Parametric Equations and Calculus Continued

Because dy/dx, is a function of t, you can use the idea for calculating the derivative repeatedly to find higher order derivatives.

Higher Order Derivatives of Parametric Curves

<u>Second Derivative:</u>

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

<u>Third Derivative:</u>

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}$$

Example 1: For the curve given by $x = \sqrt{t}$ and $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$, find the slope and concavity at the point $(2, 3)$.

$2 = \sqrt{t} \Rightarrow t = 4$ $\qquad$ $3 = \frac{1}{4}(t^2 - 4) \Rightarrow 12 = t^2 - 4 \Rightarrow 16 = t^2 \Rightarrow t = \pm 4$

$$\frac{dy}{dx} = \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} = t^{3/2} \qquad \left.\frac{dy}{dx}\right|_{t=4} = (4)^{3/2} = 8 \rightarrow \text{INCREASING}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[t^{3/2}\right]}{\frac{1}{2}t^{-1/2}} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} = 3t \qquad \left.\frac{d^2y}{dx^2}\right|_{t=4} = 3(4) = 12 \rightarrow \text{Concave Up}$$

Example 2: For the curve given by $x = t^2 - 4$ and $y = t^3 - 3t$. Identify the intervals of increase and decrease. Identify the intervals where the curve is concave up and concave down.

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} = \boxed{\frac{3(t^2-1)}{2t}}$$

$$= \frac{3}{2}t - \frac{3}{2}t^{-1}$$

CV: $t^2 - 1 = 0$ $\quad t = 0$

$t = \pm 1$

$-\infty \quad - \quad -1 \quad + \quad 0 \quad - \quad 1 \quad + \quad \infty \quad t$

$t \begin{cases} \text{INC}: (-1,0)\ (1,\infty) \\ \text{DEC}: (-\infty,-1)\ (0,1) \end{cases}$

$(x,y) \begin{cases} \text{INC}: (-3,2) \to (-4,0) \quad (-3,-2) \to (\infty,\infty) \\ \text{DEC}: (\infty,-\infty) \to (-3,2) \quad (-4,0) \to (-3,-2) \end{cases}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{3}{2}t - \frac{3}{2}t^{-1}\right]}{2t} = \frac{\frac{3}{2} + \frac{3}{2}t^{-2}}{2t}$$

$$= \frac{3}{2}t^{-2}\left[\frac{t^2+1}{2t}\right]$$

$$= \frac{3[t^2+1]}{4t^3}$$

CV: $t = 0$

$-\infty \quad - \quad 0 \quad + \quad \infty \quad t$

$t \begin{cases} \text{Up}: (0,\infty) \\ \text{Down}: (-\infty,0) \end{cases}$

$(x,y) \begin{cases} \text{Up}: (-4,0) \to (\infty,\infty) \\ \text{Down}: (\infty,-\infty) \to (-4,0) \end{cases}$

Integrals Involving Parametrically Defined Functions

When working with parametric equation integrals, we must be sure to express everything in terms of the parameter.

Recall:

$$\int_a^b F(x)dx = f(b) - f(a)$$

f is an antiderivative of F(x)

Since we have a parametric curve defined as,

$$x = f(t) \ \ and\ y = g(t)$$

Let $y = F(x) = F\big(f(t)\big) = g(t)$.

Then the definite integral becomes:

$$\int_{a = x_1}^{b = x_2} F(x)\,dx = \int_{\alpha = t_1}^{\beta = t_2} g(t) f'(t)\,dt = \int_{\alpha}^{\beta} y\,dx$$

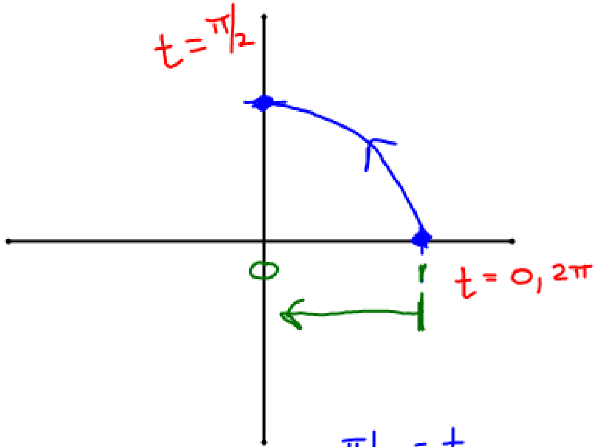
$$\frac{d}{dt}\left[x = f(t)\right]$$

$$\frac{dx}{dt} = f'(t)$$

$$dx = f'(t)\,dt$$

Example 1: Find the area of a circle defined by the equations $x = \cos t$, $y = \sin t$.

$dx = -\sin t \, dt$



$$\int_0^{2\pi} \sin(t) \cdot -\sin(t)\,dt$$

$$-\int_0^{2\pi} \sin^2 t\,dt = -\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt$$

$$= -\frac{1}{2}\int_0^{2\pi} 1-\cos(2t)\,dt$$

$$= -\frac{1}{2}\left[t - \frac{1}{2}\sin(2t)\right]_0^{2\pi}$$

$$= -\frac{1}{2}\left[(2\pi - 0) - (0-0)\right] = -\frac{1}{2}\left[2\pi\right] = \boxed{-\pi}$$

$$4\int_{0=t}^{\pi/2=t} -\sin^2 t\,dt$$

$\downarrow$

$$\int_{2\pi}^{0} \sin^2 t\,dt$$

Example 2: Find the area enclosed between the curve defined by $x = 3t + 2, y = 1 - t^2$ and the x-axis.

$dx = 3dt$

$\boxed{\text{X-INT}}$ $1 - t^2 = 0$

$t = \pm 1 \rightarrow (-1, 0)\ (5, 0)$

$$3\int_{-1}^{1} (1 - t^2)\, dt = \Big[3t - t^3\Big]_{-1}^{1} = (3-1) - (-3+1) = \boxed{4}$$

$x = 3t + 2$

$t = \frac{x-2}{3}$

$y = 1 - \left(\frac{x-2}{3}\right)^2$

$$\int_{-1}^{5} 1 - \left(\frac{x-2}{3}\right)^2 dx = \boxed{4}$$

Example 3: Find the area enclosed within the loop of the curve $x = t^2 - t, y = t^3 - 3t - 1$, if the curve crosses itself at the point $\boxed{(2, 1).}$

$dx = (2t - 1)\,dt$

$2 = t^2 - t \qquad 1 = t^3 - 3t - 1$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = -1, 2$

$$\int_{-1}^{2} (t^3 - 3t - 1)\cdot(2t - 1)\,dt$$

$$\int_{-1}^{2} (2t^4 - t^3 - 6t^2 + t + 1)\,dt$$

$$\left[\frac{2}{5}t^5 - \frac{1}{4}t^4 - 2t^3 + \frac{1}{2}t^2 + t\right] = \left(\frac{64}{5} - 4 - 16 + 2 + 2\right) - \left(-\frac{2}{5} - \frac{1}{4} + 2 + \frac{1}{2} - 1\right)$$

$$= \boxed{\frac{-81}{20}}$$