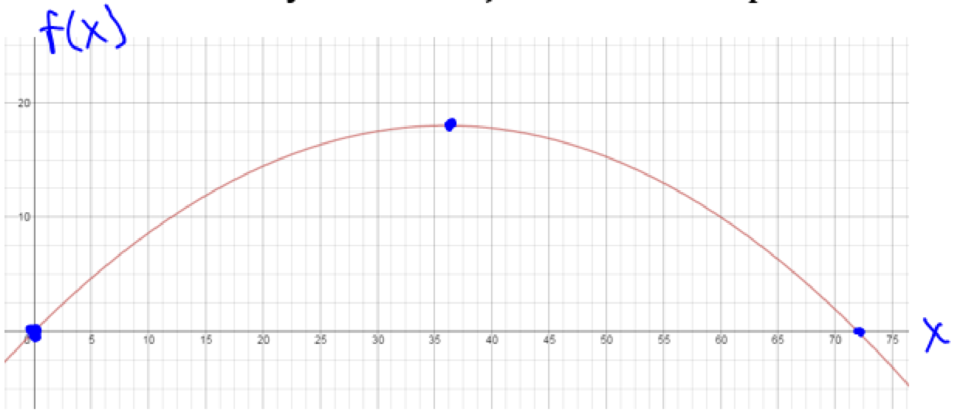
Plane Curves and Parametric Equations

Until now, we have represented a graph by a single equation involving two variables. By using a third variable to help define a function, we can obtain more information of the graphs of any curve.

Example 1: Consider the path followed by an object that is propelled into the air at an angle of 45°. If the initial velocity of the object is 48 feet per second, the object travels the parabolic path



$$\boxed{f(x) = -\frac{x^2}{72} + x}$$

Tell's us nothing about time.

What does this equation tell us?

x - horizontal distance

y - vertical distance

In order to incorporate the idea of time, we introduce a third variable, t.

Another representation for $\boldsymbol{f(x)}$: $\underline{x = 24\sqrt{2}t}$ and $\underline{y = -16t^2 + 24\sqrt{2}}$t

- The parametric equations above describe the **position of the object** in space for a particular value of **time.**
- For this particular motion problem, x and y are continuous functions of t, and the resulting path is called a **plane curve.**

Definition of Plane Curve

If f and g are continuous function of t on a given interval, then the equations

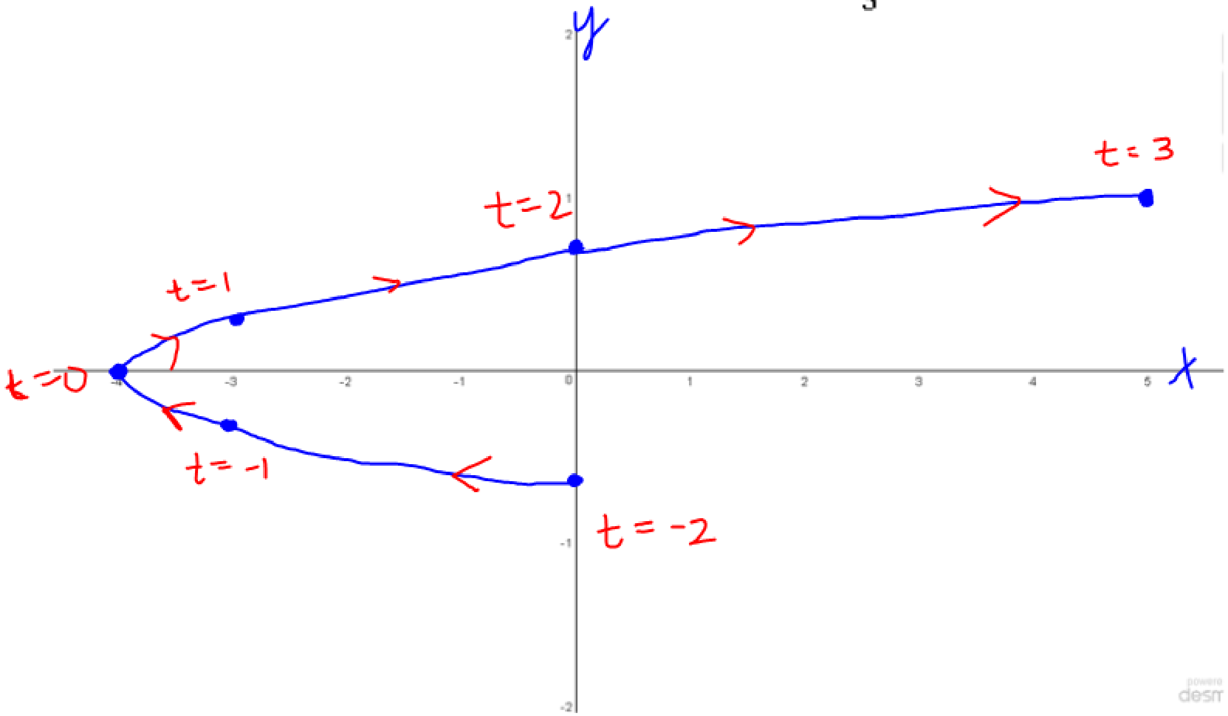
$$x = f(t) \qquad \textit{and} \qquad y = g(t)$$

