Polar Equations and Calculus

To find the slope of the tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$.
To find the slope in polar form, use the parametric representations of the polar equation:

$$x = r\cos\theta = \boxed{f(\theta)\cos\theta} \qquad \text{and} \qquad y = r\sin\theta = \boxed{f(\theta)\sin\theta}$$

Slope in Polar Form

The slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

Example 1: Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

$$r' = 8\cos\theta$$

$$\frac{dy}{dx} = \frac{(8\cos\theta)(\sin\theta) + (3+8\sin\theta)\cos\theta}{(8\cos\theta)(\cos\theta) - (3+8\sin\theta)\sin\theta} = \frac{16\sin\theta\cos\theta + 3\cos\theta}{8\cos^2\theta - 8\sin^2\theta - 3\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{16\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{\sqrt{3}}{2}\right)}{8\left(\frac{\sqrt{3}}{2}\right)^2 - 8\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right)} = \frac{\frac{11\sqrt{3}}{2}}{\frac{5}{2}} = \frac{11\sqrt{3}}{5}$$

$$x\left(\frac{\pi}{6}\right) = \left(3+8\cdot\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{7\sqrt{3}}{2}$$

$$y\left(\frac{\pi}{6}\right) = \left(3+8\cdot\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{7}{2}$$

$$y - \frac{7}{2} = \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

Example 2: Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

$r' = 2\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad dy/d\theta \to \text{Horizontal}, \quad dx/d\theta \to \text{Vertical}$$

Horizontal:

$$\frac{dy}{d\theta} = (2\sin\theta)\sin\theta + (2 - 2\cos\theta)\cos\theta$$

$$= 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta$$

$$2\sin^2\theta + 2\cos\theta - 2\cos^2\theta = 0$$

$$2(1-\cos^2\theta) + 2\cos\theta - 2\cos^2\theta = 0$$

$$-4\cos^2\theta + 2\cos\theta + 2 = 0$$

$$-2(2\cos^2\theta - \cos\theta - 1) = 0$$

$$-2(\cos\theta - 1)(2\cos\theta + 1) = 0$$

$\cos\theta = 1$ $\qquad$ $\cos\theta = -1/2$

$\theta = 0, 2\pi$ $\qquad$ $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Vertical

$$\frac{dx}{d\theta} = (2\sin\theta)\cos\theta - (2 - 2\cos\theta)\sin\theta$$

$$= 4\sin\theta\cos\theta - 2\sin\theta$$

$$2\sin\theta(2\cos\theta - 1) = 0$$

$\sin\theta = 0$ $\qquad$ $\cos\theta = 1/2$

$\theta = 0, \pi, 2\pi$ $\qquad$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

At $\theta = 0, 2\pi$, $r(\theta)$ has cusps

Hor: $y\left(\frac{2\pi}{3}\right) = 2\left(1 - \cos\frac{2\pi}{3}\right)\sin\frac{2\pi}{3} = 3\sqrt{3}/2$

$y\left(\frac{4\pi}{3}\right) = 2\left(1 - \cos\frac{4\pi}{3}\right)\sin\frac{4\pi}{3} = -3\sqrt{3}/2$

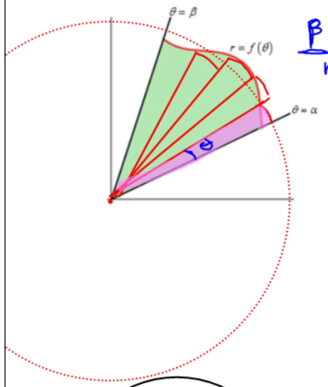
Vert: $x(\pi/3) = 2(1 - \cos\pi/3)\cos\pi/3 = 1/2$

$x(5\pi/3) = 2(1 - \cos 5\pi/3)\cos 5\pi/3 = 1/2$

$x(\pi) = 2(1 - \cos\pi)\cos\pi = -4$

Area with Polar Coordinates

The other topic concerning polar curves and calculus that is important is the idea of area. As we know from before, the integral of a curve or function represents the area between the curve and the x-axis. When working with polar equations we have to adjust this thinking.



