Representing Function by Power Series

There are several ways for finding a power series that represents a given function. Let's begin by looking at the function:

$f(x) = \frac{1}{1-x}$ This closely resembles the sum of a geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Example 1: Find a power series representation for $f(x)$ centered at 0.

$\frac{a}{1-r} = \frac{1}{1-x}$ $\quad a = 1$, $r = x$

$$\boxed{f(x) = \sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + x^3 + \cdots$$

$|x| < 1 \longrightarrow -1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$

$\boxed{x = -1}$ $\sum (-1)^n \rightarrow$ Diverge by n^{th} term test

$\boxed{x = 1}$ $\sum (1)^n \rightarrow$

- The above power series only represented our function on the interval of convergence stated above. To represent $f(x)$ on another interval, we can develop a different series.

Example 2: Find a power series representation for $f(x) = \frac{1}{1-x}$ centered -1. and centered at 2.

$(x+1)$ $(x-2)$

$$\frac{a}{1-r} = \frac{1}{1-x} = \frac{1}{2-(x+1)} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1/2}{1-\left(\frac{x+1}{2}\right)}$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n \quad (r = \tfrac{x+1}{2})$$

$$\left|\frac{x+1}{2}\right| < 1$$

$$|x+1| < 2$$

I.O.C: $-3 < x < 1$

$\boxed{x=-3}$ $\sum \frac{1}{2}(-1)^n$

$\boxed{x=1}$ $\sum \frac{1}{2}$

Both diverge by n^{th} term

$$\frac{a}{1-r} = \frac{1}{1-x} = \frac{1}{-1-(x-2)} = \frac{-1}{1-(-(x-2))}$$

$$\sum_{n=0}^{\infty} -(-(x-2))^n$$

$$|-(x-2)| < 1$$

$$|x-2| < 1$$

I.O.C: $1 < x < 3$

$\boxed{x=1}$ $\sum -1$

$\boxed{x=3}$ $\sum -(-1)^n$

Both Diverge by n^{th} term

Example 3: Find power series for $f(x) = \frac{4}{x+2}$, centered at 0 and centered at 2.

(x) $(x-2)$

$$\frac{a}{1-r} = \frac{4}{2+x} = \frac{4}{2-(-x)} = \frac{2}{1-\left(\frac{-x}{2}\right)}$$

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{2}\right)^n$$

$$\left|\frac{x}{2}\right| < 1$$

$$|x| < 2$$

I.o.c: $-2 < x < 2$

$$\frac{a}{1-r} = \frac{4}{2+x} = \frac{4}{4-(-(x-2))} = \frac{1}{1-\left(\frac{-(x-2)}{4}\right)}$$

$$\sum_{n=0}^{\infty} \left(\frac{-(x-2)}{4}\right)^n$$

$$\left|\frac{-(x-2)}{4}\right| < 1$$

$$|x-2| < 4$$

I.o.c: $-2 < x < 6$

Example 4: Find a power series, centered at 0, for $f(x) = \frac{3x-1}{x^2-1}$. $= \frac{3x-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$$3x-1 = (A+B)x + (-A+B)$$

$$A+B = 3$$
$$-A+B = -1$$
$$2B = 2$$
$$\boxed{B=1} \quad \boxed{A=2}$$

$$\frac{a}{1-r}$$

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$$

$$= \frac{2}{1+x} + \frac{1}{-1+x}$$

$$= \frac{2}{1-(-x)} + \frac{-1}{1-x}$$

$$= \sum_{n=0}^{\infty} 2(-x)^n + \sum_{n=0}^{\infty} -1(x)^n$$

$|-x| < 1$ $\quad |x| < 1$ $\quad -1 < x < 1$

$|x| < 1$ $\quad -1 < x < 1$

$$\frac{3x-1}{x^2-1} = \sum_{n=0}^{\infty} 2(-x)^n - (x)^n \text{ for } -1 < x < 1.$$

$$x^n\left[(-1)^n \cdot 2 - 1\right]$$

2014 Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Let a_n be the nth term of the Taylor series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}\, 2^{n+1}\,(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}\, 2^n\,(x-1)^n}$$
$$= \frac{-2n(x-1)}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{-2n(x-1)}{n+1}\right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

(b) The first three nonzero terms are
$2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1}\, 2^n\,(x-1)^{n-1}$ for $n \geq 1$.

