Improper Integrals: Infinite Discontinuities

Review: Identify where the function *f(x)* has infinite discontinuities. $\rightarrow$ $x \neq \underbrace{-1, 0}_{\text{INFINITE}}, \overset{\text{REMOVABLE}}{2}$

$$f(x) = \frac{x^3-8}{x^3-x^2-2x} = \frac{\boxed{(x-2)}(x^2+2x+4)}{x\boxed{(x-2)}(x+1)}$$

$$\lim_{x \to 0^-} f(x) = \frac{4}{-(\text{Really Small})} = -\infty$$

$$\lim_{x \to 0^+} f(x) = \frac{4}{\text{Really Small}} = \infty$$

When functions have infinite discontinuities, in order to evaluate integrals that occur at the discontinuities, we need to use **<u>one sided limits.</u>**

Example 1: $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{a \to 0^+} \int_a^1 x^{-1/3}\,dx = \lim_{a \to 0^+} \left[\frac{3}{2}x^{2/3}\right]_a^1$

$$= \lim_{a \to 0^+} \left[\frac{3}{2} - \frac{3}{2}a^{2/3}\right]$$

$$\frac{3}{2} - 0$$

$\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$ CONVERGES TO $\frac{3}{2}$

Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 2: $\int_{-1}^{2} \frac{dx}{x^3}$ $= \lim_{b \to 0^-} \int_{-1}^{b} x^{-3}\,dx + \lim_{a \to 0^+} \int_{a}^{2} x^{-3}\,dx$

$$\int x^{-3}\,dx$$

$$\frac{x^{-3+1}}{-3+1}$$

$$= \lim_{b \to 0^-} \left[-\frac{1}{2x^2} \right]_{-1}^{b} + \lim_{a \to 0^+} \left[-\frac{1}{2x^2} \right]_{a}^{2}$$

$$= \lim_{b \to 0^-} \left[-\frac{1}{2b^2} + \frac{1}{2} \right] + \lim_{a \to 0^+} \left[-\frac{1}{8} + \frac{1}{2a^2} \right]$$

$$-\infty + \frac{1}{2} \qquad -\frac{1}{8} + \infty$$

$\int_{-1}^{2} \frac{dx}{x^3}$ DIVERGES

Example 3: $\int_0^1 3x^2 \ln x \, dx$

$u = \ln x \qquad dv = 3x^2\,dx$

$du = \frac{1}{x}dx \qquad v = x^3$

$$= \lim_{a \to 0^+} \int_a^1 3x^2 \ln x \, dx$$

$$= \lim_{a \to 0^+} \left[x^3 \ln x - \int x^2 \, dx \right]_a^1$$

$$= \lim_{a \to 0^+} \left[x^3 \ln x - \frac{1}{3}x^3 \right]_a^1$$

$$= \lim_{a \to 0^+} \left[\left(0 - \frac{1}{3}\right) - \left(a^3 \ln a - \frac{1}{3}a^3 \right) \right]$$

$0 \cdot -\infty$

L'Hops

$$\lim_{a \to 0^+} \frac{\ln a}{a^{-3}} = \lim_{a \to 0^+} \frac{\frac{1}{a}}{\frac{-3}{a^4}} = \lim_{a \to 0^+} -\frac{a^3}{3} = 0$$

$$= \left(0 - \frac{1}{3}\right) - (0 - 0)$$

$$= -1/3$$

$\int_0^1 3x^2 \ln x \, dx$ CONVERGES TO $-\frac{1}{3}$

Example 4: $\displaystyle\int_3^9 \frac{dx}{\sqrt{x^2-9}}$

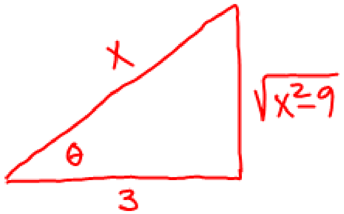
$$= \lim_{a\to 3^+} \int_a^9 \frac{dx}{\sqrt{x^2-9}}$$

