed{B}$

6. (a) Differentiable $\Rightarrow$ continuous

By IVT, since $v(0) > -1$ and $v(50) < -1$,
there must be some value on $[0,50]$
such that $v(t) = -1$.

(b) $v'(t) = a(t)$ By MVT, $a(t) = 0$ whenever $\frac{v(b)-v(a)}{b-a} = 0$ for
any subinterval $[a,b]$ on $[0,50]$.

This occurs when $v(b) = v(a)$, which happens one time.

First Derivative Test and EVT :

1. $f'(x) = (x-1)^3(-1) + (3-x)\cdot 3(x-1)^2$
$= (x-1)^2\left[(x-1)(-1) + 3(3-x)\right]$
$= (x-1)^2\left[-4x+10\right] = -2(x-1)^2(2x-5)$

CV: $x = 1$, $x = 5/2$

x	$f(x)$
1	0
$5/2$	$27/16$

$\boxed{D}$

NOTE: f(x) is a quartic function with both ends pointing down therefore maxi must occur at a critical value.

2. $f'(x) = 1 - \frac{4}{3}x^{-1/3} = 1 - \frac{4}{3\sqrt[3]{x}}$

$f'(x) = 0 \Rightarrow 1 = \frac{4}{3\sqrt[3]{x}}$

$3\sqrt[3]{x} = 4$
$\sqrt[3]{x} = 4/3 \Rightarrow x = \frac{64}{27}$

$f'(x)$: $-$ on $(-\infty, \frac{64}{27})$, $+$ on $(\frac{64}{27}, \infty)$

$\boxed{A}$

3. $f'(x) = x^2\cdot\frac{1}{x} + \ln x \cdot 2x$
$= x + 2x\ln x$
$= x(1 + 2\ln x)$

CV: $\ln x = -1/2$
$x = e^{-1/2}$

$-$ on $(0, e^{-1/2})$, $+$ on $(e^{-1/2}, \infty)$

$f(e^{-1/2}) = (e^{-1/2})^2\cdot \ln(e^{-1/2})$
$= e^{-1}\cdot -\frac{1}{2}$
$= -\frac{1}{2e}$

$\boxed{B}$

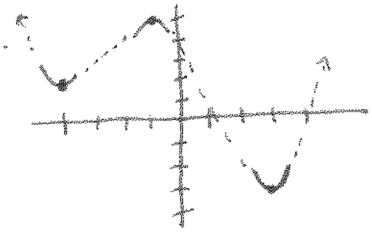
4. I. BOTH SIDES approach the same point $\boxed{\text{TRUE}}$

II. THERE IS A DEFINED POINT @ $x=a$ $\boxed{\text{TRUE}}$

III. GRAPH DOES NOT CHANGE FROM INC TO DEC $\boxed{\text{FALSE}}$

$\boxed{B}$

5.



$\boxed{B}$

6. $f'(x) = x^2 \cdot -e^{-x} + e^{-x} \cdot 2x$

$= -xe^{-x}(x-2)$

$-\infty$	$+$	0	$+$	2	$-$	∞

$\boxed{A}$

7. Using the graph of $f'(x)$, relative extrema will occur where the graph crosses x-axis

- max → above to below
- min → below to above

$\boxed{B}$

8. $f'(x) = \dfrac{3\sin(2x)}{x^2}$

$\boxed{C}$

cv: $\sin(2x) = 0$

$2x = \pi$ $\quad 2x = 2\pi$ $\quad 2x = 3\pi$ $\quad 2x = 4\pi$ $\quad 2x = 5\pi$ $\quad 2x = 6\pi$

$x = \pi/2$ $\quad x = \pi$ $\quad x = \frac{3\pi}{2}$ $\quad x = 2\pi$ $\quad x = \frac{5\pi}{2}$ $\quad x = 3\pi$

9. (A) is true by EVT

(B) is true by IVT

(D) is true by IVT

(C) could be false since MVT guarantees

$f'(c) = \dfrac{f(5) - f(-1)}{5 - -1} = \dfrac{-6}{6} = -1$

$\boxed{C}$

10. $\boxed{D}$

11. $h(t) = t^3 - 6t^2 + 20t$ $\boxed{A}$

$v(t) = 3t^2 - 12t + 20$

$v'(t) = 6t - 12$

$v'(t) = 0 \Rightarrow t = 2$

@ $t = 2$
$h(2) = 24$

$v'(t)$: $-$ on $(0, 2)$, $+$ on $(2, \infty)$

NOTE: QUESTION IS WORDED INCORRECTLY. SHOULD SAY, "... reaches it's minimum velocity."

12. $y' = 2x$ $\boxed{D}$

For a curve intersecting at right angles

$$f'(x) = \frac{-1}{2x} \Longrightarrow f(x) = -\tfrac{1}{2}\ln x$$

13. (a) Relative minimum is where $f'(x)$ changes from negative to positive (i.e. where the graph of $f'(x)$ crosses x-axis from below to above). $X = 1$

(b) Relative maximum is where $f'(x)$ changes from positive to negative (i.e. where the graph of $f'(x)$ crosses x-axis from above to below). $X = -2$

(c) Absolute maximum would be at a relative maximum, $x = -2$, of an endpoint.

$$f(-4) = f(-4) + \int_{-4}^{-4} f(x)\,dx$$

$$f(-2) = f(-4) + \int_{-4}^{-2} f(x)\,dx$$

$$f(7) = f(-4) + \int_{-4}^{7} f(x)\,dx$$

$f(7)$ would be the absolute maximum since the amount of area picked up from $\int_{-4}^{7} f(x)\,dx$ is the largest amount for the values of x considered.

