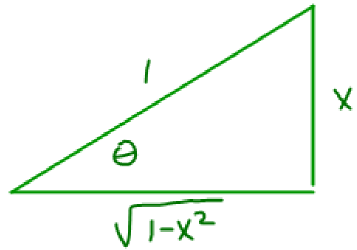
Warm Up: $\int x \arcsin x \; dx$

$u = \arcsin x \qquad dv = x\,dx$

$du = \dfrac{1}{\sqrt{1-x^2}}\,dx \qquad v = \frac{1}{2}x^2$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$



$\sin\theta = x$

$dx = \cos\theta \, d\theta$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\int 1 - \cos 2\theta\,d\theta$$

$\frac{1}{2}\left[2\sin\theta\cos\theta\right]$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \boxed{\frac{1}{2}x^2 \arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

Integration by Partial Fractions

Review: Partial Fractions – The process of taking a function and breaking into the **sum of other functions.**

Example: $\frac{5x-4}{x^2-x-2}$ $= \left[\frac{5x-4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}\right] (x-2)(x+1)$

$$5x-4 = A(x+1) + B(x-2)$$

$$5x-4 = Ax + A + Bx - 2B$$

$$5x-4 = Ax + Bx + A - 2B$$

$$5x + -4 = (A+B)x + (A-2B)$$

$$A + B = 5$$

$$-(A - 2B = -4)$$

$$3B = 9$$

$$\boxed{B = 3} \rightarrow \boxed{A = 2}$$

$$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$$

When is this useful? Consider the following integrals:

$$\int \frac{2x-1}{x^2-x-6}\,dx$$

U-SUBSTITUTION

$$\int \frac{1}{x^2-x-6}\,dx$$

PARTIAL DECOMP

∘ $\dfrac{\text{CONSTANT}}{(\text{FACTORABLE POLYNOMIAL})}$

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

PARTIAL FRACTIONS

∘ NOT

$$\int \frac{x^3-3x^2+1}{x^2+1}\,dx$$

LONG DIVISION

∘ DEG NUM $\geq$ DEG DENOM

-If the **<u>numerator is not a derivative of the denominator</u>** nor is it a **<u>constant multiple of the derivative of the denominator</u>**, you must use partial fractions.

Example1:

$$\int \frac{1}{x^2 - 5x + 6}\,dx = \int \frac{1}{(x-3)(x-2)}\,dx = \int \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$= \ln\left|\frac{x-3}{x-2}\right| + C$$

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$1 = Ax - 2A + Bx - 3B$$

$$1 = (A+B)x + (-2A - 3B)$$

$$2(A + B = 0)$$

$$-2A - 3B = 1$$

$$-B = 1$$

$$\boxed{B = -1} \rightarrow \boxed{A = 1}$$

Example 2:

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{3x+11}{(x-3)(x+2)}\,dx = \int \frac{4}{x-3} - \frac{1}{x+2}\,dx$$

$$= 4\ln|x-3| - \ln|x+2| + C$$

$$= \ln\left|\frac{(x-3)^4}{x+2}\right| + C$$

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$3x+11 = (A+B)x + (2A-3B)$$

$$3(A+B=3)$$

$$2A-3B=11$$

$$5A=20$$

$$\boxed{A=4} \rightarrow \boxed{B=-1}$$

Example 3:

$$\int \frac{x^2+4}{3x^3+4x^2-4x}dx = \int \frac{x^2+4}{x(3x^2+4x-4)}dx = \int \frac{x^2+4}{x(3x-2)(x+2)}dx$$

$$\frac{x^2+4}{x(3x-2)(x+2)} = \frac{A}{x} + \frac{B}{3x-2} + \frac{C}{x+2}$$

$$x^2+4 = A(3x-2)(x+2) + Bx(x+2) + Cx(3x-2)$$

$$x^2+4 = 3Ax^2+4Ax-4A+Bx^2+2Bx+3Cx^2-2Cx$$

$$x^2+4 = (3A+B+3C)x^2+(4A+2B-2C)x+(-4A)$$

$3A+B+3C=1$ $\quad$ $4A+2B-2C=0$ $\quad$ $-4A=4$

$\boxed{A=-1}$

$$-2(B+3C=4)$$
$$2B-2C=4$$
$$-8C=-4$$

$\boxed{C=1/2} \rightarrow \boxed{B=5/2}$

$$= \int \frac{-1}{x} + \frac{5/2}{3x-2} + \frac{1/2}{x+2}\,dx$$

$$= -\ln|x| + \frac{5}{6}\ln|3x-2| + \frac{1}{2}\ln|x+2| + C$$

$$= \ln\left|\frac{(3x-2)^{5/6}(x+2)^{1/2}}{x}\right| + C$$

Example 4: w/Long Division

$$\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{1}{x^2-1}\,dx = \int 1\,dx + \int \frac{1}{(x+1)(x-1)}\,dx$$

$$= x + \int \frac{-1/2}{x+1} + \frac{1/2}{x-1}\,dx$$

$$= x - \tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1| + C$$

$$= x - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$$\begin{array}{r|l} x^2-1 & \overline{\;x^2\;} \quad 1 \\ & -(x^2-1) \\ \hline & 1 \end{array}$$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(x+1)$$

$$1 = (A+B)x + (-A+B)$$

$$\begin{aligned} A+B &= 0 \\ -A+B &= 1 \\ \hline 2B &= 1 \end{aligned}$$

$$\boxed{B = 1/2} \rightarrow \boxed{A = -1/2}$$

Example 5: Partial Fractions Disguised

$$\int \frac{\cos x}{3\sin^2 x + 7\sin x + 4}\, dx$$