Remainder of a Taylor Polynomial

An approximation technique becomes much more valuable when an idea of the accuracy of the approximation can be determined. To measure the accuracy of approximating the function $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a remainder $R_n(x)$, defined as follows:

$$\boldsymbol{f(x) = P_n(x) + R_n(x) \qquad \rightarrow \qquad R_n(x) = f(x) - P_n(x)}$$

- The error associated with the approximation can then be represented by:

$$\textbf{Error} = \boldsymbol{|R_n(x)| = |f(x) - P_n(x)|}$$

Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!} + R_n(x)$$

where

$$R_n(x) = f^{(n+1)}(z)\,\frac{(x-c)^{n+1}}{(n+1)!}$$

Between the center, c, and the x-value of the approx

- The estimate of the remainder is known as the **Lagrange form of the remainder.**
- A useful consequence of Taylor's Theorem is that

$$|R_n(x)| \le \left|\frac{(x-c)^{n+1}}{(n+1)!}\right| max\left|f^{(n+1)}(z)\right|$$

- When applying Taylor's theorem, we are not looking for the exact value of z. Rather, we are looking for bounds of $f^{(n+1)}(z)$, which tells how large the remainder $R_n(x)$ is.

Example 1: Find the third degree Maclaurin polynomial for sin *x*. Use Taylor's theorem to approximate $\boxed{\sin(0.1)}$ and determine the accuracy of the approximation.

$$\sin x \approx P_3(x) = x - \frac{x^3}{3!} = x - \frac{1}{6}x^3$$

$$\sin(0.1) \approx P_3(0.1) = 0.1 - \frac{(0.1)^3}{6}$$

$$= \frac{599}{6000}$$

$$= 0.0998\overline{3}$$

$$|f(0.1) - P_3(0.1)| \leq \left| \frac{(0.1)^4}{4!} \cdot \max f^{(4)}(z) \right|$$

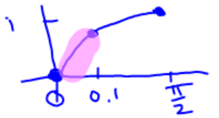
$$0 \leq z \leq 0.1$$

$$\leq \left| \left(\frac{1}{10}\right)^4 \cdot \frac{1}{24} \cdot 1 \right|$$

$$R_3(0.1) \leq \frac{1}{240{,}000}$$

$$0.00000416$$

$$f^{(4)}(x) = \sin x \rightarrow |\sin x| \leq 1$$



$$\sin(0.1) = 0.099833\overset{3}{\cancel{4}}166$$

$$-\ P_3(0.1) = 0.0998333333$$

$$0.0000000833$$

Example 2: Let f be the function $f(x) = \sqrt{x}$. Find the fourth degree Taylor polynomial about x = 4 for the function. Use Taylor's theorem to determine the accuracy of the approximation of *f(4.2)*.

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{-1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

$f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$

$f^{(5)}(x) = \frac{105}{32}x^{-9/2}$

$f(4) = 2$

$f'(4) = 1/4$

$f''(4) = -1/32$

$f'''(4) = 3/256$

$f^{(4)}(4) = -15/2048$

$$P_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1}{32}\cdot\frac{(x-4)^2}{2!} + \frac{3}{256}\cdot\frac{(x-4)^3}{3!} + \frac{-15}{2048}\,\frac{(x-4)^4}{4!}$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4$$

$P_4(4.2) = 2.049390137$

$$\left|f(4.2) - P_4(4.2)\right| \le \left|\frac{(0.2)^5}{5!}\cdot \max f^{(5)}(z)\right|$$

$4 \le z \le 4.2$

$$\le \left|\frac{1}{5^5}\cdot\frac{1}{120}\cdot f^{(5)}(4)\right|$$

$$\le \frac{1}{3125}\cdot\frac{1}{120}\cdot\frac{105}{16384}$$

$R_4(4.2) \le 0.0000000171$

$f(4.2) = 2.0493901\overset{4}{\cancel{5}}{}^{1}3$

$-\ P_4(4.2) = 2.049390137$

0.000000016

Example 3: Determine the degree of the Taylor polynomial expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than $\boxed{0.001}$.

$$f(x) = \ln x$$

$$f'(x) = x^{-1}$$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3}$$

$$f^{(4)}(x) = -3\cdot 2x^{-4}$$

$$f^{(5)}(x) = 4\cdot 3\cdot 2x^{-5}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$$

$$\left| f(1.2) - P_n(1.2) \right| \le \left| \frac{(0.2)^{n+1}}{(n+1)!} \cdot \max f^{(n+1)}(z) \right| < \frac{1}{1000}$$

$1 \le z \le 1.2$

MAX

$$\frac{1}{5^{n+1}} \cdot \frac{1}{(n+1)!}\left(\cdot \frac{n!}{z^{n+1}} \right) < \frac{1}{1000}$$

$$\frac{1}{5^{n+1}} \cdot \frac{1}{(n+1)\cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{1} < \frac{1}{1000}$$

$$\frac{1}{5^{n+1}(n+1)} < \frac{1}{1000}$$

$$1000 < 5^{n+1}(n+1)$$

$$200 < 5^{n}(n+1)$$

$$2 < n$$

Use a 3rd degree or higher

- **Example 4:** Approximate $\sqrt[3]{e}$ using a 4th degree Maclaurin polynomial. Find the associated Lagrange remainder (error bound).

$\downarrow$

$f(x) = e^x$ Approx. $f(1/3)$

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$P_4(1/3) = 1.395576132$

$e^{1/3} = 1.395612425$

$$|R_4(1/3)| \le \left|\frac{(1/3)^5}{5!} \cdot \max f^{(5)}(z)\right|$$

$\downarrow$

$0 \le z \le 1/3$ $f^{(5)}(x) = e^x$

$e^{1/3} < 8^{1/3} = 2$

$$\le \frac{1}{3^5} \cdot \frac{1}{120} \cdot 2$$

$$\le \frac{1}{14580} = 0.000068587$$

Example 5: Approximate $\tan(3/4)$ using a second degree Taylor polynomial centered at $x = \frac{\pi}{6}$. Find the associated Lagrange remainder (error bound).

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f(\pi/6) = \frac{\sqrt{3}}{3}$

$f'(\pi/6) = \frac{4}{3}$

$f''(\pi/6) = 2\left(\frac{4}{3}\right)\left(\frac{\sqrt{3}}{3}\right) = \frac{8}{9}\sqrt{3}$

$$P_2(x) = \frac{\sqrt{3}}{3} + \frac{4}{3}(x - \pi/6) + \frac{4}{9}\sqrt{3}(x - \pi/6)^2$$

$$P_2(3/4) = 0.9186766214$$

$$|R_2(3/4)| \le \left| \frac{(3/4 - \pi/6)^3}{3!} \cdot \max f^{(3)}(z) \right|$$

$\downarrow$

$\frac{\pi}{6} \le z \le \frac{3}{4}$

$f^{(3)}\left(\frac{3}{4}\right) < f^{(3)}\left(\frac{\pi}{4}\right) = 16$

22.293

$$\le \left| \frac{(3/4 - \pi/6)^3}{6} \cdot 16 \right|$$

$$\le 0.0309460370$$

2008 BC Question 3(Calculator Active)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

3. Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x - 2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$$

(b) $P_3(x) = 80 + 128(x - 2) + \dfrac{488}{6}(x - 2)^2 + \dfrac{448}{18}(x - 2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\displaystyle\max_{1.9 \le x \le 2} \left|h^{(4)}(x)\right| = \frac{584}{9}$.

Therefore,
$$\left|h(1.9) - P_3(1.9)\right| \le \frac{584}{9}\frac{|1.9 - 2|^4}{4!} = 2.7037 \times 10^{-4} < 3 \times 10^{-4}$$

$$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$$

2004 BC Question 6 (non-Calculator)

6. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(a)	$f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$ $f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$ $f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$ $P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$	4 : $P(x)$ $\langle -1 \rangle$ each error or missing term deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality
(b)	$\frac{-5^{22}\sqrt{2}}{2(22!)}$	$2 : \begin{cases} 1 : \text{magnitude} \\ 1 : \text{sign} \end{cases}$
(c)	$\left\lvert f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right\rvert \le \max\limits_{0 \le c \le \frac{1}{10}} \left\lvert f^{(4)}(c)\right\rvert \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$ $\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$	1 : error bound in an appropriate inequality