Second Derivative Test :

1. $y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x = 12x(x-1)$

$y'' = 0 \Rightarrow x = 0, 1$

y'': $+$ on $(-\infty, 0)$, $-$ on $(0, 1)$, $+$ on $(1, \infty)$

$-\infty \quad 0 \quad 1 \quad \infty$

$\boxed{B}$

2. $y = ax^3 - 6x^2 + bx - 4$

$y' = 3ax^2 - 12x + b$

$y'' = 6ax - 12$

POI. @ $(2,-2) \Rightarrow y''(2) = 0$

$0 = 12a - 12$

$1 = a$

$-2 = 1(2)^3 - 6(2)^2 + b(2) - 4$

$-2 = 8 - 24 + 2b - 4$

$b = 9$

$\boxed{D}$

3. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$

$f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$

$= -\frac{1}{4}x^{-5/2}(x-3)$

$f''(x) = 0 \Rightarrow x = 3$

$f''(x)$: $+$ on $(0, 3)$, $-$ on $(3, \infty)$

$0 \quad 3 \quad \infty$

$\boxed{C}$

4. $y = 3x^5 - 40x^3 - 21x$

$y' = 15x^4 - 120x^2 - 21$

$y'' = 60x^3 - 240x = 60x(x^2-4)$

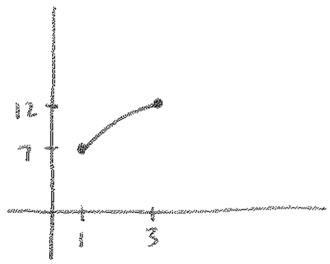
$y'' = 0 \Rightarrow x = 0, \pm 2$

y'': $-$ on $(-\infty, -2)$, $+$ on $(-2, 0)$, $-$ on $(0, 2)$, $+$ on $(2, \infty)$

$-\infty \quad -2 \quad 0 \quad 2 \quad \infty$

$\boxed{D}$

5. INCREASING, concave down function



AROC on $[1,3] = \frac{5}{2}$.

SINCE $f''(x) < 0$, AROC on $[3,5]$ must be less than $\frac{5}{2}$.

$\boxed{A}$

6. $f''(x) = 0 \Rightarrow x = a, 0, e$

$f''(x)$ changes sign $\Rightarrow x = a, 0$

$\boxed{D}$

7. $f'(x) = (x^3+2)e^x$

$f''(x) = (x^3+2)e^x + e^x(3x^2)$

$= e^x(x^3+3x^2+2)$

$f''(x) = 0 \Rightarrow x = -3.196$

$f''(x)$

− +

$-\infty$ -3.196 ∞

$\boxed{A}$

8. $f'(x) > 0 \Rightarrow$ INCREASING $f(x)$

$f''(x) > 0 \Rightarrow f'(x)$ INCREASING $\Rightarrow f(x)$ INCREASING AT AN INCREASING RATE

$\boxed{C}$

9. $f(x) = 3\sin\left(\frac{2}{3}x\right) - 4\cos\left(\frac{3}{4}x\right)$

$f'(x) = 2\cos\left(\frac{2}{3}x\right) + 3\sin\left(\frac{3}{4}x\right)$

$f''(x) = -\frac{4}{3}\sin\left(\frac{2}{3}x\right) + \frac{9}{4}\cos\left(\frac{3}{4}x\right)$

$f''(x) = 0 \Rightarrow x = 1.424, 5.761$

$f''(x)$

+ − +

0 1.424 5.761 7

max @ $x = 1.424$

$\boxed{B}$

10. $f(2)=0$

$f'(2)<0 \rightarrow$ slope of graph

$f''(2)>0 \rightarrow$ graph is concave up

$\boxed{C}$

11. $\frac{dy}{dx} > 0 \rightarrow$ increasing

$\frac{d^2y}{dx^2} > 0 \rightarrow$ concave up

$\boxed{B}$

12. (a) $f''(x)=0 \Rightarrow x=-1,1,4$

$f''(x)$ changes sign when $f'(x)$ changes from increasing to decreasing or decreasing to increasing. Therefore $x=-1,1,4$ are all points of inflection.

(b) Absolute minimum will occur at a relative minimum of an endpoint.

Relative minimum $\Rightarrow x=6$

$f(-3) = f(-3)$

$f(6) = f(-3) + \int_{-3}^{6} f(x)\,dx$

$f(7) = f(-3) + \int_{-3}^{7} f(x)\,dx$

Absolute max would occur at $x=6$ since as x increases from -2 to 6, the value of $f(x)$ decreases. $f(7) > f(6)$ since the value of $f(x)$ increases from $x=6$ to $x=7$.

(c) $h'(x) = x^2 \cdot f'(x) + f(x) \cdot 2x$

$h'(-2) = 4 \cdot f'(-2) + -4f(-2)$

$= 4(0) - 4\left(\frac{1}{2}\right) = -2$

$y - \frac{1}{2} = -2(x+2)$

13. (a) $y+1=2(x-1)$

(b) There is not enough info to determine whether there is a point of inflection at $x=1$. There is no way to determine whether $f''(x)$ changes sign.

(c) $g'(1)=1[2(-1)+2]=0$ $\qquad y=3$

(d) $g''(x)=x^2[2f'(x)+f''(x)]+2x[2f(x)+f'(x)]$

$= 2x^2f'(x)+x^2f''(x)+4xf(x)+2xf'(x)$

$= 4xf(x)+2x^2f'(x)+2xf'(x)+x^2f''(x)$

$g''(x)= 4xf(x)+2x(x+1)f'(x)+x^2f''(x)$

$g'(1)=0 \qquad g''(1)=4(-1)+2(2)(2)+0$
$= 4$

$g(x)$ has a local minimum at $x=1$ since $g'(1)=0$ and $g''(1)>0$.

