Arc Length

One of the uses of definite integrals is to helps calculate the **length of an arc** for any defined curve.

- When calculating arc lengths, the function in question must be **continuously differentiable** on a given interval. Further, given any curve, the graph must be a **smooth curve** on the given interval.

Definition of Arc Length

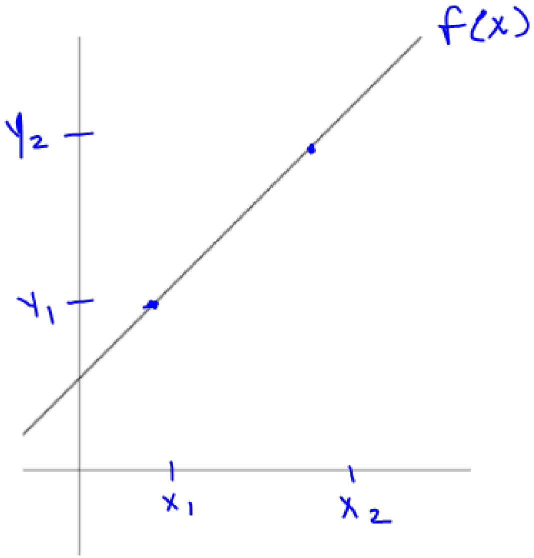
Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Similarly, for a smooth curve given by $x = f(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Example 1: Verify the distance formula for the length of a line segment.



$$f'(x) = \boxed{\frac{y_2 - y_1}{x_2 - x_1}}$$

Constant

$$\int_{x_1}^{x_2} \boxed{\sqrt{1 + \left[\frac{y_2 - y_1}{x_2 - x_1}\right]^2}}\, dx$$

Constant

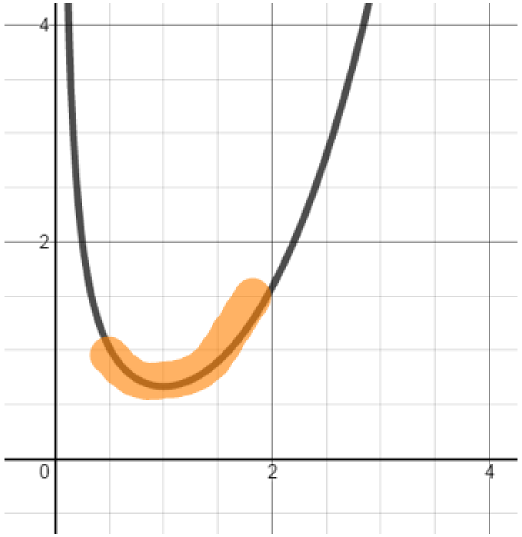
$$\left[x \cdot \sqrt{1 + \frac{(y_2 - y_1)^2}{(x_2 - x_1)^2}} \right]_{x_1}^{x_2}$$

$$\left[x \cdot \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{\sqrt{(x_2 - x_1)^2}} \right]_{x_1}^{x_2}$$

$$x_2 \cdot \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{x_2 - x_1} - x_1 \cdot \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{x_2 - x_1}$$

$$\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{\cancel{x_2 - x_1}} \left[\cancel{x_2 - x_1} \right]$$

Example 2: Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$ as shown in the graph.



$$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2} = \frac{1}{2}x^{-2}\left[x^4 - 1\right] = \frac{x^4-1}{2x^2}$$

$$\int_{1/2}^{2} \sqrt{1 + \left(\frac{x^4-1}{2x^2}\right)^2}\, dx$$

$$\int_{1/2}^{2} \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}\, dx \longrightarrow \begin{array}{c} x^8 + 2x^4 + 1 \\ (x^4+1)(x^4+1) \\ (x^4+1)^2 \end{array}$$

$$\int_{1/2}^{2} \frac{x^4+1}{2x^2}\, dx$$

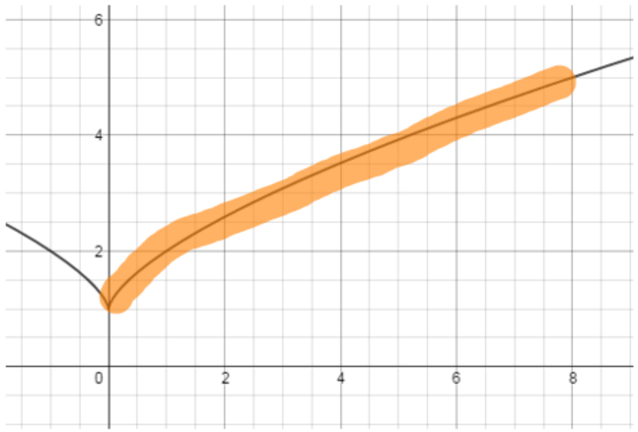
$$\frac{1}{2}\int_{1/2}^{2} x^2 + x^{-2}\, dx$$

$$\frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{2} = \boxed{\frac{1}{2}\left[\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{24} - 2\right)\right]}$$

$$= \boxed{\frac{1}{2}\left[\frac{99}{24}\right]}$$

$$= \boxed{\frac{33}{16}}$$

Example 3: Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0, 8]$.



