Taylor and Maclaurin Series

Putting together some ideas we have discussed, we can relate the power series we have talked about to the Taylor polynomials we developed to approximate functions.

Definition of Taylor and Maclaurin Series

If a function f has **derivatives of all orders** at $x = c$, then the series

$$\sum_{n=0}^{\infty} f^{(n)}(c)\,\frac{(x-c)^n}{n!}$$

is called the **Taylor series** for $f(x)$ at c. Moreover, if $c = 0$, then the series is the **Maclaurin series** for f.

- If a power series **converges to *f(x)*,** the series must be a **Taylor series**.

Common Maclaurin Series (MEMORIZE)

Function	Interval of Convergence:
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$	$-1 < x < 1$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

* On page 682, there is an expanded table of power series that provide more common power series to be familiar with.

We can use this basic list to make find power/Maclaurin series simpler.

Example 1: Use the function $f(x) = \sin(x^2)$ to form a Maclaurin Series and determine the interval of convergence.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(☺) = ☺ - \frac{(☺)^3}{3!} + \frac{(☺)^5}{5!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$-\infty < x < \infty$$

Example 2: Find the Maclaurin Series for $f(x) = \cos\sqrt{x}$. What is the interval of convergence?

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

$$\cos(\sqrt{x}) = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} + \dots$$

$$= 1 - \frac{x}{2!} + \frac{x^2}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

$$0 \le x < \infty$$

Example 3: Find the power series for $f(x) = x^2 e^{3x}$. What is the interval of convergence?

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(3x) = e^{(3x)} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n \cdot x^n}{n!}$$

$$x^2 \cdot e^{3x} = x^2\left(1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right) = x^2 \cdot \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

$$= x^2 + 3x^3 + \frac{9x^4}{2!} + \frac{27x^5}{3!} + \dots = \sum_{n=0}^{\infty} \frac{3^n \cdot x^{n+2}}{n!}$$

$e^{(3x \cdot x^2)}$

$$= x^2 + 3x^3 + \frac{9}{2}x^4 + \frac{9}{2}x^5 + \dots$$

Example 4: Use the identity: $\sin^2 x = \boxed{\frac{1-\cos 2x}{2}}$ to find the power series for $f(x) = \sin^2 x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$1 - \cos 2x = 1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)$$

$$= \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots$$

$$\frac{1-\cos 2x}{2} = \underset{n=1}{\frac{2x^2}{2!}} - \underset{n=2}{\frac{2^3 \cdot x^4}{4!}} + \underset{n=3}{\frac{2^5 \cdot x^6}{6!}} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} \cdot x^{2n}}{(2n)!}$$

$$-\infty < x < \infty$$

$$\sin^2 x = \sin x \cdot \sin x$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= x^2 - \frac{x^4}{3!} - \frac{x^4}{3!} + \frac{x^6}{5!} + \frac{x^6}{5!} + \frac{x^6}{(3!)^2} + \cdots$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$$

Example 6: Use a power series to approximate

$$\int_0^1 e^{-x^2}\,dx$$

with an error less than 0.01.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad -\infty < x < \infty$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \qquad -\infty < x < \infty$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots\right) dx$$

$$= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots\right]_0^1$$

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} + \cdots$$

Alternating Series:

$$|R_n| \le a_{n+1} < \frac{1}{100}$$

$$\frac{1}{216} < \frac{1}{100}$$

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = 0.7428571429$$

$$\int_0^1 e^{-x^2}\,dx = 0.746824132\text{8}$$

Multiplication and Division of Power Series

We can multiple and divided power series just like polynomials.

Example 7: Find the first three nonzero terms of each Maclaurin Series.

4 terms

a. $\frac{e^x}{1-x}$ $= e^x \cdot \frac{1}{1-x}$

$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + x^2 + x^3 + x^4 + \cdots\right)$

$= 1 + (x + x) + \left(x^2 + \frac{x^2}{2!} + x^2\right) + \left(x^3 + \frac{x^3}{3!} + x^3 + \frac{x^3}{2!}\right)$

$= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3$

3 terms

b. $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$

$$\begin{array}{r|l}
 & x + \frac{x^3}{3} + \frac{2x^5}{15} \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} & x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \\
 & -\left(x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots\right) \\
\hline
 & \frac{x^3}{3} - \frac{x^5}{30} + \frac{6x^7}{5040} \\
 & -\left(\frac{x^3}{3} - \frac{x^5}{6} + \frac{x^7}{72}\right) \\
\hline
 & \frac{2x^5}{15} \quad \cdots
\end{array}$$

$$\boxed{\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15}}$$