Graphical Analysis:

1. $f'>0 \rightarrow$ increasing $a<x<c$ $\qquad \boxed{A}$

$f''<0 \rightarrow$ concave down $a<x<b$

$f''>0 \rightarrow$ concave up $b<x<c$

2. $f(x) = xe^{-x^2} = \dfrac{x}{e^{+x^2}}$

$\boxed{C}$

$f(-x) = \dfrac{-x}{e^{x^2}}$

3. $\lim\limits_{x\to\infty} \dfrac{-3x^2}{\sqrt{3x^4+1}} = \lim\limits_{x\to\infty} \dfrac{\frac{-3x^2}{x^2}}{\sqrt{\frac{3x^4+1}{x^4}}} = \lim\limits_{x\to\infty} \dfrac{-3}{\sqrt{3+\frac{1}{x^4}}} = \dfrac{-3}{\sqrt{3}} = -\sqrt{3}$

$\boxed{B}$

NOTE: function has same behavoir $x \to -\infty$ since exponents are even.

4. $\boxed{A}$

5. I. False since $\int_a^c f'(x)\,dx < 0$.

II. True since $f''(x)$ is slope of $f'(x)$ and is positive

III. True since $f'(x)$ changes from $-$ to $+$ at $x = c$

$\boxed{D}$

6. $\boxed{C}$

7. $\boxed{C}$

8. (a) f has a horizontal tangent when $f'(x)=0$. Horizontals tangent are at $x=0, 2, 5$.

(b) f is increasing when $f'(x)>0$. Increasing on $(0,2)$ and $(5,8)$.

(c) f is concave up when $f''(x)>0 \Rightarrow f'(x)$ is increasing. Concave up on $(0,1)$, and $(3,7)$.

(d) Relative max when $f'(x)$ changes from $+$ to $-$. Relative max at $x=2$.

(e) P.o.I. when $f''(x)$ changes sign $\Rightarrow f'(x)$ changes slope. PoI at $x=1, 3, 7$.

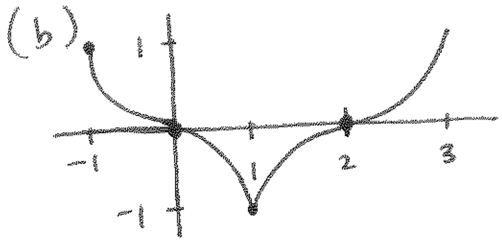
9. (a) relative extrema may occur when $f'(x)=0$ or DNE.

Relative extrema at $x=0, 1$.

at $x=0$, there is no relative extrema since $f'(x)$ does not change sign.

at $x=1$, there is a relative minimum since $f'(x)$ changes from negative to positive.

(b)



(c) $h'(x)=0 \Rightarrow f(x)=0 \Rightarrow x=0, 2$.

at $x=0$, there is a relative max since $h'(x)=f(x)$ changes from $+$ to $-$.

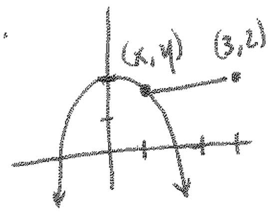
at $x=2$, there is a relative min since $h'(x)=f(x)$ changes from $-$ to $+$.

(d) $h''(x)=f'(x)$

$f'(x)$ changes sign at $x=1$, therefore there is a poi at $x=1$.

Optimization:

1.



$$dist = \sqrt{(x-3)^2+(y-2)^2}$$

$$= \sqrt{(x-3)^2 + x^4}$$

$$d = \sqrt{x^4+x^2-6x+9}$$

$$d' = \tfrac{1}{2}\left(x^4+x^2-6x+9\right)^{-1/2}\left(4x^3+2x-6\right)$$

$$= \frac{4x^3+2x-6}{2\sqrt{x^4+x^2-6x+9}}$$

$$cv: 4x^3+2x-6 = 0$$

$$x = 1$$

d': $-$ on $(-\infty, 1)$, $+$ on $(1, \infty)$

$\boxed{D}$

2.

$$A = 2x\cdot y$$

$$A = 2x\left(6-x^2\right)$$

$$A = 12x-2x^3$$

$$A' = 12-6x^2$$

$$= -6\left(x^2-2\right)$$

$$cv: \ x = \pm\sqrt{2}$$

$$A'' = -12x$$

$$A''(\sqrt{2}) < 0$$

$$A = 2(\sqrt{2})(4)$$

$$= 8\sqrt{2}$$

$\boxed{A}$

3.

$$P = x\cdot y$$

$$P = x\left(x^{-1/2}-x^{1/2}\right)$$

$$= x^{1/2}-x^{3/2}$$

$$P' = \tfrac{1}{2}x^{-1/2}-\tfrac{3}{2}x^{1/2}$$

$$= \tfrac{1}{2}x^{-1/2}\left[1-3x\right]$$

$$cv: \ x = 1/3$$

$$P'' = -\tfrac{1}{4}x^{-3/2}-\tfrac{3}{4}x^{-1/2}$$

$$= -\tfrac{1}{4}x^{-3/2}(1+3x)$$

$$P''(1/3) < 0$$

$$P\Big|_{x=1/3} = \frac{1}{3}\left(\frac{1}{\sqrt{1/3}}-\sqrt{1/3}\right)$$

$$= \frac{1}{3}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$$

$\boxed{C}$

4. $y' = \dfrac{\sin x(-\sin x) - (\cos x - m)\cos x}{(\sin x)^2}$

$\boxed{B}$

$y' = 0 \Rightarrow -\sin^2 x - \cos^2 x + m\cos x = 0$

$-1 + m\cos x = 0$ $\quad m = \sec(\pi/4) = \sqrt{2}$

$m = \sec x$

5. (a) $A = \frac{1}{2} b \cdot h$

$A = \frac{1}{2} k \cdot y = \frac{1}{2} k\sqrt{16 - x^2}$

$x = \frac{1}{2}k \Rightarrow A = \frac{1}{2}k\sqrt{16 - (\frac{1}{2}k)^2} = \frac{1}{2}k\sqrt{16 - \frac{1}{4}k^2} = \frac{1}{4}k\sqrt{64 - k^2}$

(b) $A = \frac{1}{4}k\sqrt{64 - k^2}$

$A' = \frac{1}{4}k \cdot \frac{1}{2}(64 - k^2)^{-1/2}(-2k) + (64 - k^2)^{1/2} \cdot \frac{1}{4}$

$= \frac{-1}{4}(64 - k^2)^{-1/2}\left[k^2 - 64 + k^2\right]$ $\quad$ CV: $k^2 - 32 = 0$

$k^2 = 32$

$= \dfrac{-(2k^2 - 64)}{4\sqrt{64 - k^2}} = \dfrac{-(k^2 - 32)}{2\sqrt{64 - k^2}}$ $\quad k = 4\sqrt{2}$

6.

