From yesterday:

Example 5: Partial Fractions Disguised

$$\int \frac{\cos x}{3\sin^2 x + 7\sin x + 4}\,dx = \int \frac{1}{3u^2+7u+4}\,du = \int \frac{1}{(3u+4)(u+1)}\,du$$

$u = \sin x$

$du = \cos x\,dx$

$$\frac{1}{(3u+4)(u+1)} = \frac{A}{3u+4} + \frac{B}{u+1}$$

$$1 = (A+3B)u + (A+4B)$$

$$A+3B=0$$

$$-(A+4B=1)$$

$$-B=-1$$

$$\boxed{B=1}$$

$$\boxed{A=-3}$$

$$= \int \frac{-3}{3u+4} + \frac{1}{u+1}\,du$$

$$= -\ln|3u+4| + \ln|u+1| + C$$

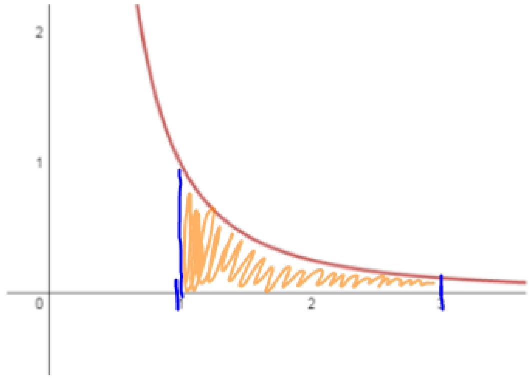
$$= \ln\left|\frac{u+1}{3u+4}\right| + C$$

$$= \ln\left|\frac{\sin x + 1}{3\sin x + 4}\right| + C$$

Previously, we have seen how to evaluate definite integrals using the Fundamental Theorem of Calculus.

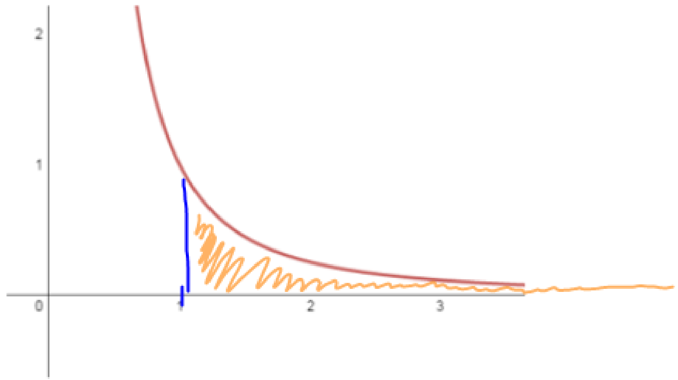
Example: $\int_1^3 \frac{1}{x^2}\,dx$

This represents the area under the curve on the interval [1,3].



The reason they are definite integrals is because **both limits of integrate are finite.** When one or both of the limits of integration are **infinite**, the integral is considered an **improper integral.** We can evaluate this type of integral using a **limit process.**

Example: $\int_1^\infty \frac{1}{x^2}\,dx$



Three types of Improper Integrals:

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

where c is any real number.

An improper integral can do two things:

- Converges – The limit of the improper integral exists (equals a number)
- Diverges – The limit of the improper integral does not exist (is infinite)

Note: For the third type, the integral on the left diverges if either of the right integrals diverges.

Example 1: $\displaystyle\int_1^\infty \frac{dx}{x}$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$\lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$\lim_{b\to\infty} \Big[\ln(b) - \ln(1)\Big]$$

$$\infty \quad - \quad 0$$

$$\int_1^\infty \frac{dx}{x} \quad \text{DIVERGES}$$

Example 2: $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$

$$\lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\,dx$$

$$\lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$

$$\lim_{b\to\infty} \Big[\arctan b - \arctan 0\Big]$$

$$\pi/2 \quad - \quad 0$$

$$\int_0^\infty \frac{1}{x^2+1}\,dx \quad \text{CONVERGES TO } \pi/2$$

Example 3: $$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx$$

$$\lim_{a\to-\infty} \int_a^0 \frac{x}{(1+x^2)^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{x}{(1+x^2)^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$$\frac{1}{2}\int \frac{1}{u^2}\,du$$

$$-\frac{1}{2u}$$

$$\lim_{a\to-\infty} \left[\frac{-1}{2(1+x^2)}\right]_a^0 + \lim_{b\to\infty} \left[\frac{-1}{2(1+x^2)}\right]_0^b$$

$$\lim_{a\to-\infty} \left[-\frac{1}{2} + \frac{1}{2(1+a^2)}\right] + \lim_{b\to\infty} \left[\frac{-1}{2(1+b^2)} + \frac{1}{2}\right]$$

$$-\frac{1}{2} + 0 \qquad 0 + \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx$$ CONVERGES TO 0

