## Arc Length: Parametric Curves

Since we can use parametric equations to represent the **path of a particle** moving in the plane, the arc length associated with the parametric equations determines the **distance traveled** by the particle along its path.

- We can develop the formula directly from the arc length formula for functions. Let $y = h(x)$, then arc length is given by:

$$s = \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2}\, dx$$

$$h'(x) = \frac{dy/dt}{dx/dt}$$

$$\frac{\Delta y}{\Delta x} \quad \frac{0}{0} \quad \begin{matrix}\Delta y \to 0 \\ \Delta x \to 0\end{matrix} \Rightarrow \frac{dy}{dx}$$

Parametric form

$$\int_{t_0}^{t_1} \sqrt{1 + \left[\frac{dy/dt}{dx/dt}\right]^2} \cdot dx \cdot \frac{dt}{dt}$$

$$\int_{t_0}^{t_1} \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{\cancel{(dx/dt)^2}}}\ \cancel{\frac{dx}{dt}} \cdot dt$$

$$\int_{t_0}^{t_1} \sqrt{(dx/dt)^2 + (dy/dt)^2}\; dt$$

**Arc Length in Parametric Form:**

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

**Example 1:** The position of a particle at time t is given parametrically by $x = t^2$ and $y = \frac{1}{3}t^3 - t$. Find the distance the particle travels between t=1 and t=2.

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = t^2 - 1$

$$\int_1^2 \sqrt{(2t)^2 + (t^2-1)^2}\, dt$$

$$\int_1^2 \sqrt{t^4 + 2t^2 + 1}\, dt$$

$$\int_1^2 \sqrt{(t^2+1)^2}\, dt$$

$$\int_1^2 t^2 + 1\, dt = \left[\frac{1}{3}t^3 + t\right]_1^2 = \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right) = \boxed{\frac{10}{3}}$$

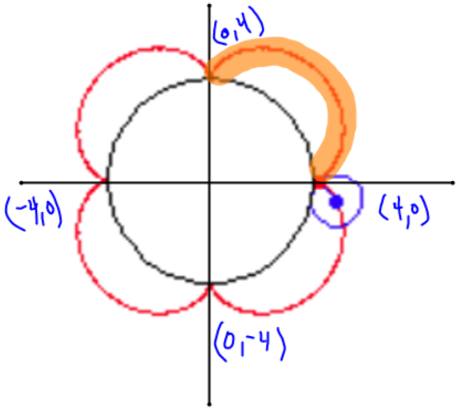
**Example 2:** A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The figure traced out by the a point on the circumference of the smaller circle is a epicycloid, given by

$$x = 5\cos t - \cos 5t \qquad y = 5\sin t - \sin 5t.$$

Find the distance traveled by the point in one complete trip about the larger circle.

$\frac{dx}{dt} = -5\sin t + 5\sin(5t)$

$\frac{dy}{dt} = 5\cos t - 5\cos(5t)$




$\boxed{\text{Start}}$

$4 = 5\cos t - \cos 5t$

$0 = 5\sin t - \sin 5t$

$t = \cancel{\pi}, \boxed{0}, \cancel{\pi}, \boxed{2\pi}$

$\boxed{\text{END}}$

$0 = 5\cos t - \cos 5t$

$4 = 5\sin t - \sin 5t$

$t = \cancel{\frac{\pi}{2}}, \boxed{\frac{\pi}{2}}, \cancel{\frac{3\pi}{2}}, \boxed{\frac{5\pi}{2}}$

$$S = \int_0^{\pi/2} \sqrt{25\sin^2 t - 50\sin t\sin(5t) + 25\sin^2(5t) + 25\cos^2 t - 50\cos t\cos 5t + 25\cos^2(5t)}\,dt$$

$$= \int_0^{\pi/2} \sqrt{50 - 50(\sin t\sin(5t) + \cos t\cos(5t))}\,dt$$

$$= 5\int_0^{\pi/2} \sqrt{2 - 2(\sin t\sin(5t) + \cos t\cos(5t))}\,dt$$

$$= 5\int_0^{\pi/2} \sqrt{2 - 2\cos(4t)}\,dt$$

$$= 5\int_0^{\pi/2} \sqrt{4\sin^2(2t)}\,dt$$

$$= 10\int_0^{\pi/2} \sin(2t)\,dt = 5\left[-\cos 2t\right]_0^{\pi/2} = 5\left[1 - -1\right] = 10$$