are called **parametric equations** and t is called the **parameter.** There is a distinction between a graph (the set of points) and a curve (the points together with their defining parametric equations).

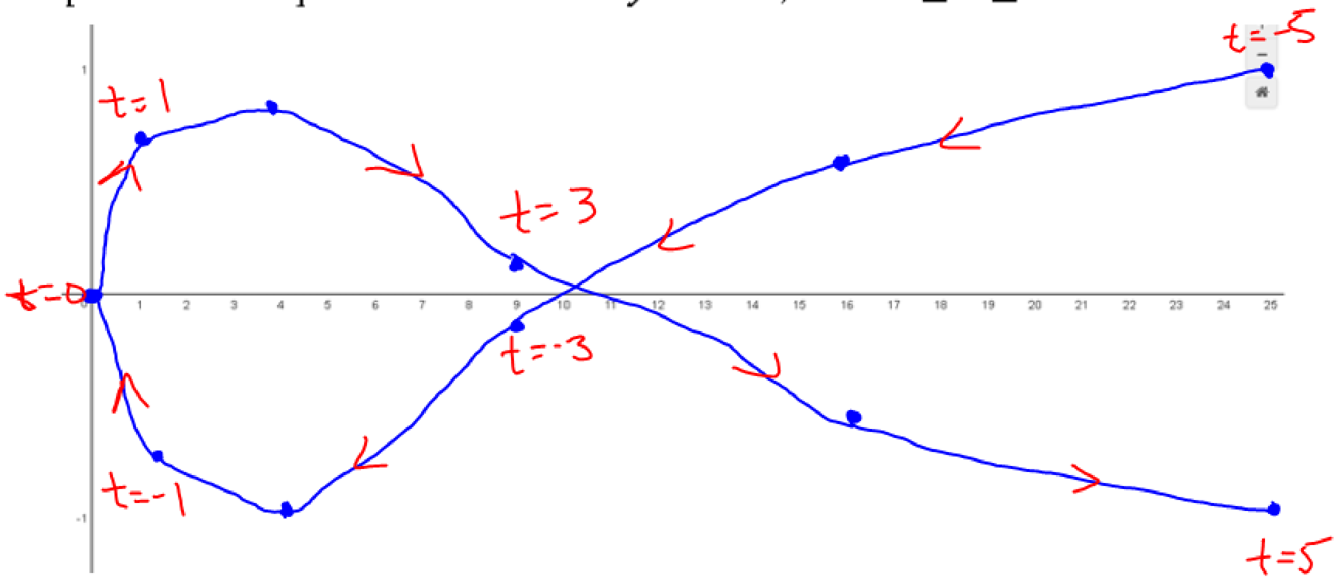
When sketching a curve represented by parametric equations, it is important to plot the coordinates using **increasing values of t**, which traces the curve in a **specific direction**.

