2013 Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $\boxed{h'(x)} = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

2013 Question 6 Scoring

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$ $P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$ $f'(0) \cdot \frac{1}{2} = 1$ $f'(0) = 2$	$2: \begin{cases} 1: \text{uses } P_1(x) \\ 1: \text{verifies } f'(0) = 2 \end{cases}$
(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$ $= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$	$3: \begin{cases} 1: \text{first two terms} \\ 1: \text{third term} \\ 1: \text{fourth term} \end{cases}$
(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$. $h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$ $Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$ $Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$ OR $h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$ $h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$ $\boxed{Q_3(x)} = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$	$4: \begin{cases} 2: \text{applies } h'(x) = f(2x) \\ 1: \text{constant term} \\ 1: \text{remaining terms} \end{cases}$

Taylor Approximations

Maclaurin polynomials are for estimating functions centered about $x = 0$. When we want to shift that estimating polynomial, we must apply a function shift to the variable x in the polynomial approximation.

Definition of n$^{\text{th}}$ Taylor Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + f'''(c)\frac{(x-c)^3}{3!} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!}$$

is called the **n$^{\text{th}}$ Taylor polynomial centered at c**.

Example 1: Find the fourth degree Taylor polynomials for $f(x) = \ln x$ centered at $c = 1$.

$f(x) = \ln x$ $\quad$ $f(1) = 0$

$f'(x) = x^{-1}$ $\quad$ $f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad$ $f''(1) = -1$

$f'''(x) = 2x^{-3}$ $\quad$ $f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad$ $f^{(4)}(1) = -6$

$$P_4(x) = 0 + 1\cdot(x-1) + -1\cdot\frac{(x-1)^2}{2!} + 2\cdot\frac{(x-1)^3}{3!} + -6\,\frac{(x-1)^4}{4!}$$

$$= (x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$n=1$ $\quad$ $n=2$

$$(-1)^{n+1}\frac{(x-1)^n}{n}$$

Example 2: Find the third degree Taylor polynomial for $f(x) = \sin(2x)$, expanded about $c = \frac{\pi}{6}$.

$f(x) = \sin(2x)$ $\qquad$ $f(\pi/6) = \frac{\sqrt{3}}{2}$

$f'(x) = 2\cos(2x)$ $\qquad$ $f'(\pi/6) = 1$

$f''(x) = -4\sin(2x)$ $\qquad$ $f''(\pi/6) = -2\sqrt{3}$

$f'''(x) = -8\cos(2x)$ $\qquad$ $f'''(\pi/6) = -4$

$$P_3(x) = \frac{\sqrt{3}}{2} + 1\cdot(x - \pi/6) + -2\sqrt{3}\,\frac{(x-\pi/6)^2}{2!} + -4\,\frac{(x-\pi/6)^3}{3!}$$

$$= \frac{\sqrt{3}}{2} + (x-\pi/6) - \sqrt{3}\,(x-\pi/6)^2 - \frac{2}{3}(x-\pi/6)^3$$

Example 3: Compute the fourth degree Taylor polynomial of $f(x) = \sqrt{1-x}$ centered at $c = -3$.

$f(x) = (1-x)^{1/2}$ $\qquad f(-3) = 2$

$f'(x) = -\frac{1}{2}(1-x)^{-1/2}$ $\qquad f'(-3) = \frac{-1}{4}$

$f''(x) = -\frac{1}{4}(1-x)^{-3/2}$ $\qquad f''(-3) = -\frac{1}{32}$

$f'''(x) = -\frac{3}{8}(1-x)^{-5/2}$ $\qquad f'''(-3) = -\frac{3}{256}$

$f^{(4)}(x) = \frac{-15}{16}(1-x)^{-7/2}$ $\qquad f^{(4)}(-3) = \frac{-15}{2048}$

$$P_4(x) = 2 + \frac{-1}{4}(x+3) + -\frac{1}{32}\frac{(x+3)^2}{2!} + \frac{-3}{256}\frac{(x+3)^3}{3!} + \frac{-15}{2048}\frac{(x+3)^4}{4!}$$

$$= 2 - \frac{1}{4}(x+3) - \frac{1}{64}(x+3)^2 - \frac{1}{512}(x+3)^3 - \frac{5}{16384}(x+3)^4$$

Example 4: Find the nth degree Taylor polynomial of $f(x) = \frac{1}{x+1}$ centered at $c = 2$.

$$f(x) = (x+1)^{-1} \qquad f(2) = \frac{1}{3}$$

$$f'(x) = -(x+1)^{-2} \qquad f'(2) = -\frac{1}{3^2}$$

$$f''(x) = 2(x+1)^{-3} \qquad f''(2) = 2 \cdot \frac{1}{3^3}$$

$$f'''(x) = -6(x+1)^{-4} \qquad f'''(2) = -6 \cdot \frac{1}{3^4}$$

$$f^{(4)}(x) = 24(x+1)^{-5} \qquad f^{(4)}(2) = 24 \cdot \frac{1}{3^5}$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = (-1)^n \cdot n!\,(x+1)^{-(n+1)} \qquad f^{(n)}(x) = (-1)^n \cdot n! \cdot \frac{1}{3^{n+1}}$$

$$P_n(x) = \frac{1}{3} + -\frac{1}{3^2}(x-2) + \cancel{2} \cdot \frac{1}{3^3} \frac{(x-2)^2}{\cancel{2!}} + \cancel{-6} \cdot \frac{1}{3^4} \frac{(x-2)^3}{\cancel{3!}} + \dots + (-1)^n \cancel{n!} \cdot \frac{1}{3^{n+1}} \cdot \frac{(x-2)^n}{\cancel{n!}}$$

$$P_n(x) = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 + \dots + (-1)^n \cdot \frac{(x-2)^n}{3^{n+1}}$$