TOTAL ARC : $4(10) = \boxed{40}$

**Example 3:** A curve is define by the parametric equations

$$x = \int_1^t \frac{\cos u}{u}\,du \qquad y = \int_1^t \frac{\sin u}{u}\,du$$

$$\frac{dx}{dt} = \frac{\cos t}{t} \qquad \frac{dy}{dt} = \frac{\sin t}{t}$$

Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

$(0,0) \Rightarrow t = 1$

$$\frac{dx}{dt} = 0$$

$$\cos t = 0$$

$$t = \frac{\pi}{2}$$

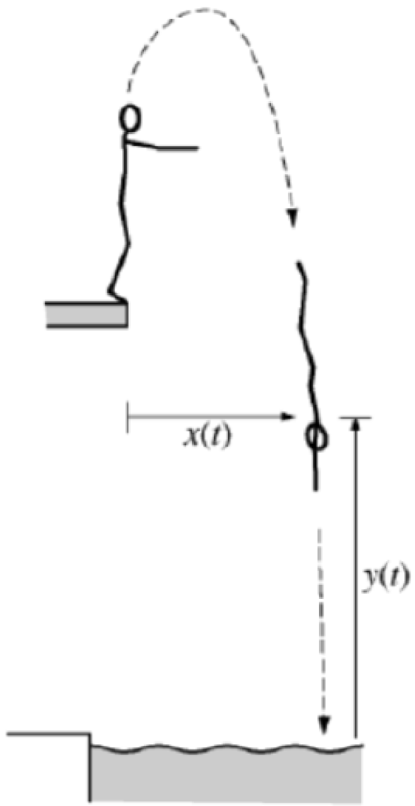
$$\int_1^{\pi/2} \sqrt{\left(\frac{\cos t}{t}\right)^2 + \left(\frac{\sin t}{t}\right)^2}\,dt$$

$$\int_1^{\pi/2} \sqrt{\frac{\cos^2 t + \sin^2 t}{t^2}}\,dt$$

$$\int_1^{\pi/2} \sqrt{\frac{1}{t^2}}\,dt$$

$$\int_1^{\pi/2} \frac{1}{t}\,dt = \Big[\ln|t|\Big]_1^{\pi/2} = \ln(\pi/2) - \ln(1) = \boxed{\ln(\pi/2)}$$

## 2009 Question 3 (Calculator Active)




Note: Figure not drawn to scale.

A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time $t$ seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

for $0 \le t \le A$, where $A$ is the time that the diver's shoulders enter the water.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find $A$, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle $\theta$, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.

(a) $\frac{dy}{dt} = 0$ only when $t = 0.36735$. Let $b = 0.36735$.

The maximum vertical distance from the water surface to the diver's shoulders is

$$y(b) = 11.4 + \int_0^b \frac{dy}{dt}\,dt = 12.061 \text{ meters.}$$

Alternatively, $y(t) = 11.4 + 3.6t - 4.9t^2$, so $y(b) = 12.061$ meters.

$3 : \begin{cases} 1 : \text{considers } \frac{dy}{dt} = 0 \\ 1 : \text{integral or } y(t) \\ 1 : \text{answer} \end{cases}$

(b) $y(A) = 11.4 + \int_0^A \frac{dy}{dt}\,dt = 11.4 + 3.6A - 4.9A^2 = 0$ when $A = 1.936$ seconds.

$2 : \begin{cases} 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^A \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 12.946$ meters

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) At time $A$, $\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=A} = -19.21913$.

The angle between the path of the diver and the water is $\tan^{-1}(19.21913) = 1.518$ or $1.519$.

$2 : \begin{cases} 1 : \frac{dy}{dx} \text{ at time } A \\ 1 : \text{answer} \end{cases}$