Example 2: Sketch the curve described by the parametric equations $x = t^2 - 4$ and $y = \frac{t}{3}$, for $-2 \le t \le 3$.

t	x	y
-2	0	-2/3
-1	-3	-1/3
0	-4	0
1	-3	1/3
2	0	2/3
3	5	1



Example 3: Sketch the curve described by the parametric equations $x = t^2$ and $y = \sin t$, for $-5 \leq t \leq 5$



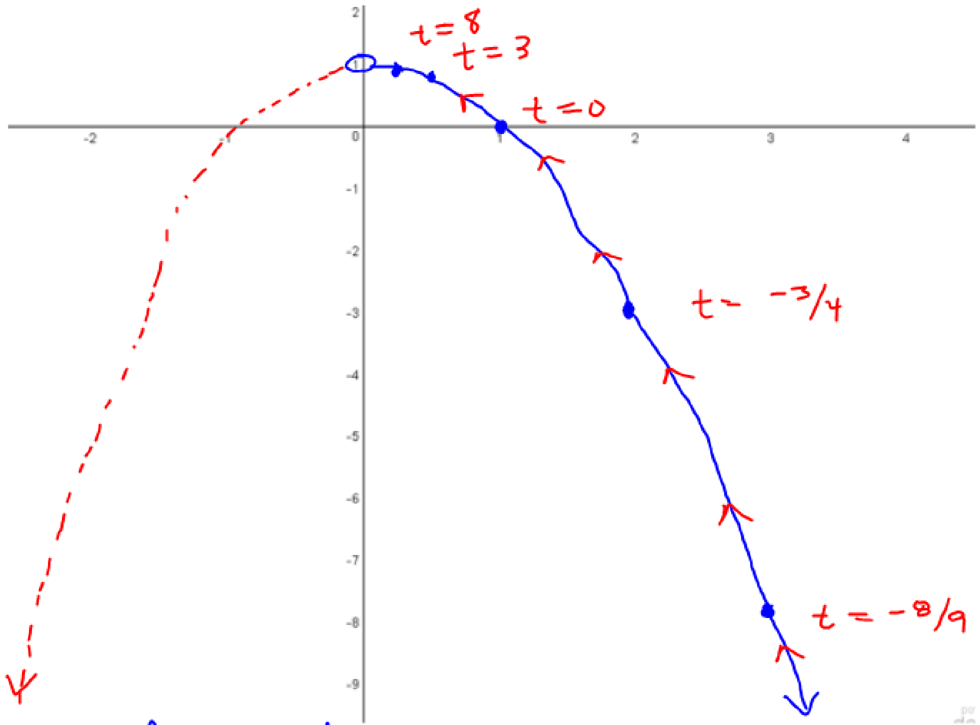
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
x	25	16	9	4	1	0	1	4	9	16	25
y	.959	.757	-.141	-.909	-.841	0	.841	.909	.141	-.757	-.959

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter.**

$t > -1$

Example 4: Sketch the curve represented by the equations $x = \frac{1}{\sqrt{t+1}}$ and $y = \frac{t}{t+1}$.

t	x	y
$-8/9$	3	-8
$-3/4$	2	-3
0	1	0
3	$1/2$	$3/4$
8	$1/3$	$8/9$



$\boxed{x}$ $\lim_{t \to \infty} \frac{1}{\sqrt{t+1}} = 0$ $\lim_{t \to -1^+} \frac{1}{\sqrt{t+1}} = \frac{1}{\text{small pos}} = \infty$

$\boxed{y}$ $\lim_{t \to \infty} \frac{t}{t+1} = 1$ $\lim_{t \to -1^+} \frac{t}{t+1} = \frac{-1}{\text{small pos}} = -\infty$

