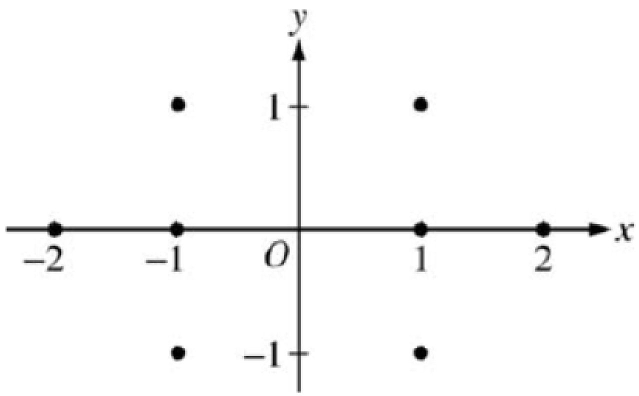
2006

5. Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.

(Note: Use the axes provided in the pink exam booklet.)

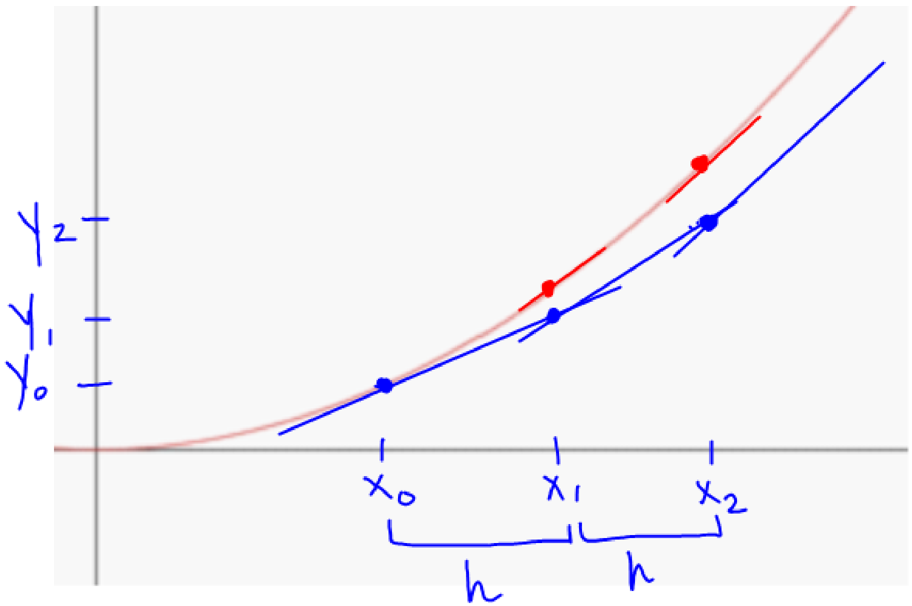


(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

Euler's Method for Solving Differentials

What happened when we can't separate the variables? We need to use Euler's method to get an approximation of a particular solution.

We are approximating the solution to the differential equation $y' = F(x, y)$. We know it passes through an initial point (x_0, y_0) and has a slope of $F(x_0, y_0)$.



$x_0 \qquad y_0$

$x_1 = x_0 + h \qquad y_1 = y_0 + h \cdot F(x_0, y_0)$

$x_2 = x_1 + h \qquad y_2 = y_1 + h \cdot F(x_1, y_1)$

$x_3 = x_2 + h \qquad y_3 = y_2 + h \cdot f(x_2, y_2)$

$\vdots \qquad \vdots$

Example 1: Use Euler's Method to approximate the particular solution of the differential equation $y' = x - y$ passing through the point (0, 1). Use a step of $h = 0.1$ and approximate the value of $\boxed{y(0.5)}$.

$x_0 = 0$ $\qquad$ $y_0 = 1$

$x_1 = 0.1$ $\qquad$ $y_1 = 1 + 0.1[0 - 1] = 0.9$

$x_2 = 0.2$ $\qquad$ $y_2 = 0.9 + 0.1[0.1 - 0.9] = 0.82$

$x_3 = 0.3$ $\qquad$ $y_3 = 0.82 + 0.1[0.2 - 0.82] = 0.758$

$x_4 = 0.4$ $\qquad$ $y_4 = 0.758 + 0.1[0.3 - 0.758] = 0.7122$

$x_5 = 0.5$ $\qquad$ $y_5 = 0.7122 + 0.1[0.4 - 0.7122] = 0.68098$

$$\boxed{y(0.5) \approx 0.681}$$

Example 2: The curve passing through $(2, 0)$ satisfies the differential equation $\frac{dy}{dx} = 4x + y$. Find an approximation to $y(3)$ using Euler's method with two equal steps. $\rightarrow h = \frac{1}{2}$

