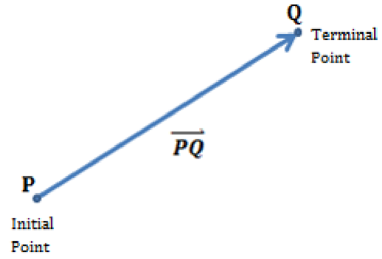
Motion Along a Curve Vectors

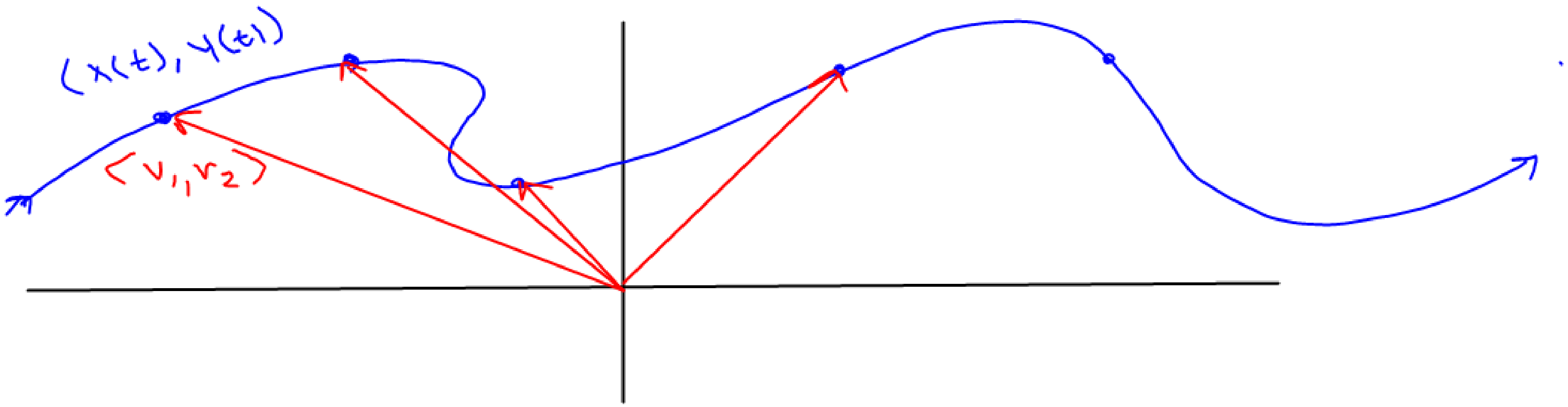
A vector is a **<u>directed line segment</u>** from an **<u>initial point</u>** to a **<u>terminal point.</u>**



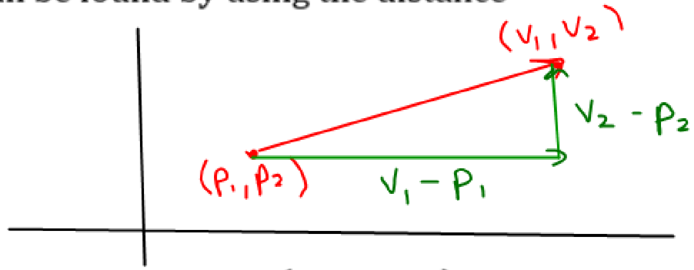
Component Form of a Vector in a Plane

If **v** is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **<u>component form of v</u>** is given by

$$\boldsymbol{v} = \langle v_1, v_2 \rangle$$



One important thing to consider is the **length (or magnitude) of the vector**. We can easily see, a vector is drawn from the initial point (p_1, p_2) to the terminal point (v_1, v_2) can be found by using the distance formula.