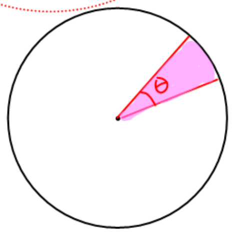
$$\frac{\beta-\alpha}{n} = \Delta\theta$$

$$\text{Area} \approx \frac{1}{2}(r_1)^2\cdot\Delta\theta + \frac{1}{2}(r_2)^2\cdot\Delta\theta + \dots + \frac{1}{2}(r_n)^2\cdot\Delta\theta$$

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2}(r_i)^2\cdot\Delta\theta$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}(r_i)^2\cdot\Delta\theta$$

$$\int_\alpha^\beta \frac{1}{2}(r)^2\, d\theta$$



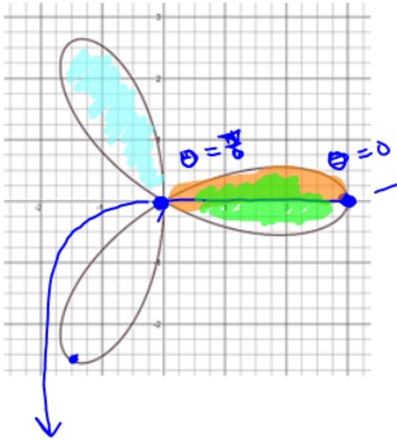
$$\text{Area} = \frac{\theta}{2\pi}\pi r^2$$
$$= \frac{1}{2} r^2\cdot\theta$$

Area in Polar Coordinates

If f is continuous on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ *and* $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

Example 3: Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.



$r = 3$ OR $r = -3$

$3\cos 3\theta = 3$ $\quad$ $3\cos 3\theta = -3$

$\cos 3\theta = 1$ $\quad$ $\cos 3\theta = -1$

$3\theta = 0, 2\pi, 4\pi$ $\quad$ $3\theta = \pi, 3\pi, -\pi$

$\theta = 0, \cancel{\frac{2\pi}{3}}, \cancel{\frac{4\pi}{3}}$ $\quad$ $\theta = \cancel{\frac{\pi}{3}}, \pi, \cancel{-\frac{\pi}{3}}$

$r = 0$

$3\cos 3\theta = 0$

$\cos 3\theta = 0$

$3\theta = -\frac{3\pi}{2}$ $\quad$ $3\theta = \frac{-\pi}{2}$ $\quad$ $3\theta = \frac{\pi}{2}$ $\quad$ $3\theta = \frac{3\pi}{2}$ $\quad$ $3\theta = \frac{5\pi}{2}$

$\theta = \frac{-\pi}{2}$ $\quad$ $\theta = \frac{-\pi}{6}$ $\quad$ $\theta = \frac{\pi}{6}$ $\quad$ $\theta = \frac{\pi}{2}$ $\quad$ $\theta = \frac{5\pi}{6}$

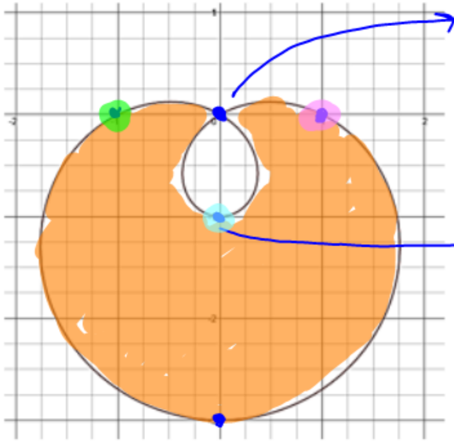
$$2\left[\frac{1}{2}\int_0^{\pi/6}[3\cos 3\theta]^2\,d\theta\right]$$

$$\frac{1}{2}\int_{-\pi/6}^{\pi/6}[3\cos 3\theta]^2\,d\theta$$

$$\frac{1}{2}\int_{\pi/2}^{5\pi/6}[3\cos 3\theta]^2\,d\theta$$

$$\frac{1}{6}\left[\frac{1}{2}\int_0^{2\pi}[3\cos 3\theta]^2\,d\theta\right]$$

Example 4: Find the area of the area of the region lying between the inner and outer loops of the curve $r = 1 - 2\sin\theta$.