$x = \frac{1}{\sqrt{t+1}}$ $y = \frac{t}{t+1}$

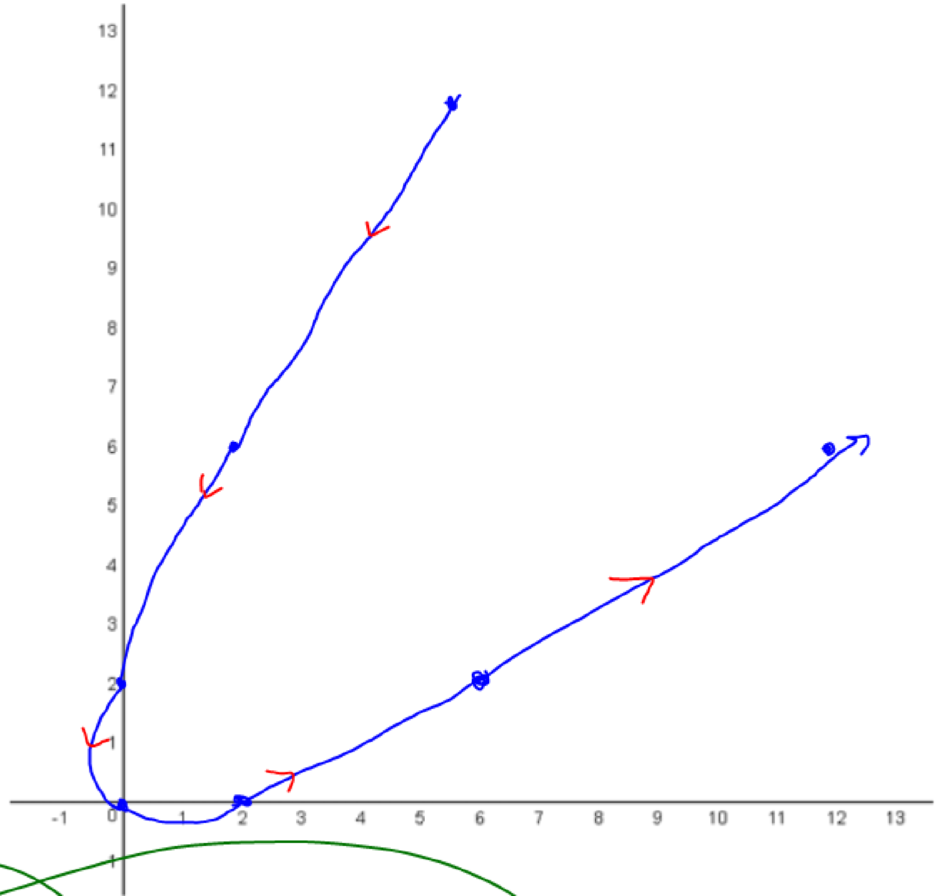
$\frac{1}{x} = \sqrt{t+1}$

$\frac{1}{x^2} - 1 = t$

$y = \dfrac{\frac{1}{x^2} - 1}{\frac{1}{x^2}} = 1 - x^2$

Example 5: Sketch the curve and write the corresponding rectangular equation by eliminating the parameter for $x = t^2 + t$ and $y = t^2 - t$