(a) $f'(x) = -2x$ $\quad y - (3 - z^2) = -2z(x - z)$

$f'(z) = -2z$

$y = -2zx + z^2 + 3$

(b) $A = \frac{1}{2}\left(\frac{z^2+3}{2z}\right)(z^2+3) = \dfrac{(z^2+3)^2}{4z}$

CV: $z^2 - 1 = 0$

$0 = -2zx + z^2 + 3$ $\quad A' = \dfrac{4z \cdot 2(z^2+3) \cdot 2z - (z^2+3)^2 \cdot 4}{16z^2}$ $\quad z = 1$

$x = \dfrac{z^2+3}{2z}$ $\quad = \dfrac{4(z^2+3)\left[4z^2 - z^2 - 3\right]}{16z^2}$

$y = z^2 + 3$

$= \dfrac{4(z^2+3)(3z^2-3)}{16z^2} = \dfrac{3(z^2+3)(z^2-1)}{4z^2}$

Tangent Line Approximation :

1. Approximation @ $x=0$

$h(0)=2$

$h'(x) = \frac{1}{3}(8+h)^{-2/3} = \frac{1}{3(8+h)^{2/3}}$

$h'(0) = \frac{1}{3(4)} = \frac{1}{12}$

$y-2 = \frac{1}{12}(x)$

$y = \frac{1}{12}x + 2$

$\boxed{C}$

2. $y(0) = 1$

$y'(x) = \frac{1}{2}(1-\sin x)^{-1/2}(-\cos x) = \frac{-\cos x}{2\sqrt{1-\sin x}}$

$y'(0) = \frac{-1}{2}$

$L(x) = 1 + \frac{-1}{2}(x)$

$L(-0.1) = 1 - \frac{1}{2}(-0.1)$

$= 1 + 0.05$

$= 1.05$

$\boxed{D}$

4. $L(x) = \frac{5}{2} + \frac{1}{2}(x-2)$

$0 = \frac{5}{2} + \frac{1}{2}(x-2)$

$0 = \frac{1}{2}x + \frac{3}{2}$

$x = -3$

$\boxed{A}$

5. $y(4) = \frac{1}{2}$

$y'(x) = -\frac{1}{2}x^{-3/2}$

$y'(4) = \frac{-1}{16}$

$L(x) = \frac{1}{2} - \frac{1}{16}(x-4)$

$L(4.1) = \frac{1}{2} - \frac{1}{16}\left(\frac{1}{10}\right)$

$= \frac{1}{2} - \frac{1}{160}$

$= \frac{79}{160}$

$\boxed{B}$

$$f(x) = 2\left(e^{\sin x}+1\right)^{-1}$$

8. (a) $f(0) = 1$

$$f'(x) = -2\left(e^{\sin x}+1\right)^{-2} \cdot e^{\sin x} \cdot \cos x$$

$$= \frac{-2\cos x \, e^{\sin x}}{\left(e^{\sin x}+1\right)^2}$$

$$f'(0) = \frac{-2}{4} = -\frac{1}{2}$$

$$y - 1 = -\frac{1}{2}x$$

$$y = -\frac{1}{2}x + 1$$

(b) $f(0.1) \approx -\frac{1}{2}(0.1) + 1 = 0.95$

(c) INVERSE $\Longrightarrow$

$$x = \frac{2}{e^{\sin y}+1}$$

$$e^{\sin y} + 1 = \frac{2}{x}$$

$$e^{\sin y} = \frac{2}{x} - 1$$

$$\sin y = \ln\left|\frac{2}{x} - 1\right|$$

$$y = \arcsin\left(\ln\left|\frac{2}{x} - 1\right|\right)$$

Differentiation Review Answers:

1. $f(x) = (x^3-7)(x^2-4x)$

$$f'(x) = (x^3-7)(2x-4) + (x^2-4x)(3x^2)$$
$$= 5x^4 - 16x^3 - 14x + 28$$

2. $f(x) = \dfrac{3x^2-x}{\sqrt{x+1}}$

$$= \frac{(x+1)^{1/2}(6x-1) - (3x^2-x)\cdot\frac{1}{2}(x+1)^{-1/2}}{(x+1)^2}$$
$$= \frac{\frac{1}{2}(x+1)^{-1/2}\left[2(x+1)(6x-1) - (3x^2-x)\right]}{(x+1)^2} = \frac{9x^2+11x-2}{2(x+1)^{5/2}}$$

3. $y = (x^4-2x+5)^{1/2}$

$$y' = \frac{1}{2}(x^4-2x+5)^{-1/2}\cdot(4x^3-2) = \frac{2x^3-1}{\sqrt{x^4-2x+5}}$$

4. $h(x) = f(x^3)$

$$h'(x) = f'(x^3)\cdot 3x^2$$
$$h''(x) = f'(x^3)\cdot 6x + 3x^2\cdot f''(x^3)\cdot 3x^2$$
$$= 6x\cdot f'(x^3) + 9x^4\cdot f''(x^3) = 3x\left[2f'(x^3) + 3x^3\cdot f''(x^3)\right]$$

5. $y = x^2\sin x + 2x\cos x$

$$y' = x^2\cos x + 2x\sin x + 2x\cdot -\sin x + 2\cos x$$
$$= x^2\cos x + 2\cos x = \cos x\,(x^2+2)$$