Magnitude of a Vector: $\sqrt{(v_1 - p_1)^2 + (v_2 - p_2)^2}$

Vectors and Parametric Curves

Previously we talked about parametric curves being defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations

$$x = f(t) \quad and \quad y = g(t)$$

We can write the parametric curve as a **Vector-Valued Function**, where the component functions are real valued function of the parameter t. Often, vector valued function are denoted as

$$r(t) = \boxed{\langle f(t), g(t) \rangle} \text{ or } r(t) = \boxed{\langle x(t), y(t) \rangle}$$

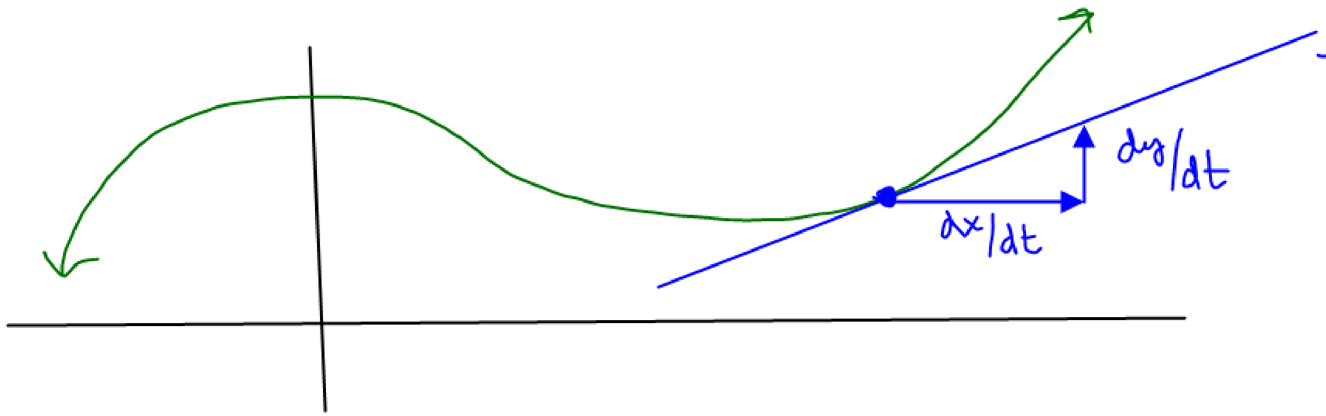
This form of a vector represents the **position vector** at any time t .

Similar to particle motion, the derivative of the position vector represents the velocity vector and the derivative of the velocity vector represents the acceleration vector.

$\langle x'(t), y'(t) \rangle$ is the **velocity vector** at any time t.

$\langle x''(t), y''(t) \rangle$ is the **acceleration vector** at any time t.

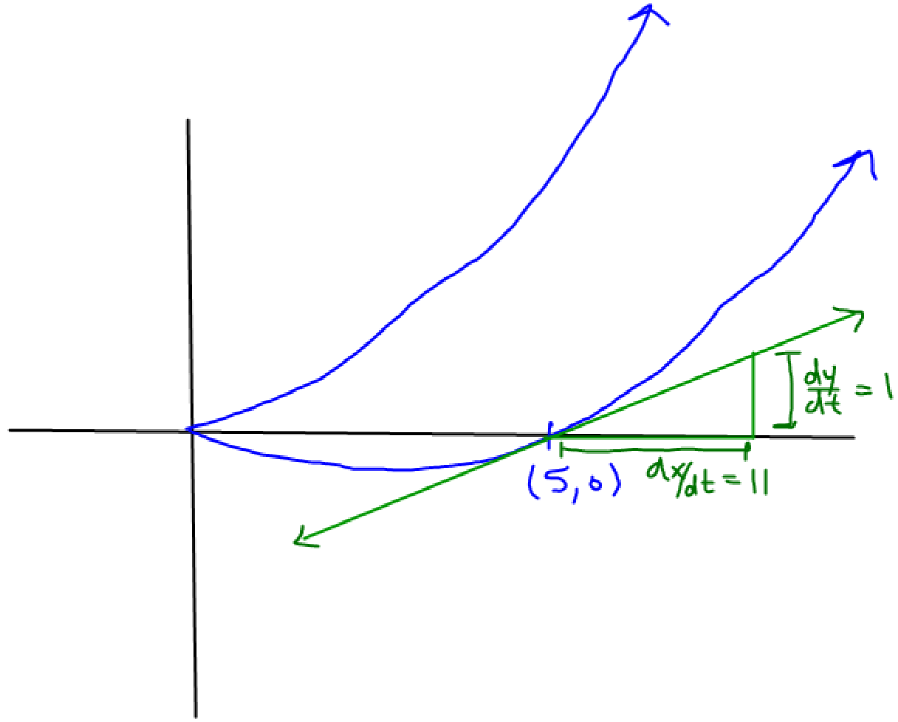
- $x'(t) = \frac{dx}{dt}$ is the rate at which the x-coordinate is changing with respect to time. This represents the **velocity of an object in the horizontal direction**.
- $y'(t) = \frac{dy}{dt}$ is the rate at which the y-coordinate is changing with respect to time. This represents the **velocity of an object in the vertical direction.**