t	x	y
-3	6	12
-2	2	6
-1	0	2
0	0	0
1	2	0
2	6	2
3	12	6



$\boxed{x = t^2 + t}$ $\boxed{y = t^2 - t}$

$x = \frac{(x-y)^2}{4} + \frac{x-y}{2}$

$y = \frac{(x-y)^2}{4} - \frac{x-y}{2}$

$x = t^2 + t$
$-(y = t^2 - t)$

$x - y = 2t$

$\frac{x-y}{2} = t$

2011 Question 1 (Calculator Active)

For $0 \le t \le 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

(a) Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

(b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

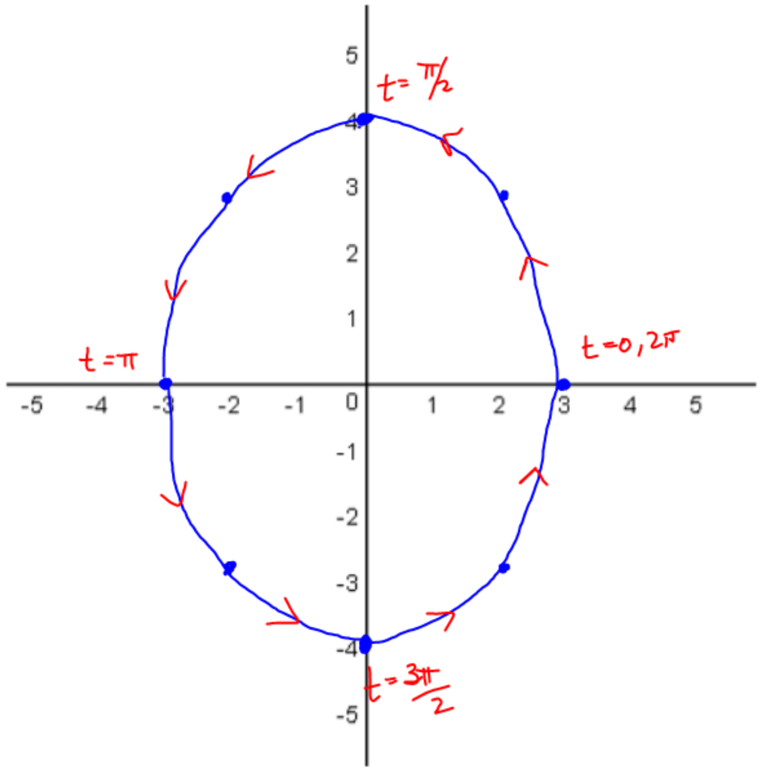
(c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

(d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

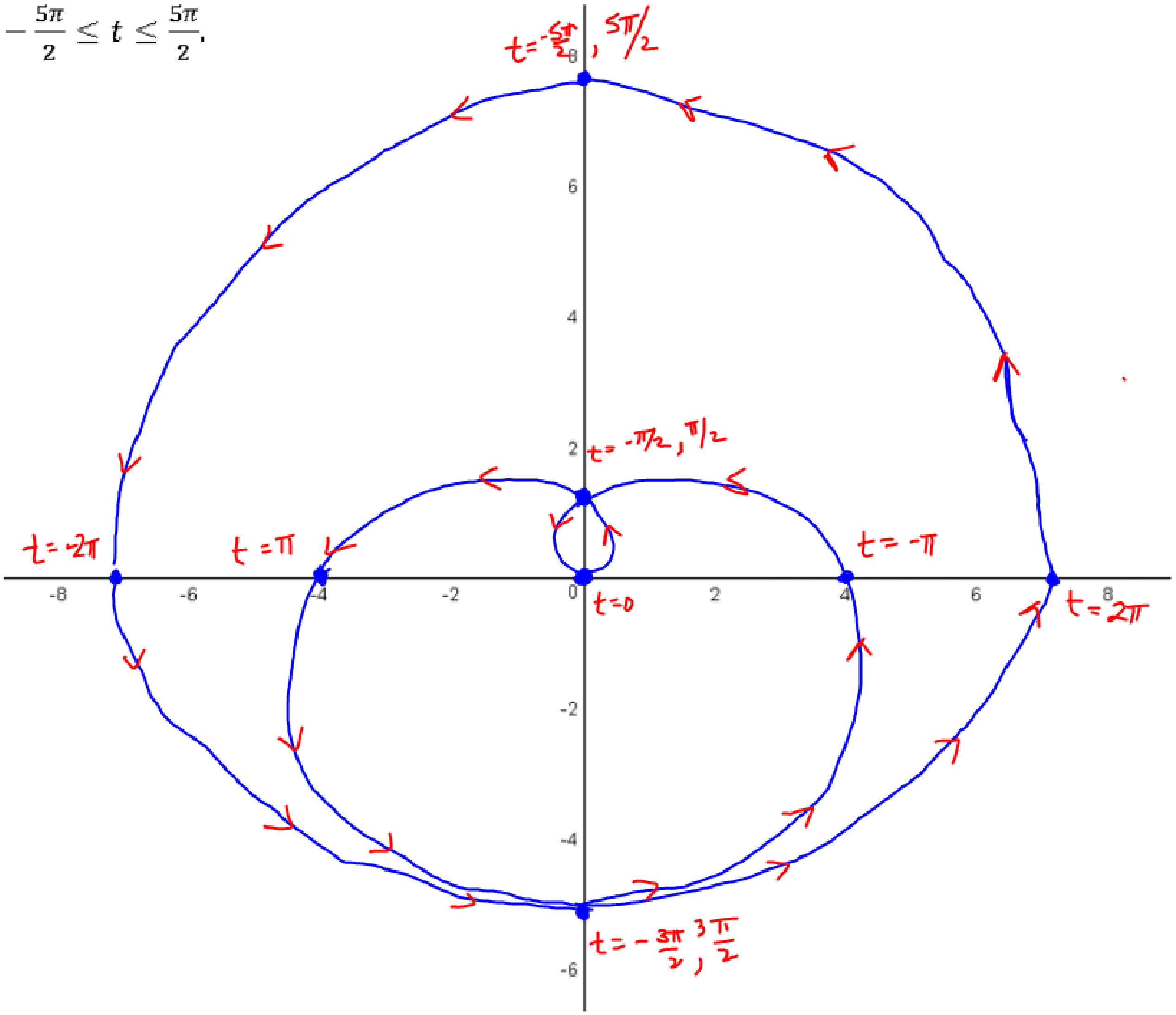
(a) $v(5.5) = -0.45337,\ a(5.5) = -1.35851$ The speed is increasing at time $t = 5.5,$ because velocity and acceleration have the same sign.	2 : conclusion with reason
(b) Average velocity $= \frac{1}{6}\int_0^6 v(t)\,dt = 1.949$	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(c) Distance $= \int_0^6 \lvert v(t) \rvert\,dt = 12.573$	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) $v(t) = 0$ when $t = 5.19552$. Let $b = 5.19552$. $v(t)$ changes sign from positive to negative at time $t = b$. $x(b) = 2 + \int_0^b v(t)\,d\,t = 14.134$ or 14.135	$3 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Example 6: Sketch the curve represented by $x = 3\cos t$ and $y = 4\sin t$, for $0 \le t \le 2\pi$.