$0 = 1 - 2\sin\theta$

$\sin\theta = 1/2$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$1 = 1 - 2\sin\theta$

$\sin\theta = 0$

$\theta = 0, \pi, 2\pi$

$-1 = 1 - 2\sin\theta$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

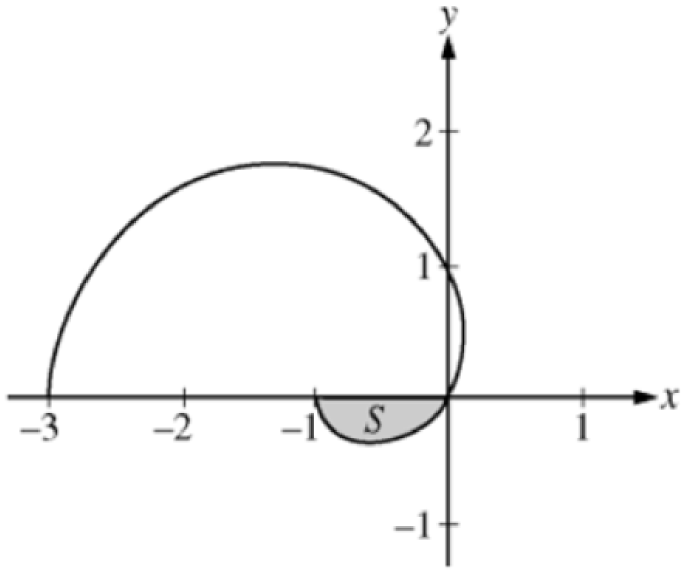
$$\int_{\pi/6}^{5\pi/6} \left[1 - 2\sin\theta\right]^2 d\theta$$

① $$\frac{1}{2}\int_0^{2\pi} \left[1 - 2\sin\theta\right]^2 d\theta - 2\left[\frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[1 - 2\sin\theta\right]^2 d\theta\right]$$

② $$\frac{1}{2}\int_{5\pi/6}^{13\pi/6} \left[1 - 2\sin\theta\right]^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[1 - 2\sin\theta\right]^2 d\theta$$

2009 Form B Question #4

The graph of the polar curve $r = 1 - 2\cos\theta$ for $\boxed{0 \le \theta \le \pi}$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.



(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

(a) $r(0) = -1;\ r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$$

$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0,\ y = 1$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$$

The tangent line is given by $y = 1 - 2x$.

$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

Area Between Polar Curves

Example 1: Find the area of the region common to the two regions bounded by the following curves.

$r = -6\cos\theta$ $\quad\quad$ $r = 2 - 2\cos\theta$

$r = 0$
$\downarrow$
$\cos\theta = 0$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

