Power Series: Interval of Convergence

Previously, we found the radius of convergence for power series. The work that we did, gave us an idea of the x-values for which a power series converged. The theorem previously used said nothing about convergence at the endpoints of the interval.

- Each endpoint must be tested separately for convergence or divergence.
- The interval of convergence can be open on both ends, closed on both ends, or open on one end and closed on the other.

Example 1: Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

$$\lim_{n\to\infty} \left| \sqrt[n]{\frac{x^n}{n}} \right|$$

$$|x| \cdot \lim_{n\to\infty} \left| \frac{1}{\sqrt[n]{n}} \right| = |x|$$

$$|x| < 1$$

$$\downarrow$$

$$R = 1$$

$\boxed{x = -1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Convergent Alternating Harmonic

$\boxed{x = 1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ Divergent Harmonic

Interval of Convergence: $-1 \le x < 1$

$[-1, 1)$

Example 2: Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{-(x+1)}{2}\right)^n$ $\quad \sum ar^n$

$$|r| = \left|\frac{x+1}{2}\right| < 1$$

$$|x+1| < 2$$

$$R = 2$$

$\boxed{x=-3}$ $\displaystyle\sum \frac{(-1)^n(-2)^n}{2^n} = \sum\left(\frac{-1\cdot -2}{2}\right)^n = \sum 1^n$

Diverges by n^{th} term test

$\boxed{x=1}$ $\displaystyle\sum \frac{(-1)^n(2)^n}{2^n} = \sum (-1)^n$

Diverges by n^{th} term test

Interval of Convergence : $-3 < x < 1$

$(-3, 1)$

Example 3: $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

$$\lim_{n\to\infty} \left| \frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n\sqrt{n}}{(2x-1)^n} \right|$$

$$\left|\frac{2x-1}{5}\right| \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \left|\frac{2x-1}{5}\right|$$

$$\left|\frac{2x-1}{5}\right| < 1$$

$$|2x-1| < 5$$

$$|x - 1/2| < \frac{5}{2}$$

$$\downarrow$$

$$R = \frac{5}{2}$$

$\boxed{x=-2}$ $\displaystyle\sum \frac{(-5)^n}{5^n\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$

Converges by Alt. Series Test

$\boxed{x=3}$ $\displaystyle\sum \frac{5^n}{5^n\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$

Divergent P-series

INTERVAL of Convergence : $-2 \le x < 3$

$[-2, 3)$

We can examine the characteristics of a power series by looking at differentiation and integration of a power series.

Properties of Functions Defined by Power Series

If the function, given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \ldots$$

1. $f'(x) = a_1 + 2 \cdot a_2(x-c) + 3 \cdot a_3(x-c)^2 + \ldots$

$$= \sum_{n=1}^{\infty} n \cdot a_n (x-c)^{n-1}$$

2. $\int f(x)dx = a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \ldots$

$$= \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1}$$

Considering the function and its derivative, how are the radius of convergence and interval of convergence related?

Example 4: Consider the function given by $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ $= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$\frac{1}{n} \cdot x^n$

Find the radius and interval of convergence for each of the following.

a. $f(x)$

I.O.C:

$-1 \le x < 1$

b. $\int f(x)dx$

$= \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots$

$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$

$\lim_{n\to\infty} \left| \frac{x^{n+2}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^{n+1}} \right|$

$|x| \cdot \lim_{n\to\infty} \left| \frac{n}{n+2} \right| = |x|$

$|x| < 1$

$\boxed{x=-1}$ $\sum \frac{(-1)^{n+1}}{n(n+1)}$ Convergent by Alt Series Test

$\boxed{x=1}$ $\sum \frac{1}{n^2+n}$ Converges by Limit comp w/ $b_n = \frac{1}{n^2}$

I.O.C: $-1 \le x \le 1$

c. $f'(x)$

$= 1 + x + x^2 + x^3 + \cdots$

$= \sum_{n=0}^{\infty} x^n$

Geometric: $|x| < 1$

$\boxed{x=-1}$ $\sum (-1)^n$

$\boxed{x=1}$ $\sum 1^n$

Diverge by n^{th} term test

I.O.C: $-1 < x < 1$

* Differentiating / Integrating a power series results in a series with an equal Radius of Convergence, but may change the interval of convergence.

2016 BC Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence.

It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by

$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1)=1,\ f'(1)=-\frac{1}{2},\ f''(1)=\frac{1}{2^2},\ f'''(1)=-\frac{2}{2^3}$ $f(x)=1-\frac{1}{2}(x-1)+\frac{1}{2^2\cdot 2}(x-1)^2-\frac{1}{2^3\cdot 3}(x-1)^3+\cdots$ $+\frac{(-1)^n}{2^n\cdot n}(x-1)^n+\cdots$	$4:\begin{cases}1: \text{first two terms}\\ 1: \text{third term}\\ 1: \text{fourth term}\\ 1: \text{general term}\end{cases}$
(b) $R=2$. The series converges on the interval $(-1,3)$. When $x=-1$, the series is $1+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$. Since the harmonic series diverges, this series diverges. When $x=3$, the series is $1-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\cdots$. Since the alternating harmonic series converges, this series converges. Therefore, the interval of convergence is $-1<x\le 3$.	$2:\begin{cases}1: \text{identifies both endpoints}\\ 1: \text{analysis and interval of convergence}\end{cases}$
(c) $f(1.2)\approx 1-\frac{1}{2}(0.2)+\frac{1}{8}(0.2)^2=1-0.1+0.005=0.905$	1 : approximation
(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0. $\lvert f(1.2)-T_2(1.2)\rvert\le\left\lvert\frac{-1}{2^3\cdot 3}(0.2)^3\right\rvert=\frac{1}{3000}\le 0.001$	$2:\begin{cases}1: \text{error form}\\ 1: \text{analysis}\end{cases}$

2005 BC Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $P_6(x) = 7 + \frac{1!}{3^2} \cdot \frac{1}{2!}(x-2)^2 + \frac{3!}{3^4} \cdot \frac{1}{4!}(x-2)^4 + \frac{5!}{3^6} \cdot \frac{1}{6!}(x-2)^6$

$$3 : \begin{cases} 1 : \text{polynomial about } x = 2 \\ 2 : P_6(x) \\ \quad \langle -1 \rangle \text{ each incorrect term} \\ \quad \langle -1 \rangle \text{ max for all extra terms,} \\ \quad\quad + \cdots, \text{ misuse of equality} \end{cases}$$

(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty} \frac{1}{2n \cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{2(n+1)} \cdot \frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n} \cdot \frac{1}{3^{2n}}(x-2)^{2n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2n}{2(n+1)} \cdot \frac{3^{2n}}{3^2 3^{2n}}(x-2)^2 \right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of} \\ \quad \text{interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis/conclusion for} \\ \quad \text{both endpoints} \end{cases}$$