6. $y = [\tan(x^3)]^2$

$y' = 2\tan(x^3)\cdot \sec^2(x^3)\cdot 3x^2$

$= 6x^2\tan(x^3)\cdot\sec^2(x^3)$

7. $y = e^{\cos x}$

$y' = e^{\cos x}\cdot -\sin x = -\sin x\, e^{\cos x}$

8. $y = 3^{\sqrt{x^2-x}}$

$y' = \ln 3 \cdot 3^{\sqrt{x^2-x}} \cdot \frac{1}{2}(x^2-x)^{-1/2}\cdot(2x-1)$

$= \dfrac{\ln 3\,(2x-1)\cdot 3^{\sqrt{x^2-x}}}{2\sqrt{x^2-x}}$

9. $y = \dfrac{\ln x}{x^2}$

$y' = \dfrac{x^2\cdot\frac{1}{x} - \ln x\cdot 2x}{x^4} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1-2\ln x}{x^3}$

10. $y = x^{\ln x}$

$\ln y = \ln x^{\ln x}$

$\frac{d}{dx}\left[\ln y = \ln x\cdot\ln x\right]$

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x}\cdot\ln x + \frac{1}{x}\cdot\ln x$

$\frac{dy}{dx} = \frac{2\ln x}{x}\cdot y = \frac{2\ln x}{x}\cdot x^{\ln x}$

11. $y^2 = x^2 - \cos(xy)$

$$2y \cdot \frac{dy}{dx} = 2x + \sin(xy) \cdot \left[x \cdot \frac{dy}{dx} + y \right]$$

$$2y \frac{dy}{dx} = 2x + x\sin(xy) \cdot \frac{dy}{dx} + y\sin(xy)$$

$$2y \frac{dy}{dx} - x\sin(xy) \cdot \frac{dy}{dx} = 2x + y\sin(xy)$$

$$\frac{dy}{dx} = \frac{2x + y\sin(xy)}{2y - x\sin(xy)}$$

12. $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3} x^{-2/3}$ $\quad f'(8) = \frac{1}{12}$ $\quad \boxed{A}$

13. $f'(x) = (x^3 - 2x + 5)(-2x^{-3} - x^{-2}) + (x^{-2} + x^{-1})(3x^2 - 2)$ $\quad \boxed{A}$

$$f'(1) = (4)(-3) + (2)(1) = -10$$

14. $$\frac{d}{dx}\left[\frac{g(x)}{x^2} \right] = \frac{x^2 \cdot g'(x) - g(x) \cdot 2x}{x^4}$$

$$\left.\frac{d}{dx}\right|_{x=2} = \frac{4 \cdot g'(2) - g(2) \cdot 4}{16} = \frac{-4-12}{16} = -1 \quad \boxed{B}$$

15. $$f(x) = \frac{x}{x - \frac{a}{x}} = \frac{x}{\frac{x^2-a}{x}} = \frac{x^2}{x^2-a}$$

$$f'(x) = \frac{(x^2-a) \cdot 2x - x^2(2x)}{(x^2-a)^2}$$

$$f'(x) = \frac{-2ax}{(x^2-a)^2}$$

$$f'(1) = \frac{-2a}{(1-a)^2} = \frac{1}{2} \quad \boxed{B}$$

$$-4a = (1-a)^2$$

$$-4a = 1 - 2a + a^2$$

$$0 = a^2 + 2a + 1$$

$$0 = (a+1)^2$$

$$a = -1$$

16. $y = x^2 \cdot f(x)$

$y' = x^2 \cdot f'(x) + f(x) \cdot 2x$

$y'' = x^2 \cdot f''(x) + f'(x) \cdot 2x + f(x) \cdot 2 + 2x \cdot f'(x)$

$= x^2 \cdot f''(x) + 4x f'(x) + 2f(x)$.

$\boxed{D}$

17. $f(x) = (x + x^{1/2})^{1/2}$

$f'(x) = \frac{1}{2}(x + x^{1/2})^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)$

$= \dfrac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}} = \dfrac{\frac{2\sqrt{x}+1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}} = \dfrac{2\sqrt{x} + 1}{4\sqrt{x^2 + x\sqrt{x}}}$

$\boxed{D}$

18. $f(x) = (x^2 - 3x)^{3/2}$

$f'(x) = \frac{3}{2}(x^2 - 3x)^{1/2}(2x - 3)$

$f'(4) = \frac{3}{2}(4)^{1/2}(5) = 15$

$\boxed{D}$

19. $f(x) = (3 - \sqrt{x})^{-1}$

$f'(x) = -1(3 - \sqrt{x})^{-2} \cdot \left(-\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x}(3 - \sqrt{x})^2}$

$f''(x) = \dfrac{2\sqrt{x}(3-\sqrt{x})^2 \cdot 0 - 1\left[2\sqrt{x} \cdot 2(3-\sqrt{x}) \cdot -\frac{1}{2}x^{-1/2} + (3-\sqrt{x})^2 \cdot x^{-1/2}\right]}{\left[2\sqrt{x}(3-\sqrt{x})^2\right]^2}$

$= \dfrac{2(3-\sqrt{x}) - \frac{(3-\sqrt{x})^2}{\sqrt{x}}}{4x(3-\sqrt{x})^4}$

$f''(4) = \dfrac{2(1) - \frac{1}{2}}{16(1)} = \dfrac{3}{32}$

$\boxed{A}$

20. $f(x) = \cos x$ $\quad f'(\pi/3) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$ $\quad \boxed{B}$

$f'(x) = -\sin x$

21. $f(x) = \sin(2x)$ $\quad \boxed{C}$

$f'(x) = \cos(2x) \cdot 2$

22. $\frac{d}{dx}\left[(\sec(\sqrt{x}))^2\right] = 2\sec(\sqrt{x}) \cdot \sec(\sqrt{x})\tan(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$ $\quad \boxed{C}$

$= \frac{\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

23. $f(\theta) = \cos\pi - \frac{1}{2}\sec\theta + \frac{1}{3}\cot\theta$

$f'(\theta) = -\frac{1}{2}\sec\theta\tan\theta + \frac{1}{3}(-\csc^2\theta)$ $\quad$ $\boxed{\text{ANSWER CHOICE NOT PRESENT}}$