Example 4: $$\int_1^{\infty} (1-x)e^{-x}\,dx$$

$u = 1-x \qquad dv = e^{-x}dx$

$du = -dx \qquad v = -e^{-x}$

$$\lim_{b\to\infty} \left[-e^{-x}(1-x) - \int e^{-x}\,dx\right]_1^b$$

$$\lim_{b\to\infty} \left[-e^{-x}(1-x) + e^{-x}\right]_1^b$$

$$\lim_{b\to\infty} \left[-e^{-x}(1-x-1)\right]_1^b$$

$$\lim_{b\to\infty} \left[x e^{-x}\right]_1^b$$

$$\lim_{b\to\infty} \left[\frac{x}{e^x}\right]_1^b$$

$$\lim_{b\to\infty} \left[\frac{b}{e^b} - \frac{1}{e}\right]$$

$$0 - \frac{1}{e}$$

$$\int_1^{\infty} (1-x)e^{-x}\,dx$$ CONVERGES TO $-\frac{1}{e}$

Example 5: $\displaystyle\int_4^{\infty} \frac{1}{x^2+5x-14}\,dx$

$$\frac{1}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2}$$

$$1 = (A+B)x + (-2A+7B)$$

$$2(A+B=0)$$
$$-2A+7B = 1$$
$$9B = 1$$
$$\boxed{B = 1/9}$$
$$\boxed{A = -1/9}$$

$$\lim_{b\to\infty} \int_4^b \frac{-1/9}{x+7} + \frac{1/9}{x-2}\,dx$$

$$\lim_{b\to\infty} \left[-\tfrac{1}{9}\ln|x+7| + \tfrac{1}{9}\ln|x-2|\right]_4^b$$

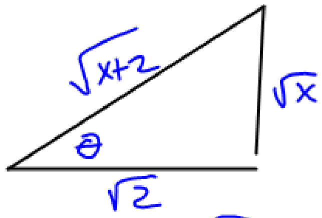
$$\lim_{b\to\infty} \left[\tfrac{1}{9}\ln\left|\frac{x-2}{x+7}\right|\right]_4^b$$

$$\lim_{b\to\infty} \left[\tfrac{1}{9}\ln\left|\frac{b-2}{b+7}\right| - \tfrac{1}{9}\ln\left|\frac{2}{11}\right|\right]$$

Since $\displaystyle\lim_{b\to\infty} \frac{b-2}{b+7} = 1$, $\displaystyle\lim_{b\to\infty} \ln\left|\frac{b-2}{b+7}\right| = \ln|1| = 0$

$\displaystyle\int_4^{\infty} \frac{1}{x^2+5x-14}\,dx$ CONVERGES to $-\tfrac{1}{9}\ln\left|\frac{2}{11}\right|$ OR $\tfrac{1}{9}\ln\left|\frac{11}{2}\right|$

Example 6: $\displaystyle\int_4^{\infty} \frac{1}{\sqrt{x^2+2x}}\,dx$



$$\tan\theta = \frac{\sqrt{x}}{\sqrt{2}}$$
$$\sqrt{x} = \sqrt{2}\tan\theta$$
$$x = 2(\tan\theta)^2$$
$$dx = 4(\tan\theta)\sec^2\theta\,d\theta$$

$$\lim_{b\to\infty} \int_4^b \frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\,dx$$

$$\lim_{b\to\infty} \int \frac{4\tan\theta\sec^2\theta}{\sqrt{2}\tan\theta\sqrt{2\tan^2\theta+2}}\,d\theta$$

$$\lim_{b\to\infty} 2\int \sec\theta\,d\theta$$

$$\lim_{b\to\infty} 2\ln|\sec\theta+\tan\theta|$$

$$\lim_{b\to\infty} \left[2\ln\left|\frac{\sqrt{x+2}}{\sqrt{2}} + \frac{\sqrt{x}}{\sqrt{2}}\right|\right]_4^b$$

$$\lim_{b\to\infty} \Bigg[\underbrace{2\ln\left|\frac{\sqrt{b+2}+\sqrt{b}}{\sqrt{2}}\right|}_{\infty} - 2\ln\left|\frac{\sqrt{6}+2}{\sqrt{2}}\right|\Bigg]$$

$\displaystyle\int_4^{\infty} \frac{1}{\sqrt{x^2+2x}}\,dx$ DIVERGES