$x_0 = 2$ $\quad y_0 = 0$

$x_1 = \frac{5}{2}$ $\quad y_1 = 0 + \frac{1}{2}\left[4(2) + 0\right] = 4$

$x_2 = 3$ $\quad y_2 = 4 + \frac{1}{2}\left[4\left(\frac{5}{2}\right) + 4\right] = 11$

$$\boxed{y(3) \approx 11}$$

Example 3: Assume that f and its derivative f' have the values given in the table, Use Euler's Method with two equal steps to approximate the value of $f(4.4)$

x	4	4.2	4.4
$f'(x)$	-0.5	-0.3	-0.1
$f(x)$	2		

$h = 0.2 = \frac{1}{5}$

$x_0 = 4 \qquad y_0 = 2$

$x_1 = 4.2 \qquad y_1 = 2 + \frac{1}{5}\left[-\frac{1}{2}\right] = \frac{19}{10}$

$x_2 = 4.4 \qquad y_2 = \frac{19}{10} + \frac{1}{5}\left[-\frac{3}{10}\right] = \frac{92}{50} = \frac{46}{25}$

$$\boxed{f(4.4) \approx \frac{46}{25} = 1.840}$$

2016

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\dfrac{d^2y}{dx^2} = 2x - \dfrac{1}{2}\dfrac{dy}{dx} = 2x - \dfrac{1}{2}\left(x^2 - \dfrac{1}{2}y\right)$	$2 : \dfrac{d^2y}{dx^2}$ in terms of x and y
(b) $\left.\dfrac{dy}{dx}\right\rvert_{(x,\,y)=(-2,\,8)} = (-2)^2 - \dfrac{1}{2}\cdot 8 = 0$ $\left.\dfrac{d^2y}{dx^2}\right\rvert_{(x,\,y)=(-2,\,8)} = 2(-2) - \dfrac{1}{2}\left((-2)^2 - \dfrac{1}{2}\cdot 8\right) = -4 < 0$ Thus, the graph of f has a relative maximum at the point $(-2, 8)$.	2 : conclusion with justification
(c) $\lim\limits_{x\to -1}(g(x) - 2) = 0$ and $\lim\limits_{x\to -1} 3(x+1)^2 = 0$ Using L'Hospital's Rule, $\lim\limits_{x\to -1}\left(\dfrac{g(x)-2}{3(x+1)^2}\right) = \lim\limits_{x\to -1}\left(\dfrac{g'(x)}{6(x+1)}\right)$ $\lim\limits_{x\to -1} g'(x) = 0$ and $\lim\limits_{x\to -1} 6(x+1) = 0$ Using L'Hospital's Rule, $\lim\limits_{x\to -1}\left(\dfrac{g'(x)}{6(x+1)}\right) = \lim\limits_{x\to -1}\left(\dfrac{g''(x)}{6}\right) = \dfrac{-2}{6} = -\dfrac{1}{3}$	$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(d) $h\left(\dfrac{1}{2}\right) \approx h(0) + h'(0)\cdot\dfrac{1}{2} = 2 + (-1)\cdot\dfrac{1}{2} = \dfrac{3}{2}$ $h(1) \approx h\left(\dfrac{1}{2}\right) + h'\left(\dfrac{1}{2}\right)\cdot\dfrac{1}{2} \approx \dfrac{3}{2} + \left(-\dfrac{1}{2}\right)\cdot\dfrac{1}{2} = \dfrac{5}{4}$	$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

Logistics Differential Equations

We have previously studied **exponential growth** models that derive from the fact that the **rate of change of a variable *y* is proportional to the value of *y*.**

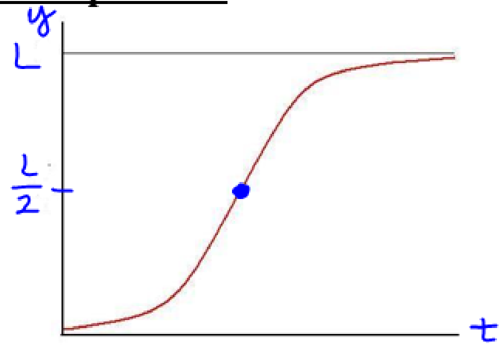
$$\frac{dy}{dt} = ky \quad \Longrightarrow \quad Solution: \quad y = Ce^{kt}$$

- Exponential growth describes something **increasing without bound.**

When talking about populations, there is often some upper boundary passed which growth cannot occur. This upper limit, ***L***, is called the **carrying capacity**, which is the maximum population *y(t)* that can be sustained/supported. We can model this with a **logistic differential equation.**

$$\boxed{\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)}$$

k - growth factor
y - population
L - carrying capacity



- $\lim_{t\to\infty} y(t) = L$
- MAX RATE OF GROWTH : $y = \frac{1}{2}L$

General Solution:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

$$\frac{dy}{dt} = k\boxed{y\left(\frac{L-y}{L}\right)}$$

$$\int \frac{L}{y(L-y)}\,dy = \int k\,dt$$

$$\frac{L}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$L = (-A+B)y + AL$$