$f'(\pi/6) = -\frac{1}{2}\sec(\pi/6)\tan(\pi/6) + \frac{1}{3}\left(-\csc^2(\pi/6)\right) = -5/3$

24. $y = \ln(\cos x)$ $\quad \boxed{A}$

$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$

25. $f(x) = \frac{1}{2}\ln x$ $\quad f'(e) = \frac{1}{2e}$ $\quad \boxed{C}$

$f'(x) = \frac{1}{2x}$

26. $y = e^{\sqrt{x^2+1}}$ $\quad \boxed{D}$

$y' = e^{\sqrt{x^2+1}} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

27. $y = x\sqrt{3+x^2}$

$y' = x \cdot \frac{1}{2}(3+x^2)^{-1/2} \cdot 2x + (3+x^2)^{1/2} = \frac{x^2}{\sqrt{3+x^2}} + \sqrt{3+x^2}$

$\boxed{C}$

$y'(1) = \frac{1}{2} + 2 = \frac{5}{2}$

$y - 2 = 5/2\,(x-1)$

$y = 5/2\,x - 1/2$

28. $f(x) = x^2 - x$

$f'(x) = 2x - 1$

$2x - 1 = 3$

$x = 2$

$f'(2) = 3$

$(2, 2)$

$y - 2 = 3(x-2)$

$y = 3x - 4$

$\boxed{C}$

29. $y = \tan x$

$y' = \sec^2 x$

$y'(\pi/6) = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$

Normal slope : $-\frac{3}{4}$

$y - \frac{1}{\sqrt{3}} = -\frac{3}{4}\left(x - \frac{\pi}{6}\right)$

$\boxed{C}$

30. $y = -\frac{2}{3}x + \frac{4}{3}$

$f'(-1) = \frac{3}{2}$

$\boxed{D}$

31. $f(x) = x^4 - x$

$f'(x) = 4x^3 - 1$

$y = 2x - k$

$4x^3 - 1 = -\frac{1}{2}$

$4x^3 = \frac{1}{2}$

$x^3 = 1/8$

$x = 1/2$

$\left(\frac{1}{2}, \frac{-7}{16}\right)$

$-\frac{7}{16} = 2\left(\frac{1}{2}\right) - k$

$\frac{23}{16} = k$

$\boxed{A}$

32. $3\left(x \cdot \frac{dy}{dx} + y\right) + 2x - 4y \cdot \frac{dy}{dx} = 0$

$3\left(\frac{dy}{dx} + 1\right) + 2 - 4\frac{dy}{dx} = 0$

$-\frac{dy}{dx} + 5 = 0$

$\frac{dy}{dx} = 5$

$\boxed{A}$

33. $x^2\cdot\frac{dy}{dx} + y\cdot 2x + 2\left(x\cdot 2y\cdot\frac{dy}{dx} + y^2\right) = 5$

$x^2\frac{dy}{dx} + 2xy + 4xy\frac{dy}{dx} + 2y^2 = 5$

$\boxed{B}$

$\frac{dy}{dx}\left(x^2+4xy\right) = 5 - 2xy - 2y^2$

$\frac{dy}{dx} = \frac{5-2xy-2y^2}{x^2+4xy}$

34. $x\frac{dy}{dx} + y + \sec^2(xy)\cdot\left(x\cdot\frac{dy}{dx}+y\right) = 0$

$x\frac{dy}{dx} + y + \sec^2(xy)\cdot x\frac{dy}{dx} + y\sec^2(xy) = 0$

$\frac{dy}{dx}\left(x + x\sec^2(xy)\right) = -y - y\sec^2(xy)$

$\frac{dy}{dx} = \frac{-y-y\sec^2(xy)}{x+x\sec^2(xy)} = \frac{-y\left(1+\sec^2(xy)\right)}{x\left(1+\sec^2(xy)\right)} = -\frac{y}{x}$ $\boxed{D}$

35. $1 + \cos(y)\cdot\frac{dy}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{1-\cos(y)} = (1-\cos y)^{-1}$

$\frac{d^2y}{dx^2} = -(1-\cos y)^{-2}\cdot \sin y\cdot\frac{dy}{dx}$

$= \frac{-1}{(1-\cos y)^2}\cdot\sin y\cdot\frac{1}{1-\cos(y)}$ $\boxed{C}$

$= \frac{-\sin y}{(1-\cos y)^3}$

36. $\frac{d}{dx}\left[g(x)\right] = \frac{1}{f'(f^{-1}(x))}$

$\boxed{C}$

$g'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(4)} = \frac{2}{3}$

37. $f(x) = 1 + \ln x$

$f(e) = 2$

$f^{-1}(2) = e$

$f'(x) = \dfrac{1}{x}$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(e)} = \frac{1}{\frac{1}{e}} = e$$

$\boxed{D}$

38. $\dfrac{d}{dx}\left[\arcsin(x^2)\right] = \dfrac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \dfrac{2x}{\sqrt{1-x^4}}$ $\boxed{D}$

39. $f(x) = \arctan(\sin x)$

$f'(x) = \dfrac{1}{1+(\sin x)^2} \cdot \cos x = \dfrac{\cos x}{1+\sin^2 x}$ $\boxed{A}$

$f'(\pi/3) = \dfrac{1/2}{1+(\sqrt{3}/2)^2} = \dfrac{1/2}{1+3/4} = \dfrac{2}{7}$

40. (a) $\lim_{x \to 0^-} f'(x) = 1$

$$f'(x) = \begin{cases} 1 & \text{for } x \le 0 \\ x+2 & \text{for } x > 0 \end{cases}$$

(b) $\lim_{x \to 0^+} f'(x) = 2$

(c) f is not differentiable at $x=0$ since the left and right hand derivatives are not equal. There would be a sharp point present at $x=0$.