$$= \lim_{a\to 3^+} \left[\ln\left| \frac{x+\sqrt{x^2-9}}{3} \right| \right]_a^9$$

$$= \lim_{a\to 3^+} \left[\ln\left| \frac{9+\sqrt{72}}{3} \right| - \ln\left| \frac{a+\sqrt{a^2-9}}{3} \right| \right]$$

$$= \ln\left|3+2\sqrt{2}\right| - \ln|1|$$

$\displaystyle\int_3^9 \frac{dx}{\sqrt{x^2-9}}$ CONVERGES TO $\ln\left|3+2\sqrt{2}\right|$



$\sec\theta = \frac{x}{3}$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\, d\theta$

$$\int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}}$$

$$\int \sec\theta\, d\theta$$

$$\ln|\sec\theta + \tan\theta|$$

$$\ln\left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right|$$

Example 5: $\int_1^5 \frac{3}{x^2-3x}\,dx = \int_1^5 \frac{3}{x(x-3)}\,dx = \lim_{b\to 3^-}\int_1^b \frac{3}{x(x-3)}\,dx + \lim_{a\to 3^+}\int_a^5 \frac{3}{x(x-3)}\,dx$

$x \neq 0, 3$

$$\frac{3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$$

$$3 = (A+B)x + (-3A)$$

$A+B=0$ $\quad -3A=3$

$\boxed{B=1}$ $\quad \boxed{A=-1}$

$$\int \frac{-1}{x} + \frac{1}{x-3}\,dx$$

$$-\ln|x| + \ln|x-3|$$

$$= \lim_{b\to 3^-}\left[\ln\left|\frac{x-3}{x}\right|\right]_1^b + \lim_{a\to 3^+}\left[\ln\left|\frac{x-3}{x}\right|\right]_a^5$$

$$= \lim_{b\to 3^-}\left[\ln\left|\frac{b-3}{b}\right| - \ln|2|\right] + \lim_{a\to 3^+}\left[\ln\left|\frac{2}{5}\right| - \ln\left|\frac{a-3}{a}\right|\right]$$

$$= \left[-\infty - \ln 2\right] + \left[\ln\left|\tfrac{2}{5}\right| + \infty\right]$$

$\int_1^5 \frac{3}{x^2-3x}\,dx$ DIVERGES

TWO METHODS

Example 6: $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

① U-SUB

$$\lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}(x+1)}$$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$du = \frac{1}{2u}dx$

$2u\,du = dx$

$\displaystyle\int \frac{2u\,du}{u(u^2+1)}$

$\displaystyle 2\int \frac{1}{u^2+1}du$

$2\arctan u$

$$= \lim_{a\to 0^+}\Big[2\arctan\sqrt{x}\Big]_a^1 + \lim_{b\to\infty}\Big[2\arctan\sqrt{x}\Big]_1^b$$

$$= \lim_{a\to 0^+}\Big[2\arctan 1 - 2\arctan\sqrt{a}\Big] + \lim_{b\to\infty}\Big[2\arctan\sqrt{b} - 2\arctan 1\Big]$$

$$= \Big[2\arctan 1 - 0\Big] + \Big[2\left(\pi/2\right) - 2\arctan 1\Big]$$

$$= \pi$$

$\displaystyle\int_0^{\infty}\frac{dx}{\sqrt{x}(x+1)}$ converges to π

② TRIG SUB

$$\lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}(x+1)}$$

$\tan\theta = \sqrt{x}$

$x = \tan^2\theta$

$dx = 2\tan\theta\sec^2\theta\,d\theta$

$\displaystyle\int \frac{2\tan\theta\sec^2\theta\,d\theta}{\tan\theta\left(\tan^2\theta+1\right)}$

$\displaystyle\int 2\,d\theta$

2θ

$2\arctan\sqrt{x}$ → From here, evaluation will be the same