t	x	y
0	3	0
$\pi/4$	$3\sqrt{2}/2$	$2\sqrt{2}$
$\pi/2$	0	4
$3\pi/4$	$-3\sqrt{2}/2$	$2\sqrt{2}$
π	-3	0
$5\pi/4$	$-\frac{3\sqrt{2}}{2}$	$-2\sqrt{2}$
$3\pi/2$	0	-4
$7\pi/4$	$\frac{3\sqrt{2}}{2}$	$-2\sqrt{2}$
2π	3	0



Example 7: Use the graphing calculator to sketch the curve described by the parametric equations $x = t\cos t$ and $y = t\sin t$, for $-\frac{5\pi}{2} \le t \le \frac{5\pi}{2}$.



Parametric Equations Continued

General Parametric Form of a Circle:

Let f *and* g be differentiable functions of t, and $a > 0$ then

$\underline{x = f(t) = \boldsymbol{a}\cos \boldsymbol{t}}$ and $\underline{y = g(t) = \boldsymbol{a}\sin \boldsymbol{t}}$ will form a circle with a radius of a units.

Example 1: Write the rectangular equation of each of the following parametric equations.

a. $x = 4\cos t \quad y = 4\sin t$

$\frac{x}{4} = \cos t \quad \frac{y}{4} = \sin t$

$\cos^2 t + \sin^2 t = 1$

$16 \cdot \left[\frac{x^2}{16} + \frac{y^2}{16} = 1\right]$

$x^2 + y^2 = 16$

b. $x = 4 + 2\cos t \quad y = -2 + 2\sin t$

$\frac{x-4}{2} = \cos t \quad \frac{y+2}{2} = \sin t$

$\cos^2 t + \sin^2 t = 1$

$\frac{(x-4)^2}{4} + \frac{(y+2)^2}{4} = 1$

$(x-4)^2 + (y+2)^2 = 4$

General Parametric Form of an Ellipse;

Let $f\ \ and\ \ g$ be differentiable functions of t, then

1. If $\underline{0 < a < b}$, then $\underline{x = a\cos t \quad and\ y = b\sin t}$ will result in an ellipse with a **<u>vertical major axis.</u>**