(d) Differentiability $\Rightarrow$ Continuity

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$2 = ab^2$

$a = \frac{2}{b^2}$

$a = \frac{2}{16}$

$\boxed{a = 1/8}$

$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$

$1 = 2a(0+b)$

$1 = 2ab$

$1 = 2\left(\frac{2}{b^2}\right)(b)$

$1 = \frac{4}{b}$

$\boxed{b = 4}$

41. $h'(x) = x \cdot f(x) \cdot g'(x) + x \cdot g(x) \cdot f'(x) + f(x) \cdot g(x)$

$h'(1) = 1(-2)(1/2) + 1(3)(1) + (-2)(3)$

$h'(1) = -4$

42. $h'(1)$ $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$ $h'(1) = f'(2) \cdot -1 = 1$

$h'(2)$ $h(x) = x\,f(x^2) \Rightarrow h'(x) = x \cdot f'(x^2) \cdot 2x + f(x^2)$ $h'(2) = 8 \cdot f'(4) + f(4) = 13$

$h'(1)$ $h(x) = (x^9 + f(x))^{-2} \Rightarrow h'(x) = -2(x^9 + f(x))^{-3}(9x^8 + f'(x))$ $h'(1) = \frac{-2}{4^3}(9+1) = -5/16$

$h'(3)$ $h(x) = \frac{f(x)}{\sqrt{g(x)}} = f(x)[g(x)]^{-1/2} \Rightarrow h'(x) = f(x) \cdot \frac{-1}{2}[g(x)]^{-3/2} \cdot g'(x) + f'(x)[g(x)]^{-1/2}$ $h'(3) = 13/16$

$h'(2)$ $h(x) = [f(2x)]^2 \Rightarrow h'(x) = 2[f(2x)] \cdot f'(2x) \cdot 2$ $h'(2) = 20$

43. (a) $f(g(x)) = 2x$

$$f'(g(x)) \cdot g'(x) = 2$$

$$g'(x) = \frac{2}{f'(g(x))}$$

(b) $$g'(x) = \frac{2}{f'(g(x))}$$

$$f'(g(x)) = 1 + [f(g(x))]^2$$
$$= 1 + [2x]^2$$

$$g'(x) = \frac{2}{1 + [f(g(x))]^2}$$

$$g'(x) = \frac{2}{1 + 4x^2}$$

44. $h(x) = f(x) \cdot g(\tan x)$

$$h'(x) = f(x) \cdot g'(\tan x) \cdot \sec^2 x + g(\tan x) \cdot f'(x)$$

$$h'(\pi/4) = (-2)(\sqrt{2}) \cdot \left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{3}{2}\right) \cdot (2)$$

$$= -4\sqrt{2} + 3$$

45. Continuous:

$$\sin \pi = a\pi + b$$

$$0 = a\pi + b$$

$$0 = -\pi + b$$

$$\boxed{\pi = b}$$

Differentiable:

$$\cos \pi = a$$

$$\boxed{-1 = a}$$

46. $h(x) = e^{f(x)}$

$h'(x) = f'(x)e^{f(x)}$

$h''(x) = [f'(x)]^2 e^{f(x)} + f''(x)e^{f(x)}$

$h''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$

$1+x^2 = [f'(x)]^2 + f''(x)$

$1+x^2 = f''(x) + [f'(x)]^2$

$\boxed{f'(x) = x}$

47. (a) $3x^2 - \left[x\cdot\frac{dy}{dx} + y\right] + 2y\cdot\frac{dy}{dx} = 0$

$3x^2 - x\frac{dy}{dx} - y + 2y\cdot\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left[2y - x\right] = y - 3x^2$

$\frac{dy}{dx} = \frac{y-3x^2}{2y-x}$

(b) $(1)^3 - y + y^2 = 3$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = -1, 2$

$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-1-3}{-2-1} = \frac{4}{3}$ $\quad y+1 = \frac{4}{3}(x-1)$

$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2-3}{4-1} = \frac{-1}{3}$ $\quad y-2 = \frac{-1}{3}(x-1)$

(c) Horizontal Tangent $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow y - 3x^2 = 0$

$y = 3x^2$

$x^3 - x(3x^2) + (3x^2)^2 = 3$

$x^3 - 3x^3 + 9x^4 = 3$

$9x^4 - 2x^3 - 3 = 0$

$x = -.710, .822$

48. (a) $2x + 2y \cdot \frac{dy}{dx} - \left[x \cdot \frac{dy}{dx} + y\right] = 0$

$2x + 2y\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

(b) $4 + y^2 - 2y = 7$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y = -1, 3$

$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{-1-4}{-2-2} = \frac{5}{4}$ $\quad y + 1 = \frac{5}{4}(x-2)$

$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3-4}{6-2} = \frac{-1}{4}$ $\quad y - 3 = -\frac{1}{4}(x-2)$

(c) Vertical Tangent $\Rightarrow$ $\frac{dy}{dx}$ = undefined $\Rightarrow$ $2y - x = 0$

$2y = x$

$y = \frac{1}{2}x$

$x^2 + \left(\frac{1}{2}x\right)^2 - x\left(\frac{1}{2}x\right) = 7$

$x^2 + \frac{1}{4}x^2 - \frac{1}{2}x^2 = 7$

$\frac{3}{4}x^2 = 7$

$x^2 = \frac{28}{3}$

$x = \pm\sqrt{\frac{28}{3}} = \pm 3.055$

49. (a) $f(2) = -1$

$f^{-1}(-1) = 2$

$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$

$\frac{d}{dx}\left[f^{-1}(-1)\right] = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(2)} = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x+1)$

(b) $h(1) = f(g(1)) = 3$

$h'(x) = f'(g(x)) g'(x)$ $\quad h'(1) = f'(-1) \cdot 2 = -4$

(c) $h(1) = 3$

$h^{-1}(3) = 1$

$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))} = \frac{1}{h'(1)} = -\frac{1}{4}$