$3 : \begin{cases} 2 : \text{first three nonzero terms} \\ 1 : \text{general term} \end{cases}$

(c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1-(-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$
$$f(x) = \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$$f(1) = 0$$
$$\ln|1| + C = 0 \Rightarrow C = 0$$
$$f(x) = \ln|2x-1| \text{ for } |x-1| < \frac{1}{2}$$

$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{antiderivative} \\ 1 : f(x) \end{cases}$

Finding a Power Series by Integration

Example 5: Find a power series for $\boxed{f(x) = \ln x,}$ centered at 1.

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$(x-1)$

$$\frac{a}{1-r} = \frac{1}{x} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (\underbrace{-(x-1)}_{r})^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$\to |x-1| < 1$

$0 < x < 2$

$$\ln x = \int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} + \underbrace{C}$$

$f(1) = 0 \Rightarrow C = 0$

$$\boxed{\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \quad \text{for} \quad 0 < x \le 2}$$

$\boxed{x=0}$ $\sum \frac{-1}{n+1}$ Divergent General Harmonic (by Limit Comp w/ $\frac{1}{n}$)

$\boxed{x=2}$ $\sum \frac{(-1)^n}{n+1}$ Convergent Alt General Harmonic

Example 6: Find a power series for $g(x) = \arctan x$, centered at 0.

$g(0) = 0$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$C = 0$

$$\frac{a}{1-r} = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$|x^2| < 1$

$|x| < 1$

$-1 < x < 1$

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$(x^2)^n \cdot x$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1}$$

$\boxed{x=-1}$ $\sum \frac{(-1)^{n+1}}{2n+1}$ → Converge by Alt Series Test

$\boxed{x=1}$ $\sum \frac{(-1)^n}{2n+1}$ →

Finding a Power Series by Differentiation

$|x| < 1$

$-1 < x < 1$

$(1-x)^{-1}$

Example 7: Find a power series for the following given that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

a. $\dfrac{1}{(1-x)^2}$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \quad \text{for} \quad -1 < x < 1}$$

b. $\dfrac{1}{1+x}$

$$\frac{a}{1-r} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

c. $\frac{1}{(1+x)^3}$

$$\frac{d}{dx}\left[\frac{1}{1+x}\right] = \frac{-1}{(1+x)^2} \qquad \frac{d}{dx}\left[\frac{-1}{(1+x)^2}\right] = \frac{2}{(1+x)^3}$$

$$\frac{d}{dx}\left[\frac{1}{1+x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=0}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$$

$$\frac{d}{dx}\left[\frac{-1}{(1+x)^2}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n\right] = \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)n x^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$$

$$\frac{1}{2}\cdot\frac{2}{(1+x)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)\cdot x^n$$

$$\boxed{\frac{1}{(1+x)^3} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^n}{2} \quad \text{for } -1<x<1}$$

d. $\frac{x^2}{(1+x)^3}$ $= x^2\cdot\frac{1}{(1+x)^3}$

$$x^2\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)(x)^n}{2}$$

$$\boxed{\frac{x^2}{(1+x)^3} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^{n+2}}{2} \quad \text{for } -1<x<1}$$

Let f be the function given by $f(x) = \frac{1}{1+2x}$.

(a)Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b)Find the radius and interval of convergence for the Maclaurin series for f.

(c)Let $P_3(x)$ be the third degree Taylor polynomial for g about $x = 0$. If $g(x) = \int_0^x f(t)dt$, then find the value of $P_3\left(\frac{1}{4}\right)$.

(d)Use the function f to find the first four non zero terms and the general term of the Maclaurin series for the function $h(x) = \frac{1}{(1+2x)^2}$.