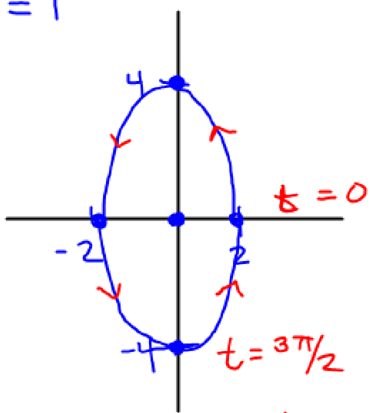
2. If $\underline{0 < b < a}$, then $\underline{x = a\cos t \quad and\ y = b\sin t}$ will result in an ellipse with a **<u>horizontal major axis.</u>**

Example 2: Write the rectangular equation of each of the following parametric equations.

a. $x = 2\cos t \qquad y = 4\sin t$

$\frac{x}{2} = \cos t \qquad \frac{y}{4} = \sin t$

$\frac{x^2}{4} + \frac{y^2}{16} = 1$



x y
$(2, 0)$

$2\cos t = 2$
$\cos t = 1$
$t = 0, 2\pi, \ldots$

$4\sin t = 0$
$\sin t = 0$
$t = 0, \pi, 2\pi \ldots$

x y
$(0, -4)$

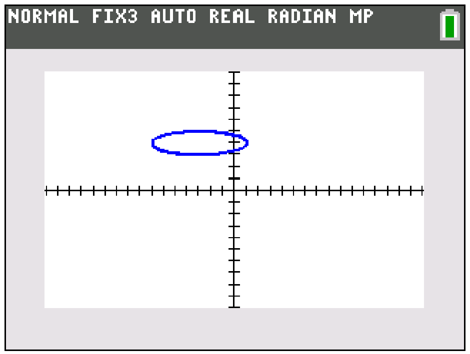
$2\cos t = 0$
$\cos t = 0$
$t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$4\sin t = -4$
$\sin t = -1$
$t = \frac{3\pi}{2}$

b. $x = -3 + 4\cos t \qquad y = 4 + \sin t$

$\frac{x+3}{4} = \cos t \qquad y - 4 = \sin t$

$\frac{(x+3)^2}{16} + \frac{(y-4)^2}{1} = 1$



What about other parametric equations involving trig?