$$\text{Slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example 1: A particle moves in the xy-plane so that at any time t, the position of the particle is given by $x(t) = t^3 + 4t^2, y(t) = t^4 - t^3$

a. Find the velocity vector when t=1.
b. Find the acceleration vector when t=2.



(a) $x'(t) = 3t^2 + 8t$ $y'(t) = 4t^3 - 3t^2$

$x'(1) = 11$ $y'(1) = 1$

Velocity Vector : $\langle 11, 1 \rangle$

(b) $x''(t) = 6t + 8 \rightarrow \frac{d^2x}{dt^2}$ $y''(t) = 12t^2 - 6t \rightarrow \frac{d^2y}{dt^2}$

$x''(2) = 20$ $y''(2) = 36$

Acc Vector : $\langle 20, 36 \rangle$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

Rectangular

Example 2: A particle moves on the curve $\boxed{y = \ln x}$ so that its x-component has derivative

parametric $\longrightarrow$ $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(\overset{x}{1}, \overset{y}{0})$. Find the position of the particle at time $t = 1$.

$$x(\overset{t}{1}) = x(0) + \int_0^1 t+1 \, dt$$

$$= 1 + \left[\tfrac{1}{2}t^2 + t\right]_0^1$$

$$= 1 + \tfrac{1}{2} + 1$$

$$\underset{t=1}{x(1)} = 5/2$$

Position:

$$\left\langle \frac{5}{2}, \ln 5/2 \right\rangle$$

Example 3: Point $P(x, y)$ moves in the xy-plane in such a way that $\frac{dx}{dt} = \frac{1}{t+1}$ and $\frac{dy}{dt} = 2t$ for $t \geq 0$.

(a) Find the coordinates of P in terms of t given that, when $t = 1$, $x = \ln 2$ and $y = 0$.

(b) Write an equation expressing y in terms of x.

(c) Find the average rate of change of y with respect to x as t varies from 0 to 4.

(d) Find the instantaneous rate of change of y with respect to x when $t = 1$.

(a)

$P(x(t), y(t))$

$(\ln(t+1), t^2-1)$

$x(t) = \int \frac{1}{t+1}\,dt = \ln|t+1| + c$

$\ln 2 = \ln 2 + c$

$0 = c$

$y(t) = \int 2t\,dt = t^2 + c$

$0 = 1 + c$

$-1 = c$

(b) $x = \ln(t+1)$

$e^x - 1 = t$

$y = (e^x - 1)^2 - 1$

(c) $t = 0 \Rightarrow (0, -1)$

$t = 4 \Rightarrow (\ln 5, 15)$

$A_{roc} = \frac{16}{\ln 5}$

(d) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{\frac{1}{2}} = 4$

or $\frac{dy}{dx} = 2(e^x - 1) \cdot e^x$

$\left.\frac{dy}{dx}\right|_{t=1} = 2(e^{\ln 2} - 1)e^{\ln 2}$

$= 4$

A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time t seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