$y = x^{2/3} + 1$

$\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$

$\int_0^8 \sqrt{1 + \left(\frac{2}{3x^{1/3}}\right)^2}\, dx$

$\int_0^8 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx$

$\int_0^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\, dx$

$u = 9x^{2/3}+4$

$du = 6x^{-1/3}dx$

$\frac{1}{18}\int_4^{40} \sqrt{u}\, du = 9.073$

$x = \pm(y-1)^{3/2}$

$\frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}$

$\int_1^5 \sqrt{1 + \left(\frac{3}{2}(y-1)^{1/2}\right)^2}\, dy$

$\int_1^5 \sqrt{1 + \frac{9}{4}(y-1)}\, dy$

$u = 1 + \frac{9}{4}(y-1)$

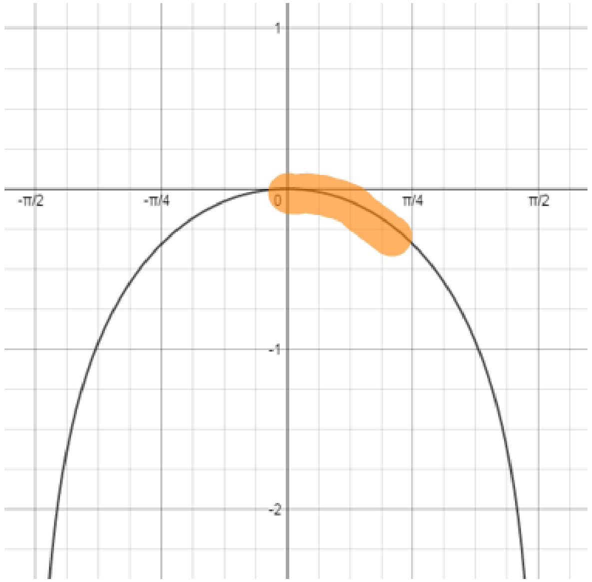
$du = \frac{9}{4}\, dy$

$\frac{4}{9}\int_1^{10} \sqrt{u}\, du$

$\frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{8}{27}\left[10^{3/2} - 1\right] = \frac{8(10\sqrt{10}-1)}{27}$

$= 9.073$

Example 4: Find the arc length of the graph of $y = \ln(\cos x)$ on the interval $\left[0, \frac{\pi}{4}\right]$.



$$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$$

$$\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$\int_0^{\pi/4} \sec x \, dx$$

$$\Big[\ln \left| \sec x + \tan x \right| \Big]_0^{\pi/4}$$

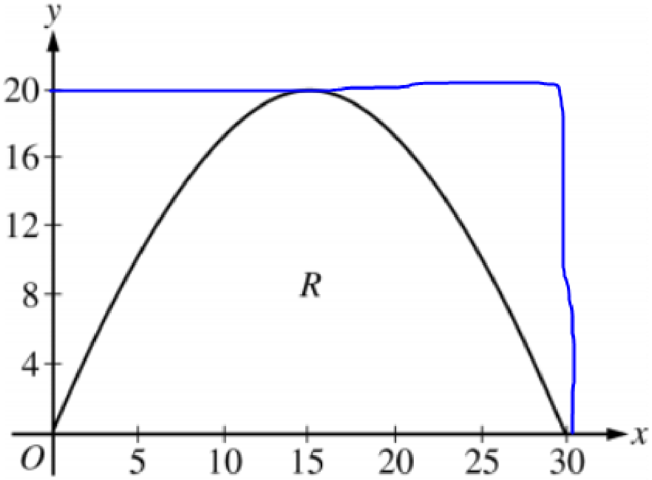
$$\ln \left| \frac{2}{\sqrt{2}} + 1 \right| - \ln \left| 1 + 0 \right|$$

$$\boxed{\ln \left| \sqrt{2} + 1 \right|}$$

2009 (Form B)

Question 1

A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.



(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

(c) Find the perimeter of the base of the cake.

(a) Area $= 30 \cdot 20 - \boxed{\int_0^{30} f(x)\, dx} = 218.028 \text{ cm}^2$

$3 : \begin{cases} 2 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \boxed{\int_0^{30} \frac{\pi}{2}\left(\frac{f(x)}{2}\right)^2 dx} = 2356.194 \text{ cm}^3$

Therefore, the baker needs $2356.194 \times 0.05 = 117.809$ or 117.810 grams of chocolate.

$3 : \begin{cases} 2 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) Perimeter $= 30 + \boxed{\int_0^{30} \sqrt{1 + (f'(x))^2}\, dx} = 81.803$ or 81.804 cm

$3 : \begin{cases} \boxed{2 : \text{integral}} \\ 1 : \text{answer} \end{cases}$