Example 3: $x = \underline{\cos 2t}$ $\boxed{y = \sin t}$ on $[0, 2\pi]$

$\boxed{x = 1}$

$\cos 2t = 1$

$2t = -4\pi, -2\pi, 0, 2\pi, 4\pi$

$t = \ldots -2\pi, -\pi, 0, \pi, 2\pi$

$\boxed{x = -1}$

$\cos 2t = -1$

$2t = -3\pi, -\pi, \pi, 3\pi$

$t = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$\boxed{y = 1}$

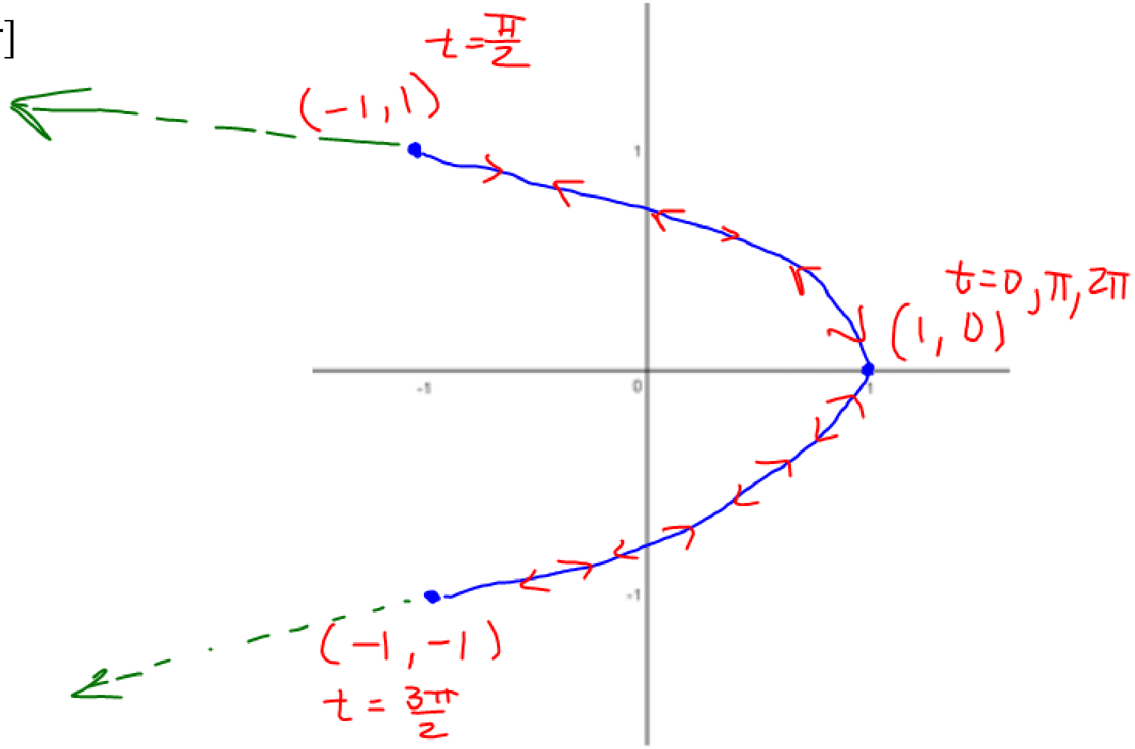
$\sin t = 1$

$t = \frac{\pi}{2}, \frac{5\pi}{2}$

$\boxed{y = -1}$

$\sin t = -1$

$t = \frac{3\pi}{2}, \frac{7\pi}{2}$



$\cos 2t = 1 - 2\sin^2 t$

$x = 1 - 2y^2$

$$\boxed{x - 1 = -2y^2}$$

Example 4: $x = \sin t \quad y = \sin 2t \quad$ on $[0, 2\pi]$

$\sin 2t = 2\sin t \cos t$

$y = 2x \cos t$

$\boxed{\frac{y}{2x} = \cos t}$

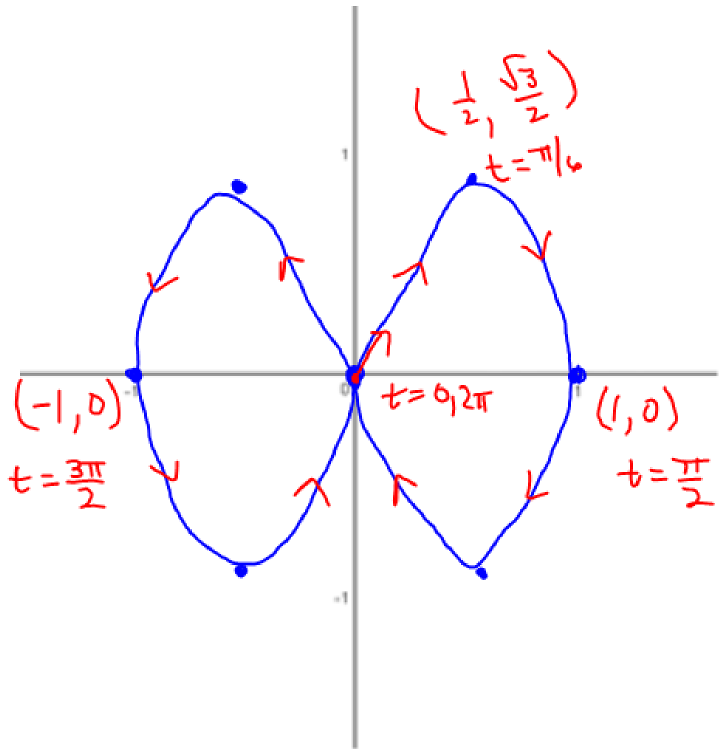
$\sin^2 t + \cos^2 t = 1$

$x^2 + \frac{y^2}{4x^2} = 1$

$\frac{y^2}{4x^2} = 1 - x^2$

$\sqrt{y^2} = \sqrt{4x^2 - 4x^4}$

$\boxed{y = \pm 2x\sqrt{1 - x^2}}$



Example 5: $x = \sec t \qquad y = \tan t \quad \text{on } [0, 2\pi]$

$1 + \tan^2 t = \sec^2 t$

$1 + y^2 = x^2$

$x^2 - y^2 = 1$