x	2	4
$f(x)$	7	13
$g(x)$	2	9
$g'(x)$	1	7
$g''(x)$	5	8

The table above gives selected values of twice-differentiable functions f and g, as well as the first two derivatives of g. If $\boxed{f'(x) = 3}$ for all values of x, what is the value of $\int_2^4 f(x)g''(x)\,dx$?

$u = f(x) \quad dv = g''(x)\,dx$

$du = f'(x)dx \quad v = g'(x)$

$$f(x)g'(x) - \int g'(x) f'(x)\,dx$$

$$f(x)\,g'(x) - 3\int g'(x)dx$$

$$\Big[f(x)g'(x) - 3g(x)\Big]_2^4 = 63$$

Integration by Parts (continued)

Sometimes when applying integration by parts, the functions that comprise the integrand require us to do more than just one process of integration. The examples below illustrate some of the difference situations that can arise when using integration by parts.

Example 1: Repeated Integration by Parts

$$\int x^2 \cdot e^x \, dx$$

$u = x^2 \qquad dv = e^x dx$

$du = 2x dx \qquad v = e^x$

$$= x^2e^x - 2\int x e^x dx$$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$

$$= \boxed{x^2e^x - 2xe^x + 2e^x + C}$$

- When making repeated use of integration by parts **do not interchange the substitutions** in successive applications.

Example 2: Tabular Method – Useful when one function **eventually differentiates to zero** and the other function **integrates forever.**

$$\int x^3 \cdot \sin x \, dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

	u	dv
+	x^3	$\sin x$
−	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
−	6	$\cos x$
+	0	$\sin x$

Example 3: Integration by Parts: Trig with Exponents

$$\int \sin^3 x\, dx = \int \sin x \sin^2 x \, dx$$

$u = \sin^2 x \qquad dv = \sin x \, dx$

$du = 2\sin x \cos x \, dx \qquad v = -\cos x$

$$= -\sin^2 x \cos x + 2\int \sin x \cdot \cos^2 x \, dx$$

$w = \cos x$

$dw = -\sin x \, dx$

$$= -\sin^2 x \cos x - 2\int w^2 \, dw$$

$$= \boxed{-\sin^2 x \cos x - \frac{2}{3}\cos^3 x + C}$$

Example 4: Solving for an Unknown Integral – Useful when both functions **differentiate/integrate forever.**

$$\int e^x \cdot \cos x \, dx$$

$u = e^x \quad dv = \cos x \, dx$

$du = e^x \, dx \quad v = \sin x$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$u = e^x \quad dv = \sin x \, dx$

$du = e^x \, dx \quad v = -\cos x$

$$= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right]$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2} \left(e^x \sin x + e^x \cos x \right) + C$$

Example 5:

$$\int e^{2x} \cdot \sin x \, dx$$

$u = \sin x \qquad dv = e^{2x}\,dx$

$du = \cos x\,dx \qquad v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx$$

$u = \cos x \qquad dv = e^{2x}\,dx$

$v = -\sin x \qquad v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx\right]$$

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\int e^{2x}\sin x\,dx = \boxed{\frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C}$$

$$\int e^{2x}\sin x\,dx$$

$u = e^{2x} \qquad dv = \sin x\,dx$

$du = 2e^{2x}\,dx \qquad v = -\cos x$

$$= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$

$u = e^{2x} \qquad dv = \cos x\,dx$

$du = 2e^{2x}\,dx \qquad v = \sin x$

$$= -e^{2x}\cos x + 2\left[e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right]$$

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$

$$5\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x$$

$$\int e^{2x}\sin x\,dx = \boxed{-\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + C}$$

Example 6: Integration by Parts: with a Trig Identity

$$\int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx = \boxed{\begin{aligned} &\int \sec x\,(\tan^2 x + 1)\,dx \\ = &\int \sec x \tan^2 x + \sec x\, dx \end{aligned}}$$

$u = \sec x \quad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx \quad v = \tan x$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

① U-SUB

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{x^2}{\sqrt{4-x^2}} \cdot x\,dx$$

$u = 4-x^2 \longrightarrow x^2 = 4-u$

$du = -2x\,dx$

$$= -\frac{1}{2}\int \frac{4-u}{\sqrt{u}}\,du$$

$$= -\frac{1}{2}\int 4u^{-1/2} - u^{1/2}\,du$$

$$= -\frac{1}{2}\left[8u^{1/2} - \frac{2}{3}u^{3/2} + C\right]$$

$$= -4(4-x^2)^{1/2} + \frac{1}{3}(4-x^2)^{3/2} + C$$

② PARTS

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int x^2 \cdot \frac{x}{\sqrt{4-x^2}}\,dx$$

$u = x^2 \quad dv = \frac{x}{\sqrt{4-x^2}}\,dx$

$du = 2x\,dx \quad v = -(4-x^2)^{1/2}$

$$= -x^2(4-x^2)^{1/2} + \int 2x(4-x^2)^{1/2}\,dx$$

$w = 4-x^2$

$dw = -2x\,dx$

$$= -x^2(4-x^2)^{1/2} - \int w^{1/2}\,dw$$

$$= -x^2(4-x^2)^{1/2} - \frac{2}{3}(4-x^2)^{3/2} + C$$