$$-A+B=0 \qquad AL = L$$

$$\boxed{B=1} \qquad \boxed{A=1}$$

$$\int \frac{1}{y} + \frac{1}{L-y}\,dy$$

$$\ln|y| - \ln|L-y| = kt + C$$

$$\ln|L-y| - \ln|y| = -kt + C$$

$$e^{\ln\left|\frac{L-y}{y}\right|} = e^{-kt+C}$$

$$\frac{L-y}{y} = Ce^{-kt}$$

$$\frac{L}{y} - 1 = Ce^{-kt}$$

$$\frac{L}{y} = \frac{1+Ce^{-kt}}{1}$$

$$y = \frac{L}{1+Ce^{-kt}}$$

- All solutions of the logistic differential equation are of the general form: $$y = \frac{L}{1+be^{-kt}}$$

Example 1: Suppose the population of bears in a national park grows according to the logistic differential equation $\boxed{\frac{dp}{dt} = 5p - 0.002p^2}$, where p is the number of bears at time t in years.

$$\frac{dp}{dt} = 5p\left(1 - \frac{p}{2500}\right)$$

$5 \to k$, $2500 \to L$

(a) Find an equation, $p(t)$, to represent the population of the bears.

(b) Find $\lim_{t\to\infty} p(t)$.

(c) Consider the initial conditions: $p(0) = 100$, $p(0) = 1500$, *and* $p(0) = 3000$. Is the population increasing or decreasing for each of the initial conditions?

(d) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

growing the fastest $\to$ MAX of $\frac{dp}{dt}$

(e) Sketch a graph of $p(t)$ for each of the initial conditions in c.

(a) $\boxed{p(t) = \frac{2500}{1 + be^{-5t}}}$

(b) $\lim_{t\to\infty} \frac{2500}{1 + \frac{b}{e^{5t}}} = \boxed{2500}$

(c) $\left.\frac{dp}{dt}\right|_{p=100} = \boxed{480 \text{ Bears/yr}}$ INC

$\left.\frac{dp}{dt}\right|_{p=1500} = \boxed{3000 \text{ Bears/yr}}$ INC

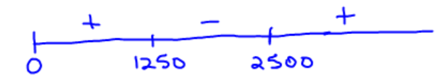
$\left.\frac{dp}{dt}\right|_{p=3000} = \boxed{-3000 \text{ Bears/yr}}$ DEC

(d) $\frac{d^2p}{dt^2} = (5 - 0.004p)\cdot\frac{dp}{dt} = (5 - 0.004p)(5p - 0.002p^2)$

$= 0.002\,p\,(2500 - 2p)(2500 - p)$

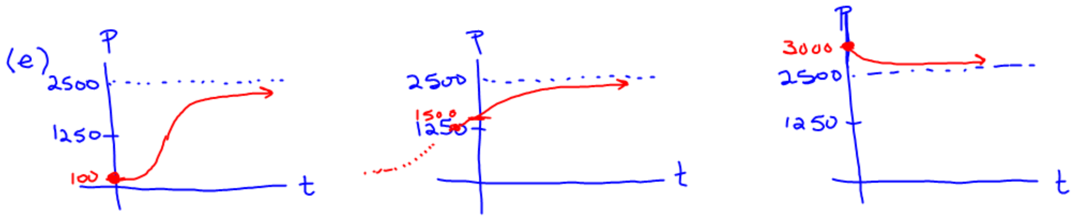
cv: $p = 0, 1250, 2500$

$\frac{d^2p}{dt^2}$



$\boxed{\text{* MAX growth at 1250 Bears}}$

(e)



Example 2: A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk.

$(0, 40)$
$(5, 104)$

(a) Write a model for the elk population in terms of t. → $E(t)$

(b) Estimate the elk population after 15 years.

(c) Find $\lim_{t\to\infty} E(t)$.

(a) $E(t) = \dfrac{L}{1+be^{-kt}} = \boxed{\dfrac{4000}{1+99e^{-0.194t}} \text{ OR } \dfrac{4000}{1+99e^{-0.195t}}}$

$40 = \dfrac{4000}{1+b}$

$1+b = 100$

$b = 99$

$104 = \dfrac{4000}{1+99e^{-5k}}$

$1+99e^{-5k} = \dfrac{500}{13}$

$99e^{-5k} = \dfrac{487}{13}$

$e^{-5k} = \dfrac{487}{1287}$

$-5k = \ln\left(\frac{487}{1287}\right)$

$k = \frac{-1}{5}\ln\left(\frac{487}{1287}\right)$

$k = 0.194$

(b) $E(15) = \boxed{625.675 \text{ or } 633.633}$

(c) $\lim_{t\to\infty} E(t) = \boxed{4000}$

Example 3:

The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is the time in years. What is $\lim_{t \to \infty} P(t)$?

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10000}\right)$$

$$\boxed{\lim_{t \to \infty} P(t) = 10000}$$

2004 Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t\to\infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential