Power Series

Previously we have been using polynomials to approximate functions. We have seen that the higher degree of the polynomial used, the more accurate the approximation becomes.

- When considering taking a polynomial's approximation **to infinity,** this is an infinite series, which **exactly** represents the function in question.

**Definition of Power Series**

If $x$ is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is called a **power series** centered at 0. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n \boxed{(x-c)}^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$$

is called a **power series centered at c**, where c is a constant.

Notice that a power series is actually a **function of x**, much different from the series we have studied previously. Depending on the value of x plugged in, the series may converge or diverge.

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$. $= a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$

- The **domain** of *f* is the set of all x values for which the power series **converges.**

- Every power series **converges at its center *c***. Consider $f(c) = a_0 + a_1(0) + a_2(0)^2 + \cdots = a_0$

**Convergence of a Power Series**

For a power series centered at c, precisely one of the following is true.

1. The series converges **only at c.**
2. There exists a real number $R > 0$ such that the series **converges absolutely for $\boxed{|x-c| < R}$**, and **diverges for $|x-c| > R$.**
3. The series converges absolutely **for all x.**

The number $R$ is the **radius of convergence** of the power series.

- If the series converges only at c, the radius of convergence is $\mathbf{R = 0}$.
- If the series converges for all x, the radius of convergence is $\mathbf{R = \infty}$.
- The set of all values of x for which the power series converges is the **interval of convergence.**

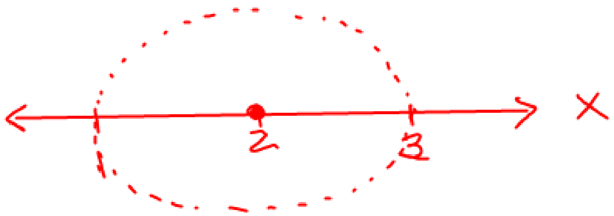
**Example 1:** Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$

Geometric $\sum ar^n \longrightarrow$ Converges when $|r| < 1$

$\sum_{n=0}^{\infty} 3(x-2)^n$ Converges when $|x-2| < 1$

- $R = 1$

★ • Interval of Convergence: $1 < x < 3$



**Example 2:** Find the radius of convergence of $\sum_{n=1}^{\infty} n!\,(2x+1)^n = \sum_{n=1}^{\infty} n! \cdot 2^n \left(x+\frac{1}{2}\right)^n$

$c = -\frac{1}{2}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\,\cancel{(2x+1)^n}\cdot(2x+1)}{\cancel{n!}\,\cancel{(2x+1)^n}}\right|$$

$$|2x+1|\cdot\lim_{n\to\infty}|n+1| = |2x+1|\cdot\infty = \infty > 1$$

$\therefore R = 0$, $\sum_{n=1}^{\infty} n!\,(2x+1)^n$ converges @ $x = \frac{-1}{2}$

**Example 3:** Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$|x^2| \cdot \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \right| = |x^2| \cdot 0 = 0 < 1$$

$$\therefore R = \infty$$

**Example 4:** Find the radius of convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{n(x+3)^n}{4^n}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n(x+3)^n}{4^n}\right|}$$

$$\lim_{n\to\infty} \left|\frac{\sqrt[n]{n}\,(x+3)}{4}\right|$$

$$\left|\frac{x+3}{4}\right| \cdot \lim_{n\to\infty} \sqrt[n]{n} = \left|\frac{x+3}{4}\right|$$

Converge when: $\left|\frac{x+3}{4}\right| < 1$

$|x+3| < 4$

$\therefore R = 4$

**Example 5:** Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{2^n}{n}(4x-8)^n \right|}$$

$$\lim_{n \to \infty} \left| \frac{2}{\sqrt[n]{n}}(4x-8) \right|$$

$$|2(4x-8)| \cdot \lim_{n \to \infty} \left| \frac{1}{\sqrt[n]{n}} \right| = |2(4x-8)|$$

Converges when: $|8(x-2)| < 1$

$|x-2| < 1/8$

$\therefore R = 1/8$

**Example 6:** Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(-1)^n \cdot 3^n} = \sum_{n=1}^{\infty} \left(-\frac{x^2}{3}\right)^n$

Root

$$\lim_{n\to\infty} \left| \sqrt[n]{\frac{x^{2n}}{3^n}} \right|$$

$$\left|\frac{x^2}{3}\right| < 1$$

Ratio

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{x^{2n}} \right|$$

$$\left|\frac{x^2}{3}\right| < 1$$

Geometric : $|r| = \left|\frac{x^2}{3}\right|$

$$\left|\frac{x^2}{3}\right| < 1$$

$$|x^2| < 3$$

$$|x| < \sqrt{3}$$

$$\therefore R = \sqrt{3}$$

**Example 7:** Consider the function $f(x) = \ln(e + x)$.

(a) Write the first four non-zero terms and the general term of the Maclaurin series of $f(x)$.

(b) What is the radius of convergence?

(c) Use the Lagrange form of the remainder to show $\left|f\left(\frac{1}{2}\right) - P_4\left(\frac{1}{2}\right)\right| < \frac{1}{160}$.

(d) Use the first three terms of that series to estimate the value of $\int_0^1 \ln(e + x)dx$.

(a) $f(x) = \ln(e+x)$ $\quad f(0) = 1$

$f'(x) = (e+x)^{-1}$ $\quad f'(0) = 1/e$

$f''(x) = -(e+x)^{-2}$ $\quad f''(0) = -1/e^2$

$f'''(x) = 2(e+x)^{-3}$ $\quad f'''(0) = 2/e^3$

$$f(x) \approx 1 + \frac{1}{e}x + \frac{-1}{e^2}\cdot\frac{x^2}{2!} + \frac{2}{e^3}\cdot\frac{x^3}{3!} + \cdots$$

$$\approx 1 + \frac{x}{e} - \frac{x^2}{2e^2} + \frac{x^3}{3e^3} + \cdots + (-1)^{n+1}\frac{x^n}{ne^n}$$

(b) $\lim\limits_{n\to\infty} \sqrt[n]{\left|(-1)^{n+1}\frac{x^n}{ne^n}\right|}$

$\lim\limits_{n\to\infty} \left|\frac{x}{\sqrt[n]{n}\cdot e}\right|$

$\left|\frac{x}{e}\right| \cdot \lim\limits_{n\to\infty}\left|\frac{1}{\sqrt[n]{n}}\right| = \left|\frac{x}{e}\right|$

$\left|\frac{x}{e}\right| < 1$

$|x| < e$

$R = e$

(c) $\left|f(\tfrac{1}{2}) - P_4(\tfrac{1}{2})\right| \le \left|\frac{(1/2)^5}{5!}\cdot \max f^{(5)}(z)\right|$

$0 \le z \le \frac{1}{2}$ $\quad f^{(5)}(x) = 24(e+x)^{-5}$

max @ $x = 0$

$f^{(5)}(0) = \frac{24}{e^5}$

$\left|R_4(1/2)\right| \le \left|\frac{1}{2^5}\cdot\frac{1}{5\cdot\cancel{4!}}\cdot\frac{\cancel{24}}{e^5}\right|$

$\left|R_4(1/2)\right| \le \left|\frac{1}{160e^5}\right| < \frac{1}{160}$

(d) $\ln(e+x) \approx 1 + \frac{1}{e}x - \frac{1}{2e^2}x^2$

$$\int_0^1 \ln(e+x)dx \approx \int_0^1\left(1 + \frac{1}{e}x - \frac{1}{2e^2}x^2\right)dx = \left[x + \frac{x^2}{2e} - \frac{x^3}{6e^2}\right]_0^1 = 1 + \frac{1}{2e} - \frac{1}{6e^2}$$

**Example 8:** Given a function $g$ such that $g(3) = 1$ and $g^{(n)}(3) = \frac{(-1)^n n!}{(2n+1)2^n}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $g$ centered at x=3.

(b) Find the radius of convergence of the Taylor Series

(c) Show that the third-degree Taylor polynomial approximates *g(4)* to within 0.01.

(a) $g(3) = 1$

$g'(3) = \frac{-1 \cdot 1}{(3) \cdot 2} = \frac{-1}{6}$

$g''(3) = \frac{1 \cdot 2!}{5 \cdot 2^2} = \frac{1}{10}$

$g'''(3) = \frac{-1 \cdot 3!}{7\ 2^3} = \frac{-3}{28}$

$g(x) \approx 1 + \frac{-1}{6}(x-3) + \frac{1}{10}\frac{(x-3)^2}{2!} + \frac{-3}{28} \cdot \frac{(x-3)^3}{3!} + \dots + \frac{(-1)^n n!}{(2n+1) \cdot 2^n} \cdot \frac{(x-3)^n}{n!}$

$\approx 1 - \frac{1}{6}(x-3) + \frac{1}{20}(x-3)^2 - \frac{1}{56}(x-3)^3 + \dots + \frac{(-1)^n}{(2n+1) \cdot 2^n}(x-3)^n$

(b) $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(2n+3) \cdot 2^{n+1}} \cdot \frac{(2n+1) \cdot 2^n}{(-1)^n (x-3)^n} \right| = \left|\frac{x-3}{2}\right| \cdot \lim_{n\to\infty} \left|\frac{2n+1}{2n+3}\right| = \left|\frac{x-3}{2}\right|$

$\left|\frac{x-3}{2}\right| < 1$

$|x-3| < 2$

$R = 2$

(c) g(x) is represented by an alternating series. So, we can use alternating series remainder.

$|R_3(4)| \le \frac{(4-3)^4}{(2(4)+1) \cdot 2^4} = \frac{1}{144} < \frac{1}{100}$