50. (a)

$$y = x^{\tan^{-1}x}$$

$$\ln y = \tan^{-1}x \cdot \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \tan^{-1}x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \left[\frac{\tan^{-1}x}{x} + \frac{\ln x}{1+x^2}\right] \cdot y = \left[\frac{\tan^{-1}x}{x} + \frac{\ln x}{1+x^2}\right] \cdot x^{\tan^{-1}x}$$

(b)

$$f(1) = 1^{\tan^{-1}(1)} = 1$$

$$f'(1) = \left[\frac{\tan^{-1}(1)}{1} + \frac{\ln(1)}{1+1^2}\right] \cdot 1^{\tan^{-1}(1)} = \frac{\pi}{4}$$

$$y - 1 = \frac{\pi}{4}(x-1)$$

51. (a)

$$A_{roc} = \frac{F(6) - F(1)}{6-1} = \frac{88 - -8}{5} = \frac{96}{5} \text{ °/month}$$

(b)

$$F'(4) \approx \frac{F(5) - F(3)}{5-3} = \frac{72 - 25}{2} = \frac{47}{2} \text{ °/month}$$

(c)

$$F'(t) = -52\cos\left(\frac{\pi t}{6} - 5\right) \cdot \frac{\pi}{6}$$

$$F'(4) = -52\cos\left(\frac{2\pi}{3} - 5\right) \cdot \frac{\pi}{6}$$

$$= 26.473 \text{ °/month}$$

L'Hospital's Rule:

1. $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2} \Rightarrow \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{e^x - 1}{2x} \Rightarrow \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{e^x}{2} \Rightarrow \dfrac{1}{2}$

$\boxed{B}$

2. $\lim\limits_{x\to 0} \dfrac{\arcsin x}{x} \Rightarrow \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

$\boxed{D}$

3. $\lim\limits_{\theta\to \pi} \dfrac{\sin\theta}{\theta - \pi} \Rightarrow \dfrac{0}{0}$

$\lim\limits_{\theta\to \pi} \dfrac{\cos\theta}{1} \Rightarrow -1$

$\boxed{A}$

4. $\lim\limits_{x\to 0^+} (\tan x)^x \Rightarrow 0^0$

$y = \lim\limits_{x\to 0^+} (\tan x)^x$

$\ln y = \lim\limits_{x\to 0^+} x \ln(\tan x)$

$= \lim\limits_{x\to 0^+} \dfrac{\ln(\tan x)}{x^{-1}}$ L'Hops

$= \lim\limits_{x\to 0^+} \dfrac{\frac{1}{\tan x} \cdot \sec^2 x}{-x^{-2}}$

$= \lim\limits_{x\to 0^+} \dfrac{-x^2}{\sin x \cos x}$

$= \lim\limits_{x\to 0^+} \dfrac{-2x}{-\sin^2 x + \cos^2 x} = 0$

$\ln y = 0$

$y = e^0 = 1$

$\boxed{D}$

5. $\lim_{x\to\infty} x^{1/x}$

$$y = \lim_{x\to\infty} x^{1/x} \Rightarrow \infty^0$$

$$\ln y = \lim_{x\to\infty} \frac{1}{x}\ln x$$

$$= \lim_{x\to\infty} \frac{\ln x}{x}$$ L'Hops

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$\ln y = 0$

$y = e^0 = 1$

$\boxed{C}$

6. $\lim_{x\to 0}\left(\frac{1}{\tan x} - \frac{1}{x}\right) \Rightarrow \infty - \infty$

$$\lim_{x\to 0} \frac{x - \tan x}{x\tan x}$$ L'Hops

$$\lim_{x\to 0} \frac{1-\sec^2 x}{x\sec^2 x + \tan x} \Rightarrow \frac{0}{0}$$ L'Hops

$$\lim_{x\to 0} \frac{-2\sec x \cdot \sec x \tan x}{2x\sec x\sec x\tan x + \sec^2 x} = \frac{0}{1} = 0$$

$\boxed{C}$

7. $\lim_{x\to 1^-}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right) \Rightarrow -\infty + \infty$

$$\lim_{x\to 1^-} \frac{2(x-1) - x(x^2-1)}{(x^2-1)(x-1)}$$

$$\lim_{x\to 1^-} \frac{-x^3+3x-2}{x^3-x^2-x+1}$$ L'Hops

$$\lim_{x\to 1^-} \frac{-3x^2+3}{3x^2-2x-1} \Rightarrow \frac{0}{0}$$ L'Hops

$$\lim_{x\to 1^-} \frac{-6x}{6x-2} = \frac{-6}{4} = \frac{-3}{2}$$

$\boxed{B}$

8. $\lim_{x\to\infty} x\left[\ln(x+3) - \ln x\right] \Rightarrow \infty - \infty$

$\lim_{x\to\infty} \dfrac{\ln(x+3) - \ln x}{x^{-1}}$ L'Hops

$\lim_{x\to\infty} \dfrac{\frac{1}{x+3} - \frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{-x^2}{x+3} + x \Rightarrow -\infty + \infty$

$\lim_{x\to\infty} \dfrac{-x^2 + x(x+3)}{x+3} = \lim_{x\to\infty} \dfrac{3x}{x+3} = 3$

9. $\lim_{x\to\infty} \left[x - \sqrt{x^2+x}\right] \Rightarrow \infty - \infty$

$\lim_{x\to\infty} x - \sqrt{x^2+x} \cdot \dfrac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}$

$\lim_{x\to\infty} \dfrac{x^2 - (x^2+x)}{x+\sqrt{x^2+x}}$

$\lim_{x\to\infty} \dfrac{-x}{x+\sqrt{x^2+x}}$ L'Hops

$\lim_{x\to\infty} \dfrac{-1}{1+\frac{1}{2}(x^2+x)^{-1/2}(2x+1)} = \lim_{x\to\infty} \dfrac{-1}{1+\frac{2x+1}{2\sqrt{x^2+x}}} = \dfrac{-1}{1+1} = \dfrac{-1}{2}$