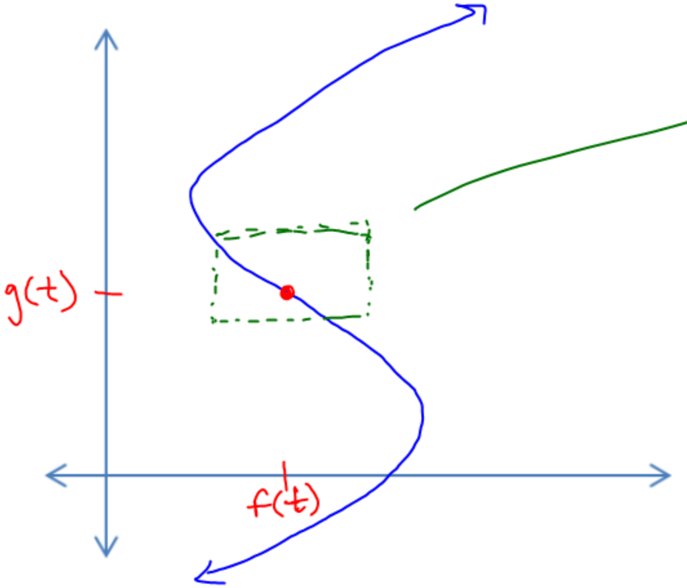
Parametric Equations and Calculus

Now that we know how to define a curve using parametric equations, it is natural to ask how we can apply calculus to parametrically defined curves. Specifically, let's look at how to differentiate parametrically defined curves

Example 1: Consider any curve defined parametrically, $x = f(t), y = g(t)$.



$$y = F(x) = F\left(f(t), g(t)\right)$$

$$\frac{d}{dt}\left[g(t) = F\left(f(t)\right)\right]$$

$$g'(t) = F'\left(f(t)\right) \cdot f'(t)$$

$$F'(x) = \frac{g'(t)}{f'(t)}$$

$$\frac{d}{dt}\left[y = F(x)\right]$$

$$\frac{dy}{dt} = F'(x) \cdot \frac{dx}{dt}$$

$$F'(x) = \frac{dy/dt}{dx/dt}$$

Parametric Form of the Derivative

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{dy/dt}{dx/dt}$$

Continuous w/ no sharp points

Example 2: Consider the curve defined by $x = \sec t$ and $y = \tan t$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Find the equation of the tangent line to the curve when $\boxed{t = \frac{\pi}{4}}$.

$x\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}}$ $\quad y\left(\frac{\pi}{4}\right) = 1$

$$\left.\begin{aligned}\frac{dy}{dt} &= \sec^2 t\\ \frac{dx}{dt} &= \sec t \tan t\end{aligned}\right\} \quad \frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{1}{\sin t} = \csc t$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2}$$

$$y - 1 = \sqrt{2}\left(x - \frac{2}{\sqrt{2}}\right)$$

Example 3: Consider the curve defined by $x = 2t - \pi \sin t$ and $y = 2 - \pi \cos t$. Find the equation of the tangent line at the point $\boxed{(0, 2)}$.

(x, y)

$\frac{dy}{dt} = \pi \sin t$

$\frac{dx}{dt} = 2 - \pi \cos t$

$\frac{dy}{dx} = \frac{\pi \sin t}{2 - \pi \cos t}$

$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{2}} = -\frac{\pi}{2}$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{\pi}{2}$

$0 = 2t - \pi \sin t$

$2t = \pi \sin t$

$t = -\frac{\pi}{2}, \frac{\pi}{2}$

$2 = 2 - \pi \cos t$

$\cos t = 0$

$t = \ldots -\cancel{\frac{5\pi}{2}}, -\cancel{\frac{3\pi}{2}}, \boxed{-\frac{\pi}{2}, \frac{\pi}{2}}, \cancel{\frac{3\pi}{2}}, \cancel{\frac{5\pi}{2}} \ldots$

$$\boxed{\begin{array}{l} y - 2 = -\frac{\pi}{2} x \\ y - 2 = \frac{\pi}{2} x \end{array}}$$

Example 4: Consider the curve defined by $x = t^2$ and $y = t^3 - 3t$. Determine where the curve has horizontal and vertical tangents. Then, find the **x** and **y** intercepts to graph the function.