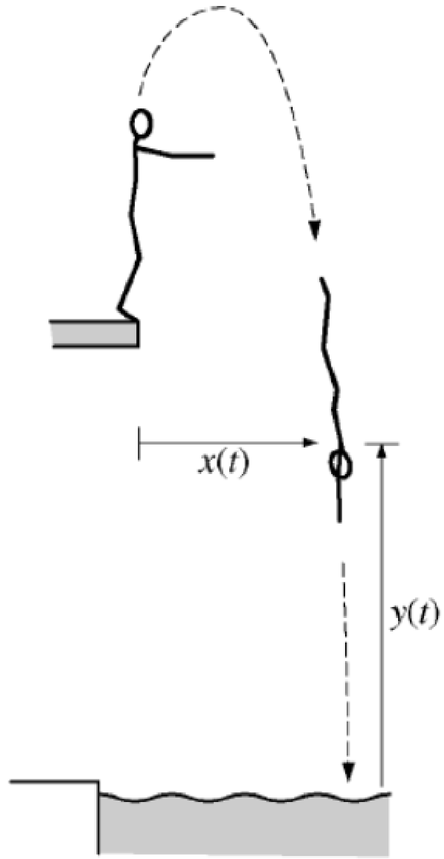
for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find A, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.



Note: Figure not drawn to scale.

Speed of a Particle Along a Curve

Given the position vector, $\langle x(t), y(t)\rangle$, of a particle, the **speed of the particle** can be found by finding the **magnitude (length) of the velocity vector.**

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example 1: A particle moves in the xy-plane so that at any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^2 + 3t, y(t) = t^3 - 3t^2$. Find the speed of the particle when t = 1 and t = 2.

$\frac{dx}{dt} = 2t + 3$ $\quad$ $\frac{dy}{dt} = 3t^2 - 6t$

$\boxed{\text{At } t=1}$

$\text{Speed} = \sqrt{5^2 + (-3)^2}$

$= \sqrt{34}$

$\boxed{\text{At } t=2}$

$\text{Speed} = \sqrt{(7)^2 + (0)^2}$

$= 7$

Example 2: An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = \sin(t^3), \frac{dy}{dt} = \cos(t^2)$. At time t = 2, the object is at the position (1, 4).

a. Find the acceleration vector for the particle at t = 2.
b. Write the equation of the tangent line to the curve at the point where t = 2.
c. Find the speed of the object at t = 2.
d. Find the position of the particle at time t = 1.

(a) $a(t) = \langle 3t^2 \cos t^3, -2t \sin t^2 \rangle$

$a(2) = \langle 12 \cos 8, -4 \sin 4 \rangle$

$\langle -1.746, 3.027 \rangle$

(b) $\left.\frac{dy}{dx}\right|_{t=2} = \frac{\cos(4)}{\sin(8)}$

$y - 4 = \frac{\cos 4}{\sin 8}(x-1)$

$y - 4 = -0.661(x-1)$

(c) $\sqrt{\sin^2(8) + \cos^2(4)}$

1.186

(d) $x(1) = 1 + \int_2^1 \sin(t^3)\,dt$

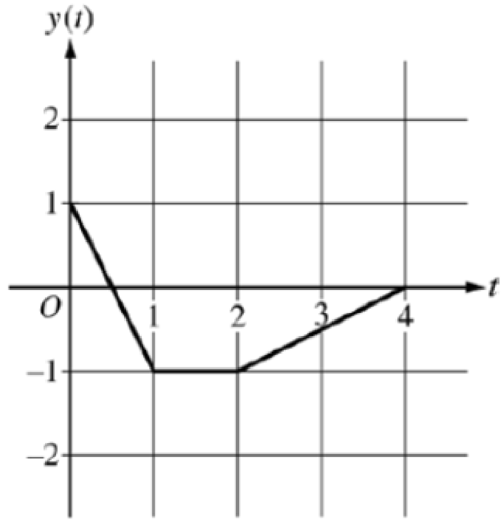
$x(1) = 0.782$

$y(1) = 4 + \int_2^1 \cos(t^2)\,dt$

$y(1) = 4.443$

$\langle x(1), y(1) \rangle = \langle 0.782, 4.443 \rangle$

2016 Question 2 (Calculator Active)



At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.