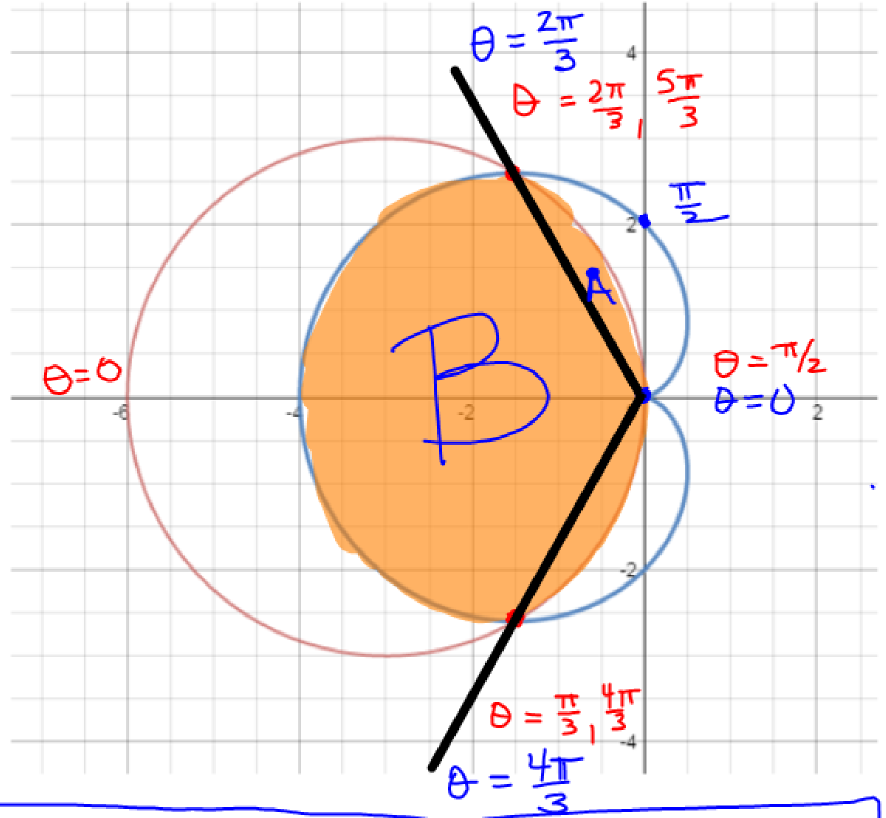
$r = 0$
$\downarrow$
$\cos\theta = 1$
$\theta = 0, 2\pi$

$r = 2$
$\downarrow$
$\cos\theta = 0$
$\theta = \pi/2$

$-6\cos\theta = 2 - 2\cos\theta$

$\cos\theta = -1/2$

$\theta = 2\pi/3, 4\pi/3$



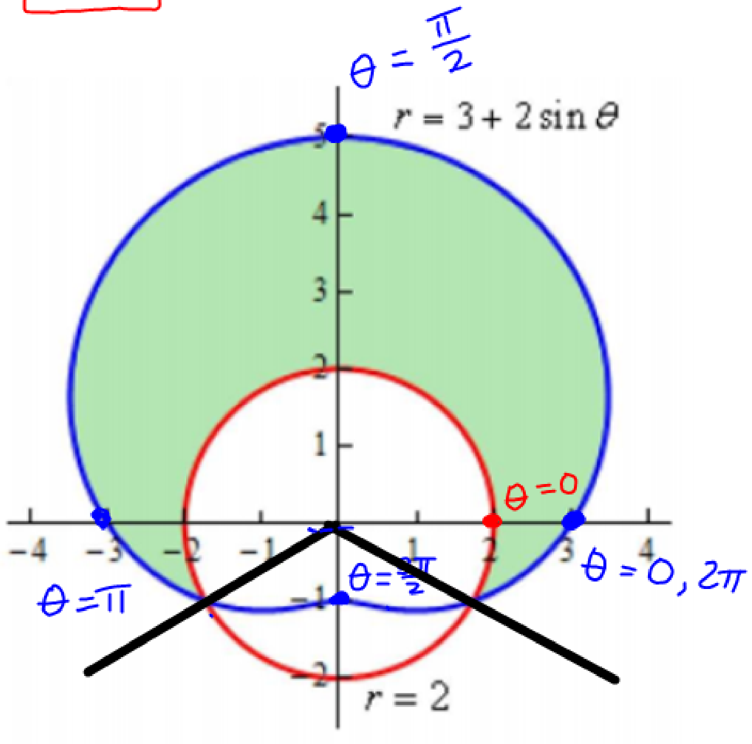
$$\boxed{2\left[\underbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3}[-6\cos\theta]^2\,d\theta}_{A}\right] + \underbrace{\frac{1}{2}\int_{2\pi/3}^{4\pi/3}[2-2\cos\theta]^2\,d\theta}_{B}}$$

Example 2: Find the area that lies inside $r = 3 + 2\sin\theta$ and outside $\boxed{r = 2.}$

$3 + 2\sin\theta = 2$

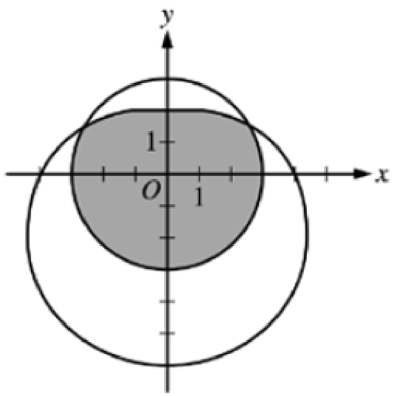
$\sin\theta = -1/2$

$\theta = \frac{-\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$



$$\boxed{\frac{1}{2}\int_{-\pi/6}^{7\pi/6} [3+2\sin\theta]^2 - [2]^2 \, d\theta}$$

Example 3: (FRQ #2 2013)



The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

(a) Area $= \boxed{6\pi} + \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4 - 2\sin\theta)^2 \, d\theta = 24.709$ (or 24.708)

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(b) $x = r\cos\theta \Rightarrow x(\theta) = (4 - 2\sin\theta)\cos\theta$

$x(t) = \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right)$

$x(t) = -1$ when $t = 1.428$ (or 1.427)

$3 : \begin{cases} 1 : x(\theta) \text{ or } x(t) \\ 1 : x(\theta) = -1 \text{ or } x(t) = -1 \\ 1 : \text{answer} \end{cases}$

(c) $y = r\sin\theta \Rightarrow y(\theta) = (4 - 2\sin\theta)\sin\theta$

$y(t) = \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right)$

Position vector $= \langle x(t), y(t) \rangle$

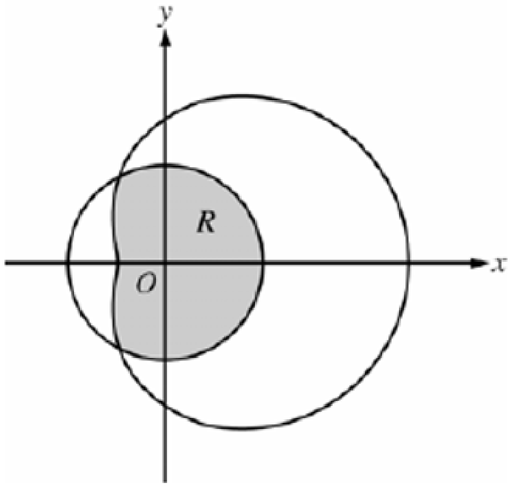
$= \left\langle \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right), \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right) \right\rangle$

$v(1.5) = \langle x'(1.5), y'(1.5) \rangle$

$= \langle -8.072, -1.673 \rangle$ (or $\langle -8.072, -1.672 \rangle$)

$3 : \begin{cases} 2 : \text{position vector} \\ 1 : \text{velocity vector} \end{cases}$

Example 4: (FRQ #3 2007)



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

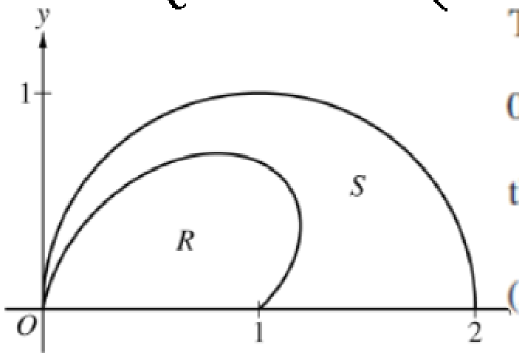
(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3+2\cos\theta)^2\,d\theta$
$= 10.370$
	4 : { 1 : area of circular sector
	2 : integral for section of limaçon
	   1 : integrand
	   1 : limits and constant
	1 : answer

(b) $\left.\frac{dr}{dt}\right\|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right\|_{\theta=\pi/3} = -1.732$ The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.	2 : { 1 : $\left.\frac{dr}{dt}\right\|_{\theta=\pi/3}$ 1 : interpretation
(c) $y = r\sin\theta = (3+2\cos\theta)\sin\theta$ $\left.\frac{dy}{dt}\right\|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right\|_{\theta=\pi/3} = 0.5$ The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.	3 : { 1 : expression for y in terms of θ 1 : $\left.\frac{dy}{dt}\right\|_{\theta=\pi/3}$ 1: interpretation

2017 Question 2 (Calc Active)



The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

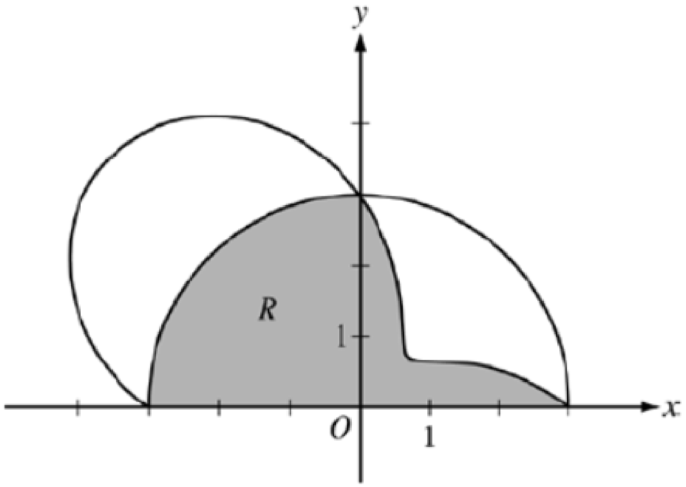
(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

(a) $\frac{1}{2}\int_0^{\pi/2}(f(\theta))^2\,d\theta = 0.648414$ The area of R is 0.648.	$2:\begin{cases}1:\text{integral}\\1:\text{answer}\end{cases}$
(b) $\int_0^k\left((g(\theta))^2-(f(\theta))^2\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left((g(\theta))^2-(f(\theta))^2\right)d\theta$ — OR — $\int_0^k\left((g(\theta))^2-(f(\theta))^2\right)d\theta = \int_k^{\pi/2}\left((g(\theta))^2-(f(\theta))^2\right)d\theta$	$2:\begin{cases}1:\text{integral expression}\\\quad\text{for one region}\\1:\text{equation}\end{cases}$
(c) $w(\theta) = g(\theta) - f(\theta)$ $w_A = \dfrac{\int_0^{\pi/2} w(\theta)\,d\theta}{\frac{\pi}{2}-0} = 0.485446$ The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.	$3:\begin{cases}1:w(\theta)\\1:\text{integral}\\1:\text{average value}\end{cases}$
(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$ $w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517). $w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.	$2:\begin{cases}1:\text{solves } w(\theta)=w_A\\1:\text{answer with reason}\end{cases}$

2014 Question 2 (Calc Active)

Question 2



The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\dfrac{dx}{d\theta}$ at $\theta = \dfrac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \dfrac{\pi}{2}$.

Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \dfrac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\dfrac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\dfrac{dr}{dt}$ at $\theta = \dfrac{\pi}{6}$.

(a) Area $= \dfrac{9\pi}{4} + \dfrac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta$
$= 9.708$ (or 9.707)

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) $x = (3 - 2\sin(2\theta))\cos\theta$
$\left.\dfrac{dx}{d\theta}\right|_{\theta=\pi/6} = -2.366$

$2 : \begin{cases} 1 : \text{expression for } x \\ 1 : \text{answer} \end{cases}$

(c) The distance between the two curves is
$D = 3 - (3 - 2\sin(2\theta)) = 2\sin(2\theta)$.

$\left.\dfrac{dD}{d\theta}\right|_{\theta=\pi/3} = -2$

$2 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{answer} \end{cases}$

(d) $\dfrac{dr}{dt} = \dfrac{dr}{d\theta} \cdot \dfrac{d\theta}{dt} = \dfrac{dr}{d\theta} \cdot 3$
$\left.\dfrac{dr}{dt}\right|_{\theta=\pi/6} = (-2)(3) = -6$

$2 : \begin{cases} 1 : \text{chain rule with respect to } t \\ 1 : \text{answer} \end{cases}$