Horizontal Tangents : $\frac{dy}{dx} = 0$

Vertical Tangents : $\frac{dy}{dx} = \text{UND}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 3t^2 - 3 = 3(t^2-1) = 0$

$t = \pm 1$

$\boxed{t=-1}$ $(1, 2)$

$\boxed{t=1}$ $(1, -2)$

$\frac{dy}{dx} = \text{UND} \Rightarrow \frac{dx}{dt} = 2t = 0$

$t = 0$

$\boxed{t=0}$ $(0, 0)$

$\boxed{\text{X-INT}}$ $y = 0 \rightarrow t^3 - 3t = 0$

$t(t^2-3) = 0$

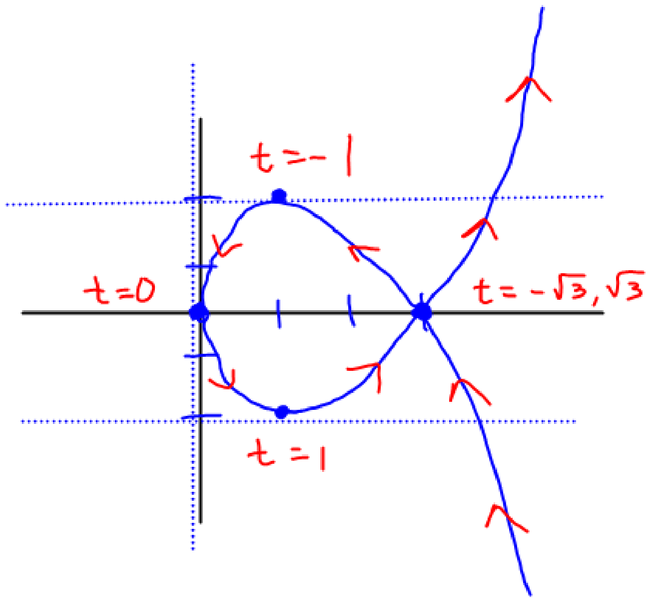
$t = \pm\sqrt{3}, 0$

$\boxed{t = -\sqrt{3}}$ $(3, 0)$

$\boxed{t = \sqrt{3}}$ $(3, 0)$

$\boxed{\text{Y-INT}}$ $x = 0 \rightarrow t^2 = 0$

$t = 0$



	$\lim_{t\to-\infty}$	$\lim_{t\to\infty}$
X	∞	∞
Y	$-\infty$	∞

Example 5: The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - 3t^2$ and $y = 2t^3 - 3t^2 - 12t$. For what values of t is the particle at rest?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \begin{array}{l} dy/dt \longrightarrow \text{Vertical Velocity} \\ dx/dt \longrightarrow \text{Horizontal Velocity} \end{array}$$

$$\frac{dy}{dx} = \frac{0}{0}$$

$$\frac{dy}{dt} = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t-2)(t+1)$$

$$\frac{dy}{dt} = 0 \Rightarrow t = -1, 2$$

$$\frac{dx}{dt} = 3t^2 - 6t = 3t(t-2)$$

$$\frac{dx}{dt} = 0 \Rightarrow t = 0, 2$$

$$\boxed{\text{At rest when } t = 2}$$

$$\downarrow$$

$$(-4, -20)$$

Example 6: A parametric curve is represented by the equations $x = t^3 - 6t$ and $y = t^2$. Find the equation of the tangent line(s) at the point(s) where the curve crosses itself.

$t_1 = A$

$t_2 = B$

$x(A) = x(B)$

$y(A) = y(B)$

$A^3 - 6A = B^3 - 6B$

$-B^3 + 6B = B^3 - 6B$

$2B^3 - 12B = 0$

$2B(B^2 - 6) =$

$B = \cancel{0},\ \pm\sqrt{6}$

$(0, 6)$

$\sqrt{A^2} = \sqrt{B^2}$

$A = \pm B$

$\cancel{A = B}$ $\boxed{A = -B}$

$dy/dt = 2t$

$dx/dx = 3t^2 - 6$

$\dfrac{dy}{dx} = \dfrac{2t}{3t^2 - 6}$

$\left.\dfrac{dy}{dx}\right|_{t=\sqrt{6}} = \dfrac{2\sqrt{6}}{12} = \dfrac{\sqrt{6}}{6}$

$\left.\dfrac{dy}{dx}\right|_{t=-\sqrt{6}} = \dfrac{-2\sqrt{6}}{12} = \dfrac{-\sqrt{6}}{6}$

$$\boxed{\begin{array}{l} y - 6 = \dfrac{\sqrt{6}}{6}x \\ y - 6 = -\dfrac{\sqrt{6}